





# Performance-Based Optimization Method for Structural Topology and Shape Design

by

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*In memory of my parents, who encouraged me to seek  
the best—the global optimum.*

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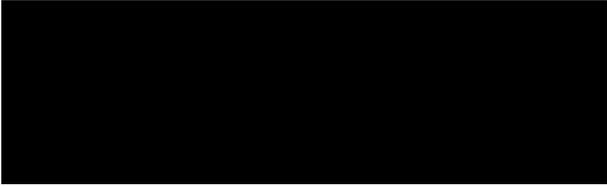
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## DECLARATION

This is to certify that the work presented in this thesis was carried out by the candidate, except where specifically indicated, and has not been submitted to any other university for any other degree.



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## LIST OF PUBLICATIONS

Based on this research work, the candidate has written the following papers, which have been published in international journals and conference proceedings.

### Journal Articles

1. Liang, Q. Q., Xie, Y. M. and Steven, G. P., "Optimal selection of topologies for the minimum-weight design of continuum structures with stress constraints," *Journal of Mechanical Engineering Science*, Proceedings of the Institution of Mechanical Engineers, UK, Part C, 1999, 213(C8), 755-762.
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## ABSTRACT

Continuum topology and shape optimization has received considerable attentions in recent years. However, no performance-based optimality criteria are used in existing continuum topology optimization methods to obtain the global optimum. In addition, existing methods mainly focus on theoretical aspects rather than practical applications. This thesis presents the systematic development of a Performance-Based Optimization (PBO) method for topology and shape design of continuum structures subject to stress, displacement and overall stiffness constraints, and its practical applications to structural engineering.

In the PBO method, the effective stress levels, virtual strain energy density and strain energy density of elements are proposed as element removal criteria for structures with stress, displacement and overall stiffness constraints, respectively. A set of performance indices is developed for evaluating the performance of structural topologies and shapes and used to monitor the optimization process. The maximization of performance indices is proposed as performance-based optimality criteria. These performance-based optimality criteria can be incorporated in any continuum topology optimization methods to obtain globally optimal designs.

The PBO method is extended to automatically generating optimal strut-and-tie models for the design and detailing of structural concrete, which includes reinforced and prestressed concrete structures. Moreover, the PBO method formulated on the basis of overall stiffness performance criteria is proposed for the optimal topology design of bracing systems for multistory steel building frameworks.

The PBO method is verified by existing analytical solutions as well as experimental evidence. It is shown that the proposed PBO method is a rational, efficient and reliable design tool for practicing engineers in structural design, especially in generating optimal strut-and-tie models in structural concrete and bracing systems for multistory steel building frameworks. The PBO method for strut-and-tie modeling would significantly improve the performance of concrete structures, and thus is suitable for inclusion in concrete model codes, such as the Asian Concrete Model Code and AS 3600.

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## NOTATIONS

|              |                                                                    |
|--------------|--------------------------------------------------------------------|
| $b$          | width of a member                                                  |
| $b_0$        | initial width of a structure                                       |
| $C$          | mean compliance or strain energy of a structure                    |
| $C^*$        | prescribed limit of the mean compliance imposed on a structure     |
| $\Delta C$   | change of strain energy of a structure due to element removal      |
| $c_e$        | strain energy of the $e$ th element                                |
| $D$          | depth of a structure                                               |
| $[D_e]$      | matrix of material elastic constants                               |
| $[D_e^s]$    | matrix of scaled material elastic constants                        |
| $E$          | Young's modulus of material                                        |
| $E_c$        | Young's modulus of concrete                                        |
| $E_e$        | current Young's modulus of the $e$ th element                      |
| $E_0$        | original Young's modulus of the $e$ th element                     |
| $f_c$        | compressive design strength of concrete                            |
| $f_c'$       | compressive cylinder strength of concrete at 28 days               |
| $f_{cm}$     | mean value of compressive strength of concrete at the relevant age |
| $f_{yp}$     | yield strength of steel tendons                                    |
| $f_{yr}$     | yield strength of steel reinforcement                              |
| $\{F_j\}$    | virtual unit load vector                                           |
| $[K]$        | stiffness matrix of a structure                                    |
| $[K_r]$      | stiffness matrix of the resulting structure                        |
| $[\Delta K]$ | change of stiffness matrix due to element removal                  |
| $[k_e]$      | stiffness matrix of the $e$ th element                             |
| $L$          | span of beam                                                       |
| $m$          | total number of displacement constraints                           |
| $n$          | total number of elements                                           |

|           |                                                                                 |
|-----------|---------------------------------------------------------------------------------|
| $p$       | total number of loading cases                                                   |
| $[P]$     | load vector                                                                     |
| $PI$      | performance index                                                               |
| $PI_{cp}$ | performance index for bending plates with mean compliance constraint            |
| $PI_{cs}$ | performance index for plane stress structure with mean compliance constraint    |
| $PI_{dp}$ | performance index for bending plates with displacement constraints              |
| $PI_{ds}$ | performance index for plane stress structures with displacement constraints     |
| $PI_{ss}$ | performance index for plane stress structures with stress constraints           |
| $Q^1$     | vector of the lowest strain energy density of elements under load case 1        |
| $Q^q$     | vector of the lowest strain energy density of elements under load case $q$      |
| $R$       | element removal ratio                                                           |
| $R_0$     | initial element removal ratio                                                   |
| $R_i$     | incremental element removal ratio                                               |
| $R_j$     | element removal ratio at the $j$ th steady state                                |
| $s_e$     | virtual strain energy of the $e$ th element                                     |
| $t_e$     | thickness of the $e$ th element                                                 |
| $t_e^s$   | scaled thickness of the $e$ th element                                          |
| $u_j$     | absolute value of the $j$ th constrained displacement                           |
| $u_j^*$   | prescribed limit of $u_j$                                                       |
| $u_{0j}$  | the $j$ th constrained displacement that is the most critical in initial design |
| $u_{ij}$  | the $j$ th constrained displacement that is the most critical in current design |

|                               |                                                                       |
|-------------------------------|-----------------------------------------------------------------------|
| $u_j^l$                       | the $j$ th constrained displacement under load case $l$               |
| $u_j^{*l}$                    | prescribed limit of $u_j^l$                                           |
| $\{u\}$                       | nodal displacement vector                                             |
| $\{\Delta u\}$                | change of displacement vector                                         |
| $\{u_e\}$                     | displacement vector of the $e$ th element under real loads            |
| $\{u_{ej}\}$                  | displacement vector of the $e$ th element under the virtual unit load |
| $\{u_j\}$                     | displacement vector of a structure under virtual unit load            |
| $V_i$                         | volume of the current design at the $i$ th iteration                  |
| $V_0$                         | volume of the initial design domain                                   |
| $v_e$                         | volume of the $e$ th element                                          |
| $W$                           | total weight of a structure                                           |
| $w_e$                         | weight of the $e$ th element                                          |
| $W_0$                         | actual weight of the initial design                                   |
| $W_0^s$                       | scaled weight of the initial design                                   |
| $W_i$                         | actual weight of the current design at the $i$ th iteration           |
| $W_i^s$                       | scaled weight of the current design at the $i$ th iteration           |
| $x_e$                         | design variable                                                       |
| $x_e^s$                       | scaled design variable                                                |
| $\sigma$                      | stress vector of elements                                             |
| $\sigma^s$                    | scaled stress vector of elements                                      |
| $\sigma^*$                    | prescribed stress limit                                               |
| $\sigma_{\max}^{\text{VM}}$   | maximum von Mises stress of an element in a structure                 |
| $\sigma_{i,\max}^{\text{VM}}$ | maximum von Mises stress of an element in the current design          |
| $\sigma_{0,\max}^{\text{VM}}$ | maximum von Mises stress of an element in the initial design domain   |
| $\varphi$                     | scaling factor                                                        |
| $\beta_j$                     | weighting parameter                                                   |

|             |                                                                                                  |
|-------------|--------------------------------------------------------------------------------------------------|
| $\nu$       | Poisson's ratio                                                                                  |
| $\rho$      | density of material                                                                              |
| $\zeta_e$   | virtual strain energy density of the $e$ th element                                              |
| $\zeta_e^m$ | virtual strain energy density of the $e$ th element under multiple constraints and loading cases |
| $\zeta_e$   | strain energy density of the $e$ th element under one loading case                               |
| $\zeta_e^j$ | strain energy density of the $e$ th element under the $j$ th loading case                        |

# Chapter 1

## INTRODUCTION

### 1.1 TOPOLOGY AND SHAPE OPTIMIZATION

The objective of structural optimization is to maximize the utilization of limited material resources. The increasing realization of the scarcity of raw materials and environmental considerations has demanded lightweight, low cost and high performance construction. In addition, the need to simplify the designer's tasks by automating the design process has become increasingly apparent with advances in high-speed computers. These growing demands have recently attracted much attention in the field of structural optimization. Topology and shape optimization is the selection of the best configurations for the design of structures. Since the topology and shape are changing during the optimization process, it is difficult to develop algorithms that produce optimal configurations. Therefore, topology and shape optimization is regarded as the most challenging design task. On the other hand, it is also considered as the most economically rewarding design task because it can result in much more material savings than the pure sizing optimization (Prager and Rozvany 1977; Kirsch 1993a).

Various structural optimization methods have been developed for the layout design of structures in the past few decades (Haftka and Gürdal 1992). These methods can be

classified into two categories, namely analytical methods and numerical methods. Analytical methods search for optimal configurations of structures using the mathematical theory of calculus and variational methods. They are suitable for studying the fundamental behavior of the material layouts of structural components and simple skeletal structures under a single loading. Analytical methods cannot be used to deal with the topology optimization of complex practical problems. Numerical methods generate optimal designs automatically in an iterative manner using mathematical programming and optimality criteria. Numerical methods can be used to solve large-scale practical design problems.

Structural optimization techniques are effective tools for improving the performance of structures in terms of the material efficiency in carrying applied loads. However, the performance of optimized designs is often limited to optimization methods used. It is of importance to realize that the formulation of a design problem in structural optimization significantly affects the results. Incomplete and improper problem formulation may lead to poor or meaningless designs. It is realistic to minimize the weight or cost of a structure subject to geometry and behavioral constraints, such as stress, displacement, mean compliance, frequency and buckling load constraints. This is because behavioral constraints are usually prescribed in the national design codes of practice (Rozvany et al. 1995). Unfortunately, some structural optimization methods use the behavioral quantity such as the compliance as the objective function and a somewhat arbitrarily chosen material volume as the constraint to search for optimal configurations. Optimization methods based on such a problem formulation may not result in minimum weight designs. The reason for this is that the arbitrarily specified material volume may not be the minimum value that is needed for supporting applied loads. Moreover, it is difficult for the designer to decide what percentage of the prescribed material volume should be sufficient to produce an optimum of a real world engineering design problem.

It is difficult to develop appropriate termination criteria that can be used in optimization algorithms to obtain the global optimum. The number of iterations or the percentage difference in the weight of a structure between the consecutive iterations has been used

as the termination criterion (Morris 1982; Seireg and Rodriguez 1997). The prescribed material volume has commonly been employed in topology optimization algorithms as the termination criterion (Bendsøe and Kikuchi 1988; Suzuki and Kikuchi 1991; Diaz and Bendsøe 1992; Tenek and Hagiwara 1993; Ramm et al. 1994; Gea 1996; Swan and Kosaka 1997; Youn and Park 1997; Krog and Olhoff 1999). In these approaches, using different amounts of material as the constraint can result in different locally optimal designs or even worst designs. Behavioral constraints have been employed in continuum topology optimization as the termination criterion, which is the only driving force for determining final results (Chu et al. 1996; Yang 1997). Moreover, no objective functions and constraints are used to control the optimization process in some topology optimization methods (Baumgartner et al. 1992; Xie and Steven 1993; Hinton and Sienz 1995; Zhao et al. 1998). The result satisfying these termination criteria mentioned above might not be the global optimum for a given design space. In addition, although these optimization methods can generate many topologies in the optimization process, it is impossible to decide which topology produced at a particular iteration is the optimum due to the lack of performance indicators. Therefore, there is a strong need to develop performance-based optimality criteria, which can be used to monitor the optimization process so that the global optimum can be identified from the optimization history.

There have been a large number of structural topologies and shapes in the literature produced by various structural optimization methods, such as those in the books by Bendsøe and Mota Soares (1993) and by Steven et al. (1998). However, it has been found that different optimization methods often provide different topologies and shapes for the same design problem. Unfortunately, little work has been undertaken to evaluate the performance of structural topologies and shapes generated by different optimization methods. Since there are no rational rules for measuring the efficiency of topologies and shapes and optimization methods used, it is difficult for the designer to select the best topologies and shapes for the design of engineering structures. Therefore, performance indices that objectively evaluate the efficiency of structural topologies and shapes could be extremely useful for design engineers and researchers.

Performance indices have been used to assist the selection of materials and geometry for the optimal design of structures by several researchers. The methodology for developing performance indices for assisting the selection of materials and cross-section shapes for mechanical components has been proposed by Ashby (1992). Weaver and Ashby (1996) have applied these performance indices to the optimal selection of materials and cross-section shapes for mechanical components. This approach has recently been extended by Burgess (1998a, 1998b) to the derivation of performance indices known as form factors for optimizing the structural layout of simply supported truss structures and beams under a single load with strength or stiffness constraints. Burgess has used these performance indices to compare the efficiency of the MBB beam obtained by various structural optimization methods. However, it is difficult to extend this approach to continuum structures because the objective function can no longer be expressed by the separable functional, geometrical and material parameter functions, as would be the case for single components and truss structures. An attempt to derive a performance index for measuring the efficiency of structural topologies has also been undertaken by Querin (1997). However, his performance index does not consider any behavioral constraints, and thus cannot be used for optimizing continuum structures with stress, displacement and stiffness constraints.

Recently, building codes of practice have been changing the focus from prescriptive to performance-based design approach, such as the Asian Concrete Model Code (ACMC) (ICCMC 1999). In the ACMC, the structural performance of a design is quantified by the performance index. As discussed previously, the goal of topology and shape optimization is often to improve the structural performance of the design in terms of the efficiency of material usage in carrying applied loads. Incorporating the performance-based design concept into the topology and shape optimization of continuum structures can overcome problems in continuum topology optimization methods. Therefore, the performance-based optimization (PBO) method developed in this thesis is of significant practical importance.

## 1.2 STRUCTURAL OPTIMIZATION IN CIVIL ENGINEERING

A study on the theoretical development and practice of structural optimization conducted by Cohn (1993) shows that over 150 books and 2,500 papers on structural optimization had been published since 1960. Most of the published work deals with mathematical aspects of structural optimization rather than practical applications. *There is a clear gap between the development of structural optimization theory and its practical application to aeronautical, mechanical and civil engineering industries*, as pointed out by Lev (1981), Ashley (1982), and Cohn (1993). One major reason for the gap between the theory and practice of structural optimization is the priority of mathematical over engineering aspects (Cohn and Dinovitzer 1994). The mathematical complexity of structural optimization methods is one of the main obstacles for structural designers to apply optimization techniques to practice even if they have the motivation to use them. Although simple topology optimization methods (Rodriguez and Seireg 1985; Mettheck and Burkhardt 1990; Yang and Chuang 1994; Xie and Steven 1997) for continuum structures have been developed recently, their incomplete problem formulation and the lack of performance indices prevent their practical applications to civil engineering.

The aeronautical, mechanical and automotive industries might have realized the potential of modern structural optimization techniques, but they are still viewed by civil engineers as an academic exercise, which is far from practical relevance. *Structural optimization techniques could become more attractive to practicing civil engineers if they are developed not only for saving materials but also for simplifying the designer's tasks by automating the major design process.*

A survey carried out by Cohn and Dinovitzer (1994) shows that steel is the most used material in published optimization examples due to its homogenous property, which is easiest to model in the elastic and plastic ranges. Concrete structures are less illustrated in the literature because the nonlinear nature of concrete such that it has a considerable strength in compression and a very low strength in tension. Moreover, reinforced

concrete is a composite material that causes difficulty in modeling for optimization. However, reinforced concrete is an economical construction material, which is widely used in the construction of large infrastructures. Although extensive research has been undertaken on the shear behavior of structural concrete, it is still difficult to understand due to its complex nature. Strut-and-tie modeling is a rational approach for the design of structural concrete (Marti 1985; Schlaich et al. 1987). However, conventional methods for developing strut-and-tie models in structural concrete usually involve a trial-and-error iterative process based on the designer's intuition and past experience. It is a challenging task for the designer to select an appropriate strut-and-tie model for a structural concrete member with complex geometry and loading conditions from many possible equilibrium configurations. Therefore, it is one of the objectives of this thesis to provide concrete designers with a rational, efficient and reliable design tool for automatically generating optimal strut-and-tie models in structural concrete.

It is clear that the gap between the development of structural optimization theory and its practical applications to civil engineering does not appear to have been reduced in the last two decades. *The challenge in structural optimization is to transform continuum topology optimization from an exotic and fruitless academic exercise into a rational and efficient design tool for practicing civil engineers.* The research work presented in this thesis is to answer this challenge.

### **1.3 AIMS OF THIS RESEARCH WORK**

The aims of this research work are given as follows:

- Develop performance indices for evaluating the performance of structural topologies and shapes subject to stress, displacement, and mean compliance constraints.
- Rank the efficiency of structural topologies and shapes generated by different structural optimization methods using performance indices.

- Develop performance-based optimality criteria that can be incorporated in any continuum topology optimization methods to obtain globally optimal designs.
- Develop the Performance-Based Optimization (PBO) method into an efficient and reliable tool for topology and shape design of continuum structures with stress, displacement and overall stiffness constraints.
- Develop the PBO method into an automated design tool for generating optimal strut-and-tie models for the design and detailing of structural concrete, which includes reinforced and prestressed concrete structures. Develop optimal strut-and-tie models in structural concrete for the Asian Concrete Model Code.
- Develop the PBO method for the topology design of bracing systems for multistory steel building frameworks.

## 1.4 LAYOUT OF THIS THESIS

*Chapter 2* presents a literature review on structural optimization. Research work on the truss topology optimization, and continuum topology and shape optimization is reviewed. The development of performance indices for structural optimization is highlighted.

The Performance-Based Optimization (PBO) method for the fully stressed topology design of continuum structures subject to maximum allowable stress constraints is presented in *Chapter 3*. In the proposed PBO method, element removal criteria are based on the effective stress levels of elements. The stress constraint is imposed on the maximum von Mises stress of elements. A performance index developed using the scaling design approach is used to monitor the optimization process, from which the optimal topology can be identified. Examples are provided to demonstrate the validity of the proposed method for fully stressed topology design.

*Chapter 4* describes the Performance-Based Optimization method for topology and shape design of continuum structures for displacement performance. Element removal

criteria are derived by undertaking a design sensitivity analysis on the change in constrained displacements due to element removal. Performance indices are developed for evaluating the performance of topologies and shapes for plane stress structures and for plates in bending. Maximizing the performance indices in the design space is proposed as performance-based optimality criteria. Performance indices developed are employed to compare the performance of structural topologies and shapes generated by different optimization methods. The effectiveness of the proposed method is shown through its applications to plane stress structures and bending plates under various loading conditions.

In *Chapter 5*, a Performance-Based Optimization approach is developed for the minimum-weight topology design of continuum structures for overall stiffness performance. The lowest strain energy density of elements is derived as element removal criteria. Performance indices are developed for optimization of plane stress continuum structures and bending plates. Performance-based optimality criteria are proposed as maximizing the performance indices in the optimization process. Examples are provided to show the validity of the method.

The PBO method is extended and proposed for automatically generating optimal strut-and-tie models in structural concrete in *Chapter 6*. The strut-and-tie modeling of structural concrete is transformed into the topology optimization problem of continuum structures. The PBO method for continuum structures with displacement constraints is used to develop optimal strut-and-tie models in reinforced concrete deep beams with openings, continuum beams, beams with various span-to-depth ratios and corbels. The PBO method is also utilized to investigate the effect of prestressing forces on optimal strut-and-tie models in nonprestressed, partially prestressed and fully prestressed concrete beams. Optimal strut-and-tie models in low-rise concrete shearwalls are developed by using the PBO method for continuum structures with mean compliance constraint. Results obtained are compared with existing analytical solutions and experimental observations.

*Chapter 7* presents the extension of the PBO method formulated on the basis of system performance criteria to optimal topology design of bracing systems for multistory steel building frameworks under multiple lateral loading conditions. The unbraced framework is initially designed by using commercial standard steel sections from databases under the strength constraints. The optimal topology of a bracing system for the multistory framework is then generated by gradually eliminating inefficient material from a continuum design domain that is used to stiffen the framework. Two design examples are provided to illustrate the efficiency of the design optimization procedure.

*Chapter 8* gives conclusions on the development of the Performance-Based Optimization method for topology and shape design of continuum structures with stress, displacement and overall stiffness constraints and on its applications to real world structural engineering problems. Significant achievements in this research work are summarized. Further research in this field is recommended.

## Chapter 2

### LITERATURE REVIEW

#### 2.1 INTRODUCTION

Structural optimization has been an important and active research area for over one hundred years. Modern structural optimization couples the finite element analysis with mathematical programming techniques or optimality criteria methods to search for optimal configurations for the layout design of structures. With advances in high-speed computers, modern structural optimization has the potential to become an automated design tool for practicing engineers in aeronautical, mechanical, automotive and civil engineering industries. Extensive research and development work has been undertaken in this multidisciplinary field during the last four decades. Several survey papers have appeared in the literature (Topping 1983; Ding 1986; Haftka and Grandhi 1986; Kirsch 1989; Topping 1993; Rozvany et al. 1995).

This chapter is to review the development of structural optimization. Although this thesis focuses on continuum topology and shape optimization, the early development of structural optimization for discrete structures will also be reviewed in order to understand important issues in structural optimization. Published work on the topology optimization of discrete structures will firstly be reviewed. Extensive reviews are then

devoted to the topology and shape optimization of continuum structures and associated problems. Finally, the study on the performance-based design concept for structural optimization is highlighted.

## 2.2 TRUSS TOPOLOGY OPTIMIZATION

Truss topology optimization is to select the best geometry of a truss, which has the minimum material for supporting the applied loads while satisfying certain constraints. The first analytical work on structural optimization was undertaken by Maxwell (1890). Michell (1904) developed the basic optimal layout theory for the minimum-weight design of trusses subject to stress constraints under a single load condition. Michell structures are statically determinate and impractical trusses that usually consist of an indefinitely large number of members. However, the significance of Michell's work is justified by the fact that it provides important insights into the optimal layouts of structural systems. In addition, Michell structures can be used as reference solutions for evaluating the efficiency of structural layouts obtained by using numerical methods.

Michell structures have extensively been studied by other researchers (Cox 1956; Hemp 1958,1973; Chan 1960; Owen 1975; Parkes 1974). The methods for graphical construction of strain fields have been proposed by Chan (1960). The superposition principle has been used to derive Michell trusses under two alternative load conditions (Hemp 1973; Nagtegaal and Prager 1973; Spillers and Lev 1971). Rozvany and Hill (1978) derived the Michell trusses and optimal grillages under four alternative loading conditions using the superposition concept. Prager (1974,1978) proposed techniques for evaluating the mass efficiency of near optimal trusses and modified design criteria for members of a discretized Michell structure.

To overcome the impracticalities of Michell structures, the ground structure approach was proposed for the topology optimization of trusses (Chan 1960; Hemp 1964; Dorn et al. 1964). In this approach, redundant members are removed from a highly connected

structure to achieve an optimal topology using a mathematical programming technique in an iterative process. The optimal design of a pin-jointed structure is obtained by maximizing the external virtual work while satisfying strain constraints. The investigation of the effects of ground structure grid on the optimal layout conducted by Dorn et al. (1964) indicates that the ground structure grid has a significant effect on the weight and topology of optimal structures. Their work leads to another optimization method for optimizing structural layouts where the joint coordinates and the member cross-sectional areas are treated as design variables.

The optimal layout theory has been developed as a generalization of Michell's work by Prager and Rozvany (1977), Rozvany (1976,1984), and Rozvany and Wang (1983). This theory deals with the layout optimization of gridlike structures. Optimal layouts for the minimum-weight design of gridlike structures can be determined by solving the primal problem or the dual problem. The basic concepts of Prager and Rozvany's layout theory are the Prager-Shield theory of optimal plastic design (Prager and Shield 1967) and the structural universe (Rozvany 1981,1984).

Nonlinear mathematical programming techniques have been used by Schmidt (1960) to solve the topology optimization problems of trusses with nonlinear inequality constraints under multiple loading cases. In this approach, the joint coordinates are treated as design variables and the minimum weight design of structures is stated as a nonlinear mathematical programming problem. This work is significant because it contributed an idea of combining the nonlinear mathematical programming with the finite element analysis to an optimization scheme so that optimal designs can be generated automatically. Modern structural optimization evolved from this early work.

Dobbs and Felton (1969) used a nonlinear programming algorithm to modify the member areas of an initial ground structure under the consideration of elastic compatibility and multiple loading conditions. Members whose cross-sectional areas are close to zero are deleted from the structure. Remaining members are modified until the optimum is reached. However, the deletion of members from the ground structure

cannot be proved theoretically. Moreover, members can only be deleted from the structure where buckling constraints are not considered. Using this approach, buckling constraints are imposed on the final optimal structure. Methods based on such an optimization process are heuristic and no mathematical formulation is given to obtain the global optimum.

Russell and Reinschmidt (1971) employed a two-step iterative optimization procedure to solve the optimal design problem of truss structures. In this approach, compressive limiting stresses are firstly estimated by assuming the slenderness ratios of members. After each linear programming solution, member sizes are modified and the required compressive limiting stresses are calculated on the basis of member forces obtained from linear programming results. The convergence of the solution can be improved by using commercial standard rolled steel sections from databases. This procedure was improved by Reinschmidt and Russell (1974) by considering compatibility, buckling and discrete rolled sections. The final design obtained by using this procedure is usually fully stressed but not necessarily minimum weight.

Majid and Elliott (1973a, 1973b) proposed a method for truss topology optimization using a ground structure. In this approach, the weight of a structure with stress, displacement and buckling constraints under multiple loading cases is minimized using a steepest descent-alternate mode algorithm. The structure only needs to be analyzed once in the whole optimization process using the formulated theorems of structural variation. The theorems and influence coefficients are applied to the sensitivity analysis, which provides information on which member should be removed from the ground structure. The theorems are also used to indicate the effect of member removal on the stability of the structure. This method is applicable only to small size structural systems because of its efficiency.

Barnes et al. (1977) and Topping (1978) applied the fully stressed design stress-ratio method to a highly connected ground structure in which many of the member cross-sectional areas are reduced to zero. For structures under one loading case with the same

stress constraints on tension and compression members, the final topology obtained using this approach will be statically determinate and will be the same as that generated by using linear programming methods. For structures under multiple loading cases and with different stress constraints, the final structural layout will be statically indeterminate.

Topping (1978) compared structural topologies obtained by using the fully stressed design stress-ratio method with those generated by linear programming techniques. The results show that final topologies produced by using the stress-ratio method are not always minimum weight designs. This is because only stress constraints but no objective function are considered in the optimization process in the stress-ratio method. However, the linear programming approach does not consider compatibility. This problem can be overcome by resizing the topology obtained by the linear programming technique using the stress-ratio method (Reinschmidt and Russell 1974; Topping 1978). This optimization procedure of resizing the structural layout obtained by the linear programming using the stress-ratio method is very useful in dealing with practical problems.

Saka (1980) presented a method for topology optimization of trusses with stiffness, stress and buckling constraints under multiple loading cases. In this method, member cross-sectional areas, nodal coordinates and joint displacements are treated as design variables. The direct differentiation is used to linearize the constraints. The results show that the efficiency of topologies obtained by this technique is better than that produced by methods not treating nodal coordinates as design variables.

Kirsch (1982) proposed an optimal design procedure for skeletal structures to reduce the number of exact analyses during the optimization process by using the high quality explicit approximations of the structural behavior and the scaling design approach. Explicit behavior models are introduced for the moving lines in the design space. The structural behavior is evaluated for each design in a given line that is generated by the mathematical programming technique. The feasible design path is traced by scaling the

current design with respect to the most critical constraint imposed on the structure. The optimal topology of the structure can be determined from the feasible design path. The scaling design approach can be applied to structural optimization to monitor the reduction of the weight of the structure when the stiffness matrix of the structure is the linear function of the design variables such as member cross-sectional areas (Morris 1982).

Ringertz (1985) proposed an optimization procedure for trusses with stress and displacement constraints under a single loading condition. After performing the structural analysis on the initial design, the structure is uniformly scaled based on the calculated stresses and displacements to obtain the feasible design. Member cross-sectional areas are then modified for the given stresses and displacements by using the linear programming algorithm. The nonlinear programming technique is employed to optimize the cross-sectional areas of members in the resulting topology. The main drawback of this optimization procedure is that the resulting topology is only optimal for the prescribed displacement limits and may not be the global optimum. This is because the method provides different locally optimal designs for different displacement limits.

Rozvany (1989) proposed a continuum-type optimality criteria (COC) method for optimizing large structural systems with stress and displacement constraints based on the earlier work of Rozvany and Ong (1986). This COC methodology was reformulated as the discretized continuum-type optimality criteria (DCOC) method by using the finite element formulation (Zhou and Rozvany 1992,1993). The COC/DCOC is shown to be efficient for large structural systems subject to stress and displacement constraints and have been extended to natural frequency, local buckling and system stability constraints. Optimality criteria methods in layout optimization are limited to topology optimization but not for geometry optimization that allows for movable nodal joints.

The simultaneous analysis and design (SAND) method has been applied to truss topology optimization by Bendsøe et al. (1991), Achtziger et al. (1992), Achtziger

(1993), Ben-Tal and Bendsøe (1993), and Bendsøe and Ben-Tal (1993). In the SAND approach, extremum requirements based on energy theorems are used to replace some of the analysis equations. Since the SAND method employs a single optimization algorithm that combines the extremum problem and cost minimization problem, it avoids the repeated analysis and thus is computationally highly efficient. However, the efficiency of the SAND method is currently limited to the compliance design problems. For large structural systems with stress and displacement constraints, the computational cost will be prohibitively high (Sankaranarayanan et al. 1992).

## 2.3 CONTINUUM TOPOLOGY AND SHAPE OPTIMIZATION

*Continuum topology and shape optimization is the selection of the best configurations for the design of continuum structures.* Continuum topology optimization allows for holes in the interior of a design domain to be created. On the other hand, continuum shape optimization only allows for inefficient material to be removed from the boundaries of a structure. The difficulty arising from topology and shape optimization of continuum structures is that topology and shape are changing during the optimization process.

The shape optimization problem of continuum structures has been solved by Zienkiewicz and Campbell (1973) using a sequential programming technique. Modern continuum topology and shape optimization evolved from this earlier work. Haftka and Grandhi (1986) have presented a survey on structural shape optimization in which the boundary variation method has extensively been used. The boundary variation method is implemented by using the mesh moving schemes to express the shape of a given design. The coordinates of nodal points of the finite element model are treated as design variables. Special techniques for maintaining the regularity of the finite element model are required to obtain a sound optimal shape in the boundary variation method (Kikuchi et al. 1986). The sensitivity analysis for shape optimization has been studied by Rousselet and Haug (1983) and by Haug et al. (1986).

In shape optimization using the boundary variation method, the finite element model is changing during the optimization process so that remeshing the model is required at each iteration. To avoid these, Bendsøe and Kikuchi (1988) proposed a homogenization-based optimization (HBO) method for the topology and shape design of continuum structures using a fixed design domain. In the HBO method, topology and shape optimization is transformed to a material redistribution problem using composite material with microstructures. Effective material properties are computed using the theory of homogenization. For the stiffness design, the mean compliance of a structure is used as the objective function while the constraint is imposed on the material volume. Further development of the HBO method was undertaken by Bendsøe (1988,1989), Suzuki and Kikuchi (1991) and Thomsen (1991). The HBO method has been extended by Diaz and Bendsøe (1992) and Bendsøe et al. (1995) for the topology design of continuum structures under multiple loading conditions. Tenek and Hagiwara (1993) applied the HBO method to plates in bending. The dynamic problems of continuum structures were solved by Diaz and Kikuchi (1992), Ma et al. (1993,1995) and Krog and Olhoff (1999) using the homogenization-based design concept. The HBO method gives a rigorous mathematical formulation for the topology design problem, but its mathematical complexity may be beyond the understanding of practicing engineers. The main drawback of the HBO method is that the final design depends on a somewhat arbitrarily chosen material volume, which is the only constraint. As a result of this, the topology optimized by the HBO method is only “optimal” for the specified material volume and it does not mean that it is a minimum weight design.

Oda (1977) proposed a geometric approach to two-dimensional shape optimization by the utilization of the finite element analysis. No formal mathematical optimization algorithm is used in the geometric approach. The shape is modified based on the stresses obtained from the results of the finite element analysis. The cycle of finite element analysis and shape modification is repeated until the optimal shape is obtained. Oda and Yamazaki (1977,1979) extended this approach to problems of axisymmetric solids and under body forces. Umetani and Hirai (1978) presented a growing-reforming procedure

for shape optimization of structural beams with displacement constraints under multiple loading conditions.

Rodriguez and Seireg (1985) developed a rule-based approach for topology and shape optimization of continuum structures. The objective of this approach is to seek the optimal shape that maximizes the utilization of the material and with the most uniform stress distribution without violating the maximum allowable stress and the continuity of the shape. This is achieved by using an elimination scheme in which elements with relatively low magnitude of stress are removed from the design domain in an iterative manner after the finite element analysis. Wu (1993) extended the rule-based approach to topology optimization of two-dimensional continuum structures under dynamic loading. The objective function is defined as the ratio of volume reduction normalized to the initial volume to the stress range normalized to the allowable stress (Seireg and Rodriguez 1997). In the rule-based optimization approach, the final topology is determined when no further improvement is possible or the prescribed percentage of remaining elements is reached or the objective function is maximized or the prescribed maximum number of iterations is executed. Obviously, the main drawback of the rule-based optimization approach is that none of these termination criteria can guarantee the optimum is obtained.

A zero-one discrete variable optimization program for the topology and shape design of continuum structures with stress, displacement and stiffness constraints under multiple loading conditions has been developed by Atrek (1989) by utilizing the element removal concept. This program is capable of removing material from inside the design domain as well as from immediate boundaries. Only the most critical constraint imposed on the structure is considered in a given time in deriving the optimal shape. Stress constraints are applied on the von Mises equivalent stresses of elements. The zero-one decision-making scheme has also been used for topology design by Bendsøe (1989) and Rozvany et al. (1991,1992, 1994).

Mattheck and Burkhardt (1990) proposed a computer-aided shape optimization (CAO) method for reducing the notch stresses and lightweight design of structural components based on biological growth. The CAO seeks the optimal design with a constant von Mises stress at the surface of a growing structure (Mattheck 1998). Baumgartner et al. (1992) presented the soft kill option (SKO) approach for topology optimization of continuum structures by simulating adaptive bone mineralization. In this method, the Young's modulus is treated as design variables. The design domain is firstly analyzed by undertaking a finite element analysis, which provides von Mises stress distribution in the domain. The local E-modulus is then set equal to the stress computed at the particular place. This means that the more highly loaded region becomes harder, and the less loaded region becomes softer. The cycle of the finite element analysis and E-modulus redistribution is repeated in an iterative process. Consequently, the actual load-bearing region is characterized by the variation in its modulus and the non-load-bearing region can be removed from the design domain. The limitation of this method is that it does not involve objective function and stress constraints in the optimization process. Therefore, the minimum weight design cannot be guaranteed without a performance index as an indicator of material efficiency.

Xie and Steven (1993,1994a) proposed an evolutionary structural optimization (ESO) procedure for topology and shape design of continuum structures based on the fully stressed design and element removal concepts. In this approach, by gradually removing lowly stressed elements from the design domain after each finite element analysis, the remaining structure evolves towards an optimum. The ESO method has been extended to structural frequency optimization of continuum structures by Xie and Steven (1994b, 1996, 1997). The frequency of a structure can be shifted towards a desired direction by removing part of the material from the design based on the sensitivity analysis. The ESO method is also a discrete variable optimization approach, which involves a zero-one decision-making scheme. The limitation of the ESO method is that the final design is determined when the prescribed number of iterations is reached or the specified amount of material that allows to be removed from the design is reached. This is similar to the volume constraint used in the homogenization-based optimization method. Moreover,

no objective function and constraints are used in the evolutionary optimization process. Therefore, it is difficult for the designer to decide which topology generated in the evolutionary path is the optimum.

Chu et al. (1996) extended the ESO method for continuum structures with displacement constraints. The change of constrained displacements due to element removal is used as the sensitivity number for element elimination. Elements with the lowest sensitivity numbers are gradually removed from the design domain to obtain the maximum stiffness design. The optimization process is terminated when one of the constrained displacements reaches the prescribed limit. The only criterion used to derive the optimum is the displacement constraint. It should be noted that for two-dimensional continuum structures, displacement limits could be easily satisfied by uniformly changing the thickness of elements. The drawback of this optimization procedure is that no criterion is used to obtain the global optimum for the given design space.

The ESO method has been extended by Querin et al. (1998) to a so-called bi-directional evolutionary structural optimization (BESO) method based on the idea of adding and removing elements. Further application of the BESO for stiffness optimization is given by Yang et al. (1998). In the BESO approach, the material property number of all elements in the ground structure is firstly set to zero. This means that these elements are still stored in the data file but they do not physically exist as part of the structure and the finite element model for the solution. The initial design is constructed by the designer, who has to use necessarily elements one by one to connect applied loads and supports. After elements are added to the initial design up to a specified percentage of elements (e.g. 50%) referred to the physically non-existing ground structure, elements are removed and added simultaneously until the termination condition is satisfied. The original idea of the BESO method is to reduce the computational cost of the ESO method. However, it will take more time to obtain the solution using the BESO than the ESO for some cases since the designer has to spend the time on setting up the initial design, which has to grow up big enough to be “killed”. In addition, the adding-and-

removing process is actually a forward-and-backward process. This method may not work well for practical structures with complex geometry and loading conditions.

The density function approach for continuum topology optimization has been proposed by Mlejnek and Schirmacher (1993) and Yang and Chuang (1994). This approach uses the material density of each finite element as the design variable to solve the topology optimization problem. The effective material properties are computed by using the assumed relationships between the material density and Young's modulus without considering their microstructures. Yang (1997) extended the density function approach to general topology optimization problems where compliance, displacements and natural frequencies are treated as constraints. The shortcoming of this approach is that it gives no theoretical proof of the relationships between the material density and property. In addition, the constraint is the only driving force in determining the final design, which may not be the global optimum. Gea (1996) presented a microstructure-based design domain method, which provides a closed-form expressions for the effective Young's modulus and shear modulus in terms of phase properties and volume fractions. However, the method by Gea uses material volume as the constraint to determine the final design.

Swan and Kosaka (1997) presented a continuum topology optimization method for the minimum compliance problem of linearly elastic structures. In this approach, the effective material properties of the mixtures without microstructure are calculated using the classical Voigt-Reuss mixing rule. The limitation of this method is that it cannot guarantee a global optimum since the material volume is used as the constraint.

## **2.4 PERFORMANCE INDICES FOR STRUCTURAL OPTIMIZATION**

Performance indices have been used to assist the optimal selection of structural topologies and shapes by several researchers. Boiten (1963) used a performance index to

optimize the energy storage devices. A methodology for deriving performance indices for the selection of the material and section shapes of structural components has been proposed by Ashby (1992). In his method, the performance of a structural component is expressed by the objective function, which can usually be described by the separable functional, geometrical and material property functions. Design variables such as cross-sectional areas are eliminated by substituting the constraint equation into the objective function. The optimal selection of the shape design of a structural component is independent of the functional requirements and material used. Therefore, shape optimization can be undertaken without solving the whole objective function or knowing all details of functional and material parameters in advance. Minimizing the weight of a structural component is achieved by maximizing the geometrical parameter function. The performance index can be obtained from the group of geometrical parameters. This procedure is efficient for deriving performance index for optimizing the shapes of structural components and provides an important insight into the development of performance indices for evaluating the efficiency of structural topologies and shapes. Weaver and Ashby (1996) have used this approach to select material and section shapes.

Burgess (1998a, 1998b) extended the approach outlined by Ashby (1992) to derive performance indices known as form factors for ranking the mass efficiency of structural layouts for simply supported trusses and beams with stiffness and strength constraints under a single load. For trusses and beams, the mass is expressed by the separable constraint, geometrical and material parameters. Burgess found that the mass of a skeletal structure for strength design is proportional to the applied load, the span and the ratio of material density to the material ultimate strength. The mass is inversely proportional to the performance index, which can be used to calculate the minimum mass required for the optimal material layout. The performance index is a dimensionless number that depends on the topology of the structure. Burgess has also employed such a performance index to evaluate the efficiency of the MBB beam produced by different structural optimization methods. However, it is difficult to extend this approach to optimization of truss-like structures under multiple loading cases and discretized

continuum structures. This is because the objective function can no longer be expressed by the separable functional, geometrical and material parameter functions, as would be the case for structural elements and trusses under a single load.

Querin (1997) presented a performance index for measuring the efficiency of structural topologies and shapes for any type of structures. This performance index was derived from the Michell type pin-jointed framework based on the fully stressed design concept. However, this performance index is only valid for fully stressed trusses with a single span under a single load condition and can be simply derived from the objective function for strength design given by Burgess (1998b). The mathematical formulation of the performance index for discretized continuum structures is incorrect. In addition, this performance index does not consider any type of constraints. Therefore, it cannot objectively measure the efficiency of topologies and shapes for truss structures under multiple loading conditions and for discretized continuum structures.

The ESO method presented by Zhao et al. (1998) is based on the strain energy principle. In this approach, which element should be removed for the design is determined by the contribution of the strain energy within the element. This method does not involve any objective function and constraint in the optimization process. An indicator defined using the work done by the external loads is employed in the ESO to indicate the efficiency of material layouts for continuum structures. However, this indicator does not take any constraint into account so that its application is very limited.

## Chapter 3

# TOWARDS FULLY STRESSED TOPOLOGY DESIGN

### 3.1 INTRODUCTION

In the design of aeronautical, mechanical and civil engineering structures, the engineer needs to specify the exact topology of a structure for a given design space and loading condition. In the absence of an efficient topology design tool, the selection of topology in current design practice usually involves a trial-and-error process based on the designer's intuition and previous experience. The automation of the design process and optimal designs are motivated by the considerations of limited material resources, technological competition and environmental issues. Structural topology optimization, which is the selection of the best topology for the design of a structure, is an important part of the design process. Continuum topology optimization has recently attracted considerable attentions in the field of structural optimization. It has the potential to become an automated design tool for practicing engineers.

The fully stressed design has traditionally been used as one of the optimality criteria for the optimal design of skeletal structures. However, the fully stressed design procedure may not lead to minimum-weight designs since no objective function is involved in the optimization algorithm. Due to its simplicity and fast convergence, the fully stressed

design is still used in other optimization procedures as a starting point for searching for optimal designs. The fully stressed design concept has been adopted in topology and shape optimization of continuum structures. Oda (1977) presented the two-dimensional shape optimization method in which the shape is modified based on the stresses of elements obtained from the finite element analysis. The rule-based approach by Rodriguez and Seireg (1985) seeks optimal topology designs with the most uniform stress distribution at minimum weight by using the element elimination concept. Baumgartner et al. (1992) proposed topology optimization as the Young's modulus distribution problem based on local stress levels. Moreover, Xie and Steven (1993) presented an evolutionary optimization procedure, which utilizes the fully stressed design and element removal concepts. However, these stress-based continuum topology optimization approaches suffer the same problem as the fully stressed design does because no performance-based optimality criteria are used to obtain globally minimum-weight designs.

This chapter presents a Performance-Based Optimization (PBO) method for fully stressed topology designs of two-dimensional linearly elastic continuum structures. In the proposed approach, the finite element method is used as the modeling and analytical tool for calculating stresses of elements. The performance objective is to seek fully stressed topology designs with minimum consumption of material and acceptable stress levels. The performance-based design concept is incorporated in continuum topology optimization. A performance index is proposed to monitor the optimization process and used as a termination criterion. Maximizing the performance index is proposed as performance-based optimality criteria. Some of the results have recently been reported in the work of Liang et al. (1999a, 1999b).

### **3.2 PERFORMANCE-BASED CONCEPT AND OBJECTIVE**

The performance-based design has become a popular design concept in structural engineering profession in recent years. The structural design codes of practice in many

countries are currently changing from prescriptive specifications to performance-based provisions for technical, economical, social and environmental reasons. The performance-based design is to design a structure or structural component that can perform physical functions in a specified manner throughout its design life. The intent is to provide owners and designers with the capability to select different performance objectives for different structures. Performance objectives are qualitatively expressed by non-engineering terms, which can be easily understood by the owners and community.

*The performance-based optimal design is to design a structure or structural component that can perform physical functions in a specified manner throughout its design life at minimum cost or weight.* The cost-performance objective is of practical importance, but it is usually difficult to construct an appropriate cost-objective function that depends on many parameters. Therefore, the minimum weight for required design specifications is frequently used as an objective function in structural optimization since it is readily quantified. The advantages of minimum-weight structures are low material cost, high technical performance and low environmental impact.

To be a minimum-weight design, all parts of a structure should be loaded equally and safely. This means that the design should be fully stressed within the maximum allowable stress level. It has been found that biological components always grow into a state of constant stress on their surface (Mattheck and Burkhardt 1990). Therefore, the performance objective of topology design for strength is to seek the fully stressed design at minimum weight while satisfying allowable stress constraints. This can be expressed in the mathematical form as follows:

$$\text{minimize } W = \sum_{e=1}^n w_e(t) \quad (3.1)$$

$$\text{subject to } \sigma_{\max}^{\text{VM}} \leq \sigma^* \quad (3.2)$$

where  $W$  is the total weight of the structure,  $w_e$  is the weight of the  $e$ th element,  $t$  is the thickness of all elements,  $n$  is the total number of elements,  $\sigma_{\max}^{\text{VM}}$  is the maximum von

Mises stress of an element in the structure under applied loads and  $\sigma^*$  is the maximum allowable stress.

The stresses of elements are localized and highly nonlinear with respect to the changing topology in an optimization process. The maximum stress may shift from element to element in the optimization process. This leads to difficulty in imposing the maximum stress constraint on a particular element. To simplify the formulation, only the global maximum stress constraint on the von Mises stress is considered in the proposed method. In order to achieve the performance objective, underutilized elements should be removed from the discretized structure. Hence, every element in a structure is treated as a design variable. The element thickness has a significant effect on the structural weight as well as the state of stress in elements because the stiffness matrix of a plane stress continuum structure is a linear function of its thickness. In the design of a plane stress structure, its thickness needs to be specified by the designer. Therefore, the thickness is also treated as one of the design variables. However, the simultaneous topology and sizing optimization of continuum structures will be very complicated and computationally highly expensive. As a result of this, only the uniform sizing of element thickness is considered in the proposed method.

### **3.3 ELEMENT REMOVAL CRITERIA BASED ON EFFECTIVE STRESS LEVEL**

Topology optimization of continuum structures is the most complicated problem in structural optimization. To solve the topology optimization problem, an initial design domain is usually used as a starting point for deriving the optimum (Bendøe and Kikuchi 1988; Xie and Steven 1993). The design domain concept is similar to the ground structure concept used in truss topology optimization. In the design domain approach, a design domain without violating any geometric constraints is discretized into fine finite elements. Under applied loads, it is found that the stress distribution of elements in the design domain is not uniform. This means that some of the elements are

not effective in carrying loads. Thus, these lowly stressed elements should be eliminated from the design domain so as to achieve the performance objective.

The equivalent stress of an element that represents its stress level in plane stress conditions can be evaluated by using the von Mises stress criteria for isotropic materials. Different element elimination criteria can be used to define the standard for elimination. In the rule-based approach by Rodriguez and Seireg (1985), elements that possess the von Mises stress values below a certain level of the average stress of elements are removed from the structure. By implying this criterion, the efficiency of a structure can be gradually improved. However, since the average stress of elements is used for elimination, the stress distribution in the final topology is still not uniform when no more elimination is possible. Further modification is often needed in order to achieve a better design. The maximum von Mises stress of elements in a continuum design domain can also be employed as criteria for element removal (Xie and Steven 1993). Element removal criteria can be expressed by

$$\sigma_{i,e}^{\text{VM}} < R_j \sigma_{i,\max}^{\text{VM}} \quad (3.3)$$

where  $\sigma_{i,e}^{\text{VM}}$  is the von Mises stress of the  $e$ th element at the  $i$ th iteration,  $\sigma_{i,\max}^{\text{VM}}$  is the maximum von Mises stress of an element in the structure at the  $i$ th iteration and  $R_j$  is the *element removal ratio* at the  $j$ th steady state. All elements that satisfy Eq. (3.3) are removed from the structure. The cycle of the finite element analysis and the element removal is repeated by using the same  $R_j$  until no more elements can be removed from the structure at the current state. In order to continue the optimization process, the element removal ratio  $R_j$  is increased by an *incremental removal ratio* ( $R_i$ ). The element removal ratio can be expressed by

$$R_j = R_0 + (j - 1)R_i \quad (j = 1, 2, 3, 4, \dots) \quad (3.4)$$

where  $R_0$  is the *initial removal ratio*.

The optimal topology of a continuum structure under one loading case is iteratively generated by using element removal criteria described in Eq. (3.3). For structures subject to multiple loading cases, only those elements that satisfy Eq. (3.3) for all load cases are removed from the design domain at each iteration. This criterion for multiple loading cases generates an optimal design that can perform the required function under all loading conditions.

The optimal design can also be generated by gradually eliminating a small number of elements with the lowest von Mises stress from a continuum design domain. A loop can be set up to count these lowly stressed elements until they make up the prescribed amount, which is the element removal ratio times the total number of elements in the initial design domain. The design is iteratively modified by removing these lowly stressed elements at each iteration until the optimum is obtained. It should be noted that the number of elements to be removed at each iteration must be sufficiently small in order to achieve a smooth solution. The elimination of a large number of elements from a design domain may cause discontinuity and the model may become singular. The initial element removal ratio  $R_0 = 1\%$  and incremental removal ratio  $R_i = 1\%$  are found to be typical for use in engineering practice.

### 3.4 ELEMENT ELIMINATION TECHNIQUE

The topology design of a continuum structure is to determine the optimal layout of a given isotropic material in the design space. It needs to determine which regions should be filled with material and which regions should be void. Thus, the design problem becomes a discrete *zero-one* problem, to which there are no direct solutions. The methodology presented here is to search for the optimal solution using the design/redesign scheme in an iterative manner. The topology of a continuum structure is modified by gradually removing underutilized portions from the design domain. In the optimization algorithm, the state of an element in the model is represented by the binary integer *zero or one*. The integer *zero* represents that the element is deleted from the

model whilst the integer *one* indicates that the element is remained in the model. In the structural analysis, the state of an element is represented by the Young's modulus as follows:

$$E_e = \begin{cases} E_0, & \text{if } \sigma_{i,e}^{\text{VM}} \geq R_j \sigma_{i,\max}^{\text{VM}} \\ 0, & \text{if } \sigma_{i,e}^{\text{VM}} < R_j \sigma_{i,\max}^{\text{VM}} \end{cases} \quad (3.5)$$

where  $E_e$  is the current Young's modulus of the  $e$ th element and  $E_0$  is the original Young's modulus of the  $e$ th element. The elimination of elements from a design domain can be done by setting their material property number to zero. These deleted elements are not assembled in the global stiffness of the structure.

### 3.5 PERFORMANCE-BASED OPTIMALITY CRITERIA

Structural topology is changing during the optimization process. The performance of the resulting topology at each iteration needs to be assessed in order to obtain the optimum. In the performance-based design, the performance of a design is quantified by using the performance index (*PI*). Structural responses such as stress and displacement are used as performance indices to evaluate the structural performance of a design. However, it is not sufficient to use structural responses alone as performance indices for evaluating the performance of optimized designs. The performance objective of topology optimization is to minimize the weight of a structure while its structural responses are maintained within acceptable limits. Therefore, the minimum material that can support applied loads without violating behavioral constraints should be used as a measure of the performance of optimal designs.

The method presented by Burgess (1998a, 1998b) for deriving performance indices is only valid for simple trusses and beams under a single load. It is difficult to extend this approach to optimize truss structures under multiple load cases and discretized

continuum structures because the objective function can no longer be expressed by the separable functional, geometrical and material parameter functions. Moreover, the form factors given by Burgess can only be used to evaluate the efficiency of trusses with a single span under a single load as the load and span are present in the objective function. In the present study, a methodology based on the scaling design concept is proposed for developing performance indices, which can be used to evaluate the performance of structural topologies and shapes with various constraints.

The scaling design concept has been used in structural optimization after each iteration to obtain the best feasible constrained design (Kirsch 1982; Morris 1982). The advantages of scaling the design are that it can trace the history of the reduction in the weight of a structure after each iteration and pick the most active constraints. This method can be applied to structural optimization when the stiffness matrix of a structure is a linear function of design variables. By scaling the design, the scaled design variable is expressed by

$$x_e^s = \varphi x_e \quad (3.6)$$

in which  $x_e^s$  is the scaled value of the design variable such as the element thickness of the  $e$ th element,  $\varphi$  is a scaling factor that is same for all elements, and  $x_e$  is the actual design variable of the  $e$ th element. The force-displacement relationship in the finite element formulation can be written as

$$\frac{1}{\varphi} [K^s] \{u\} = \{P\} \quad (3.7)$$

where  $[K^s]$  is the stiffness matrix of the scaled structure and is calculated by using the scaled design variable  $x_e^s$ . The equilibrium equation for the scaled design can be expressed in terms of the scaled design variable by

$$[K^s]\{u^s\} = \{P\} \quad (3.8)$$

From Eqs. (3.7) and (3.8), one obtains

$$\{u^s\} = \frac{1}{\varphi}\{u\} \quad (3.9)$$

From the expressions of the strain-displacement and stress-strain relations in terms of the scaled design variable, the scaled stress vector can be derived as

$$\{\sigma^s\} = \frac{1}{\varphi}\{\sigma\} \quad (3.10)$$

in which  $\{\sigma\}$  is the stress vector of elements. Obviously, in order to satisfy the stress constraint imposed on a structure, the actual design needs to be scaled by

$$\varphi = \frac{\sigma_{\max}^{\text{VM}}}{\sigma^*} \quad (3.11)$$

where  $\sigma_{\max}^{\text{VM}}$  is the maximum von Mises stress of an element in the structure and  $\sigma^*$  is the prescribed stress limit.

For linear elastic plane stress problems, the stiffness matrix of a structure is a linear function of the element thickness that is one of the design variables. By scaling the design with respect to the maximum allowable stress constraint, the scaled weight of the initial design domain can be represented by

$$W_0^s = \left( \frac{\sigma_{0,\max}^{\text{VM}}}{\sigma^*} \right) W_0 \quad (3.12)$$

in which  $W_0$  is the actual weight of the initial design domain and  $\sigma_{0,\max}^{\text{VM}}$  is the maximum von Mises stress of an element in the initial design domain under applied loads. In an iterative optimization process, the scaled weight of the current design at the  $i$ th iteration can be expressed by

$$W_i^s = \left( \frac{\sigma_{i,\max}^{\text{VM}}}{\sigma^*} \right) W_i \quad (3.13)$$

where  $W_i$  is the actual weight of the current design at the  $i$ th iteration and  $\sigma_{i,\max}^{\text{VM}}$  is the maximum von Mises stress of an element in the current design at the  $i$ th iteration.

The performance index for evaluating the efficiency of the resulting topology at the  $i$ th iteration is proposed as

$$PI_{ss} = \frac{W_0^s}{W_i^s} = \frac{(\sigma_{0,\max}^{\text{VM}} / \sigma^*) W_0}{(\sigma_{i,\max}^{\text{VM}} / \sigma^*) W_i} = \frac{\sigma_{0,\max}^{\text{VM}} W_0}{\sigma_{i,\max}^{\text{VM}} W_i} \quad (3.14)$$

If the material density is uniformly distributed within the structure, the performance index can be written in terms of the volume of the structure as

$$PI_{ss} = \frac{\sigma_{0,\max}^{\text{VM}} V_0}{\sigma_{i,\max}^{\text{VM}} V_i} \quad (3.15)$$

where  $V_0$  is the volume of the initial design domain and  $V_i$  is the volume of the current design at the  $i$ th iteration.

It can be seen from Eq. (3.15) that the performance index is a dimensionless number, which measures the performance of a structural topology in terms of material usage and the uniformity of stresses. The performance index reflects the changes in the weight and the maximum stress levels in the structure in an optimization process. For the initial

design domain, the performance index is equal to unity. The performance of a structural topology is improved by eliminating lowly stressed elements from the structure. Since the performance index is inversely proportional to the weight of the current design, the performance objective of minimizing the weight of a structure with stress constraints can be achieved by maximizing the performance index in the optimization process. Therefore, the *performance-based optimality criteria* can be proposed as follows

$$\text{maximize } PI_{ss} = \frac{\sigma_{0,\max}^{\text{VM}} W_0}{\sigma_{i,\max}^{\text{VM}} W_i} \quad (3.16)$$

The optimal topology that corresponds to the maximum performance index can be identified from the performance index history. The higher the value of the performance index, the better the performance of the topology. The stress limit is eliminated from the performance index formulas. This indicates that the optimal topology for the minimum-weight design of a plane stress continuum structure is unique for any value of prescribed stress limits. The maximum allowable stress constraint is easily satisfied by uniformly changing the thickness of elements.

The performance index formulas proposed herein do not involve the loads and geometrical parameters such as the span. This illustrates that the optimal topology of a plane stress continuum structure is independent of the scale of the loads and the structure. The performance index measures the structural response (maximum stress) and material efficiency (weight of the structure). Therefore, it is very convenient for the designer to use this performance index to evaluate the performance of an optimized design. The performance index can also be used to compare the performance of structural topologies produced by different structural optimization methods.

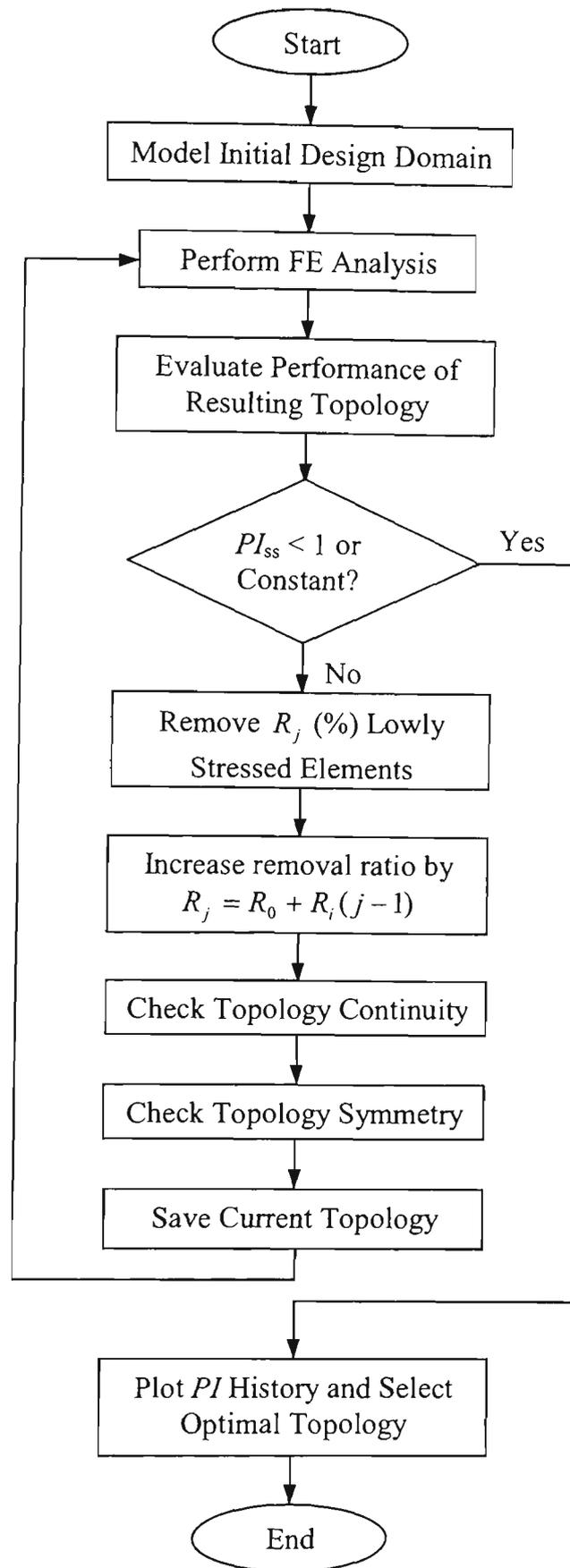
The performance index proposed herein can be incorporated in any stress-based structural optimization method to identify the optimum. When optimization methods employ principal stresses as optimization criteria, maximum principal stresses should be used to calculate the performance index in Eq. (3.14). The proposed performance index

can also be incorporated in truss topology optimization methods using ground structures when the cross-sectional areas of members are design variables. This is because the design can be uniformly scaled with respect to the cross-sectional area of members in the optimization process (Kirsch 1982).

### 3.6 PERFORMANCE OPTIMIZATION PROCEDURE

Structural topology optimization is actually a performance-improving process, which couples the finite element analysis and the element elimination scheme. The performance optimization process involves the modeling of an initial design, structural analysis, performance evaluation, element elimination, checking the model connectivity and the termination criterion. A flow chart is presented in Fig. 3.1 to show the main steps of the performance-based optimization procedure, which is described as follows:

- (1) Model a continuum design domain with fine finite elements. The material properties, applied loads and boundary conditions are specified. Non-design regions can be specified by assigning their material property to a different number from the design domain.
- (2) Perform a linearly elastic finite element analysis on the structure.
- (3) Evaluate the performance of the resulting topology using Eq. (3.14). For a structure under multiple load cases, the highest values of  $\sigma_{0,\max}^{\text{VM}}$  and  $\sigma_{i,\max}^{\text{VM}}$  of an element in the structure under all load cases should be used to calculate the  $PI_{ss}$ .
- (4) Eliminate elements that satisfy Eq. (3.3). For multiple loading cases, only elements that satisfy Eq. (3.3) for all load cases are removed from the structure at each iteration. The initial removal ratio  $R_0$  and the incremental removal ratio  $R_i$  are specified before carrying out the optimization.



**Fig. 3.1** Flowchart of PBO procedure for fully stressed topology designs

- (5) Increase the element removal ratio  $R_j$  such that  $R_j = R_0 + R_i(j - 1)$ .
- (6) Check topology connectivity. It is considered that two elements are connected if they have at least one common edge. Elements that are not connected with others are treated as singular elements, which will be deleted from the model.
- (7) Check the symmetry of the resulting topology with an initially symmetrical condition.
- (8) Save current topology. The data for resulting topologies at each iteration is saved to files. This will allow all topologies obtained at each iteration to be displayed at later states.
- (9) Repeat step (2) to (8) until the performance index is kept constant or less than unity.
- (10) Plot the performance index history and select the optimal topology from the optimization history.

The design is gradually modified by using the above optimization procedure. The performance index is used to monitor the performance improving history and as the termination criterion. When lowly stressed elements are deleted from the design, the performance index will increase from unity to the maximum value. After the performance index reaches the peak, it may keep constant in later iterations if the design is fully stressed. When the stress distribution in the design is uniform, no more elements can be removed from the design according to Eq. (3.3). However, the fully stressed designs of discretized continuum structures can only be obtained in some special cases. In most cases, elements can be continuously removed from a structure because of the non-uniformity of stresses in the structure. Consequently, the performance index in the final stage will be less than unity. If the performance index of a resulting topology is less than unity, its performance is lower than that of the initial design domain. Therefore, the

optimization process can be terminated when the performance index is less than unity or kept constant. It is apparent that the fully stressed condition cannot be used as the only optimality criterion for continuum structures in the proposed approach.

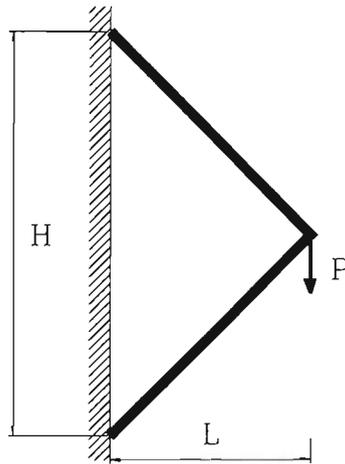
### 3.7 EXAMPLES

The PBO method is used to solve the topology optimization problems of various continuum structures in this section. It is assumed that the strength of the structure dominates the design so that the stress constraint is considered. The magnitude of stress limits might have significant influence on the weight of the final design but not on the optimal topology. As a result of this, the topology optimization process could be conveniently divided into two steps. The first step is to generate the optimal topology of a continuum structure using the PBO method regardless the magnitude of stress limit. The second step is to size the optimal topology obtained by uniformly changing the thickness of the structure in order to satisfy the stress constraint. Only the first step is considered in the following examples.

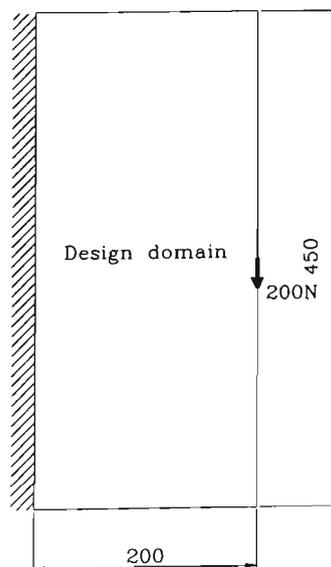
#### 3.7.1 Two-Bar Frame Structure

To verify the PBO method, the optimal design problem of a two-bar frame structure shown in Fig. 3.2 is solved by using the performance optimization procedure. The optimal height  $H$  of the two-bar frame structure can be obtained as  $H = 2L$  by using the analytical method if the structure is assumed to be a truss for the minimum-weight design. A continuum design domain that is larger than the size  $L \times 2L$  as shown in Fig. 3.3 is used as a starting point to derive the optimal two-bar frame structure. The design domain is discretized into a  $32 \times 72$  mesh using four-node plane stress elements. The left side of the design domain is fixed. A point load of 200 N is applied to the centre of

the free end. The Young's modulus  $E = 200$  GPa, the Poisson's ratio  $\nu = 0.3$  and the thickness of elements  $t = 1$  mm are used. The plane stress condition is assumed. The



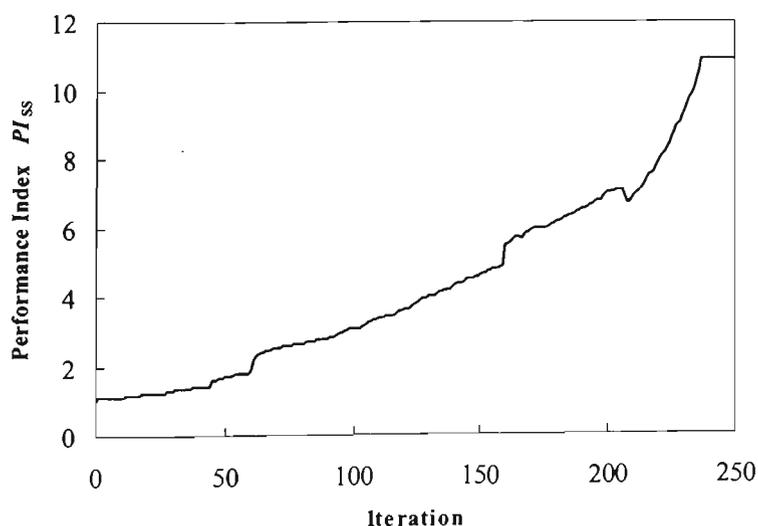
**Fig. 3.2** Two-bar frame structure



**Fig. 3.3** Design domain for the two-bar frame structure

initial removal ratio  $R_0 = 1\%$  and the incremental removal ratio  $R_i = 1\%$  are used in the optimization process.

The performance index history of the structure is presented in Fig. 3.4. It can be seen that at the initial iteration, the performance index is equal to unity because no elements have been removed at this stage. By gradually eliminating lowly stressed elements from the structure, the performance index gradually increases from unity up to the maximum value of 10.86. This means that the scaled weight of the initial design domain is 10.86 times that of the optimal design obtained. It can be observed from Fig. 3.4 that the maximum performance index is constant in later iterations. This indicates that the distribution of element effective stresses within the optimal topology is approximately uniform. In other words, the optimal topology generated by the PBO method is a fully stressed design at minimum weight. The performance index considering the maximum allowable stress constraint can indicate not only the optimum but also the uniformity of stresses within the optimum.



**Fig. 3.4** Performance index history of the two-bar frame structure

The topology optimization history of the two-bar frame structure is presented in Fig. 3.5. The optimal topology that corresponds to the maximum performance index evolves towards a two-bar frame structure, where its optimal height is exactly two times of its span as shown in Fig. 3.5(c). This proves that the performance-based optimization method is a reliable design tool for continuum structure with strength constraints. The capability of the performance index in selecting the optimal topology from the optimization history is demonstrated in Table 3.1, where a comparison of material volumes required for the initial design domain and three topologies shown in Fig. 3.5 for various stress limits is presented. It can be seen from the table that material volumes of the optimal topology are always less than those of the rest for each stress limits. This

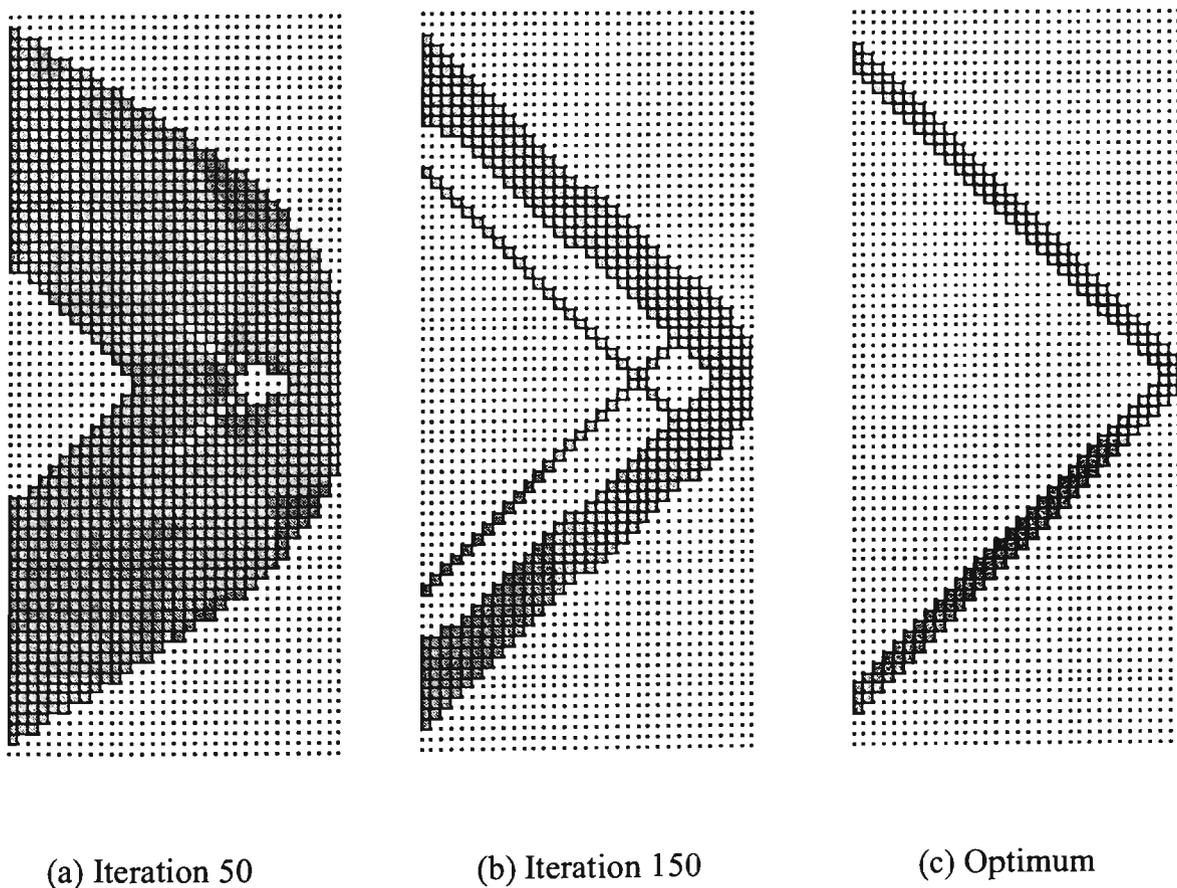


Fig. 3.5 Topology optimization history of the two-bar frame structure

illustrates that the topology shown in Fig. 3.5(c) is the best topology, which is independent of the prescribed stress limits.

**Table 3.1** Material volumes required at different iteration for various stress limits

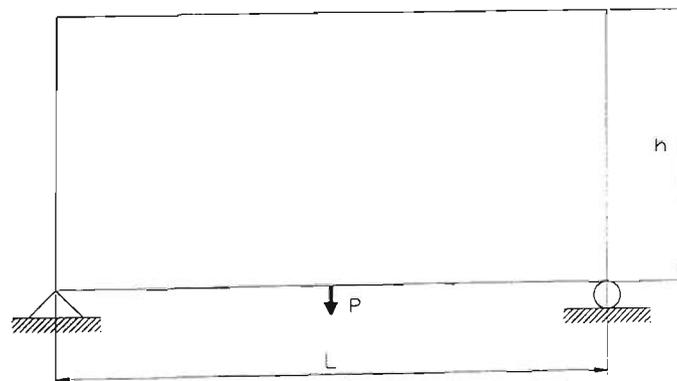
| $\sigma^*$<br>(MPa) | $V_0^s$<br>(mm <sup>3</sup> ) | $V_{50}^s$<br>(mm <sup>3</sup> ) | $V_{150}^s$<br>(mm <sup>3</sup> ) | $V_{\text{optimal}}^s$<br>(mm <sup>3</sup> ) | $PI_{\text{ss. max}} = \frac{V_0^s}{V_{\text{optimal}}^s}$ |
|---------------------|-------------------------------|----------------------------------|-----------------------------------|----------------------------------------------|------------------------------------------------------------|
| 100                 | 18855                         | 11117                            | 4086                              | 1735                                         | 10.86                                                      |
| 150                 | 12570                         | 7412                             | 2724                              | 1157                                         | 10.86                                                      |
| 250                 | 7542                          | 4447                             | 1634                              | 694                                          | 10.86                                                      |

A similar solution to this two-bar frame structure has been obtained by Xie and Steven (1993). However, no performance index was used in their approach to indicate the optimum. This means that the designer has to select one from hundreds of topologies generated in the evolutionary optimization process as the “optimum” for design according to his or her desire. Such a trial-and-error selection process would be cumbersome for the designer when dealing with real world engineering design problems. Although optimization methods can be examined by comparing results with analytical solutions, such as this example, this can only be done for simple structures. For practical problems with complex geometry and loading conditions, no classical solutions would be available. Therefore, the performance-based optimality criteria proposed here are extremely useful tool for structural designers for assisting the selection of optimal topologies in structural design.

### 3.7.2 Michell Type Structures with Height Constraints

In practice, the design space is often limited and significantly affects the optimal configurations of a structure. For example, the height of a beam in a building has to be limited in order to satisfy functional requirements. In addition, the designer should know how to select an initial design domain for deriving the optimal structure in a given design space. The design generated by design domain methods is optimum in the sense of the given design domain. The effect of geometrical restrictions such as height constraints is investigated in the section.

The design domain for the simply supported Michell type structures with various height constraints is illustrated in Fig. 3.6. In case (a), the design domain with  $h/L = 1/2$  is divided into  $100 \times 50$  mesh using four-node plane stress elements and  $R_0 = 1\%$  and  $R_i = 0.5\%$  are used in the optimization process. In case (b), the design domain with  $h/L = 1/4$  is divided into  $100 \times 25$  mesh. In case (c), the design domain with  $h/L = 1/8$  is divided into  $100 \times 13$  mesh. In case (d), a  $100 \times 9$  mesh is used for the structure with  $h/L = 1/12$ . The Young's modulus  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and the thickness of elements  $t = 2$  mm are assumed in the analysis. A point load  $P = 400$  N is applied to the structure.  $R_0 = 1\%$  and  $R_i = 1\%$  are used for cases (b) to (d). The plane stress condition is assumed for all cases.



**Fig. 3.6** Design domain for Michell structures with height constraints

The performance index history of case (a) is shown in Fig. 3.7. It can be seen that the performance index increases with the elimination of lowly stressed elements from the structure. However, further element removal from the optimal design eventually leads to the collapse of the structure, which is indicated by the sharp decrease of the performance index. This means that there are still lowly stressed elements in the optimal topology. However, the uniformity of stresses in the optimal design has ultimately been maximized. The stress distribution in the initial design domain of a continuum structure is hardly uniform owing to the stress concentration in the regions of loading and supports. The objective of the proposed method is to generate the optimal topology with the most uniform stress distribution and minimum weight. This example shows that a minimum-weight design is not necessarily a fully stressed design.

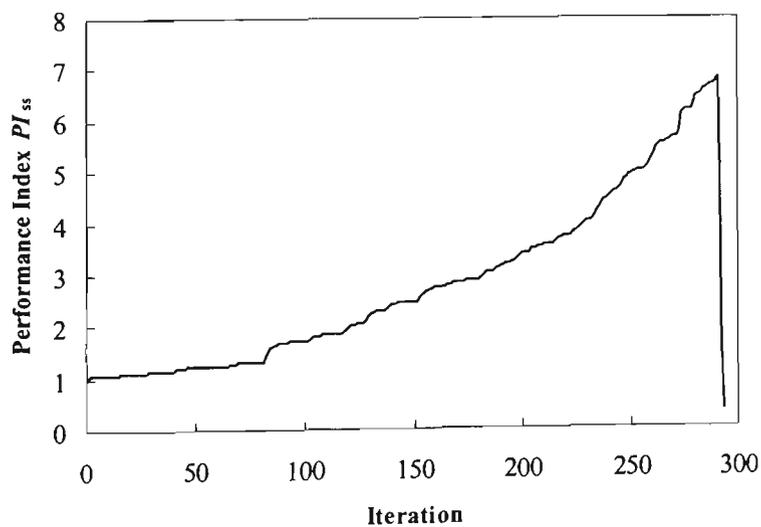
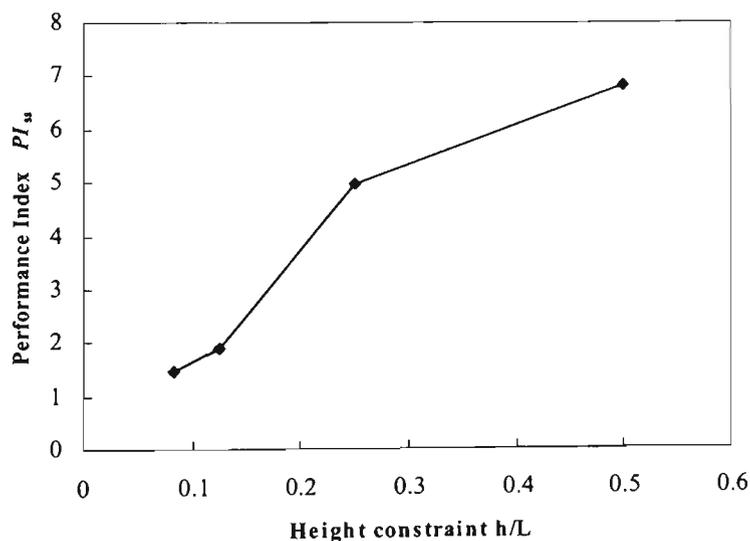
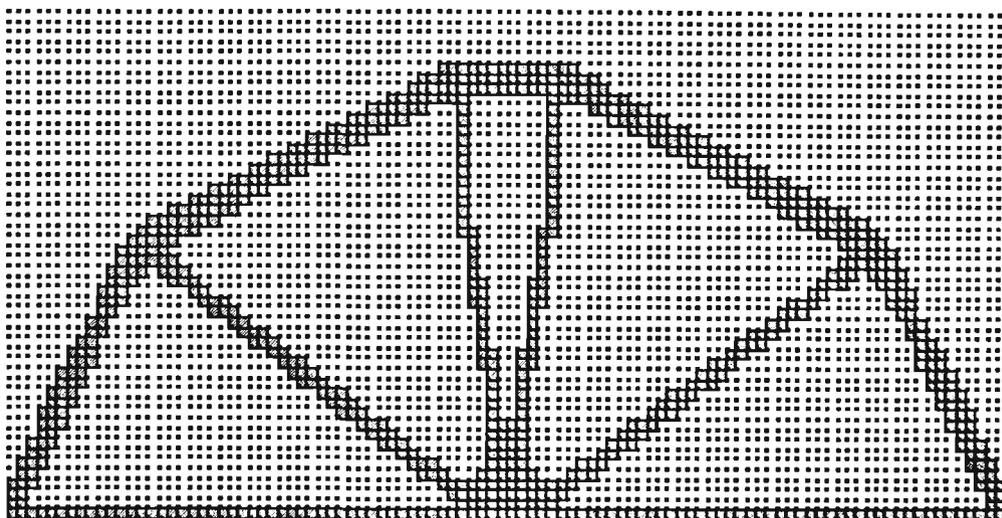


Fig. 3.7 Performance index history of case (a)

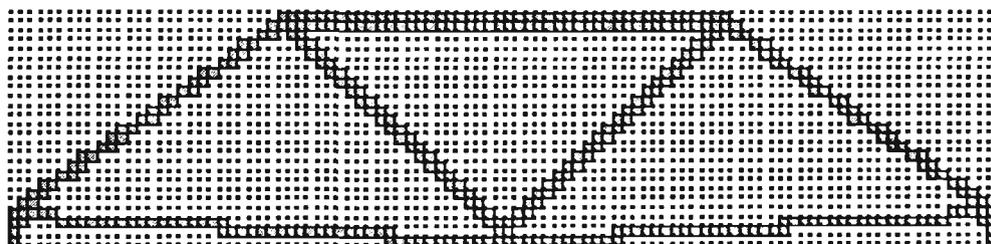
The effect of height constraints on the performance index of Michell type structures is illustrated in Fig. 3.8. It is seen that the performance index increases with the increase in the height when compared with the initial design domains. The maximum performance indices for cases (a) to (d) are 6.8, 4.97, 1.89 and 1.44. This illustrates that the performance of structural topologies can be improved by increasing the height of the initial design domain. Optimal topologies obtained for each case are shown in Fig. 3.9. The optimal topologies shown in Figs. 3.9 (a) and (b) exhibit truss like structures that can be designed as trusses. When  $h \ll L$ , the optimal topology as shown in Fig. 3.9(d) evolves to a continuum structure from which not much material could be removed.



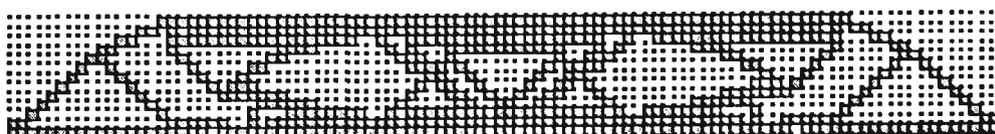
**Fig. 3.8** Effect of height constraints on performance indices



(a) Optimal topology ( $h/L=1/2$ ),  $PI_{ss} = 6.8$



(b) Optimal topology ( $h/L=1/4$ ),  $PI_{ss} = 4.97$



(c) Optimal topology ( $h/L=1/8$ ),  $PI_{ss} = 1.89$



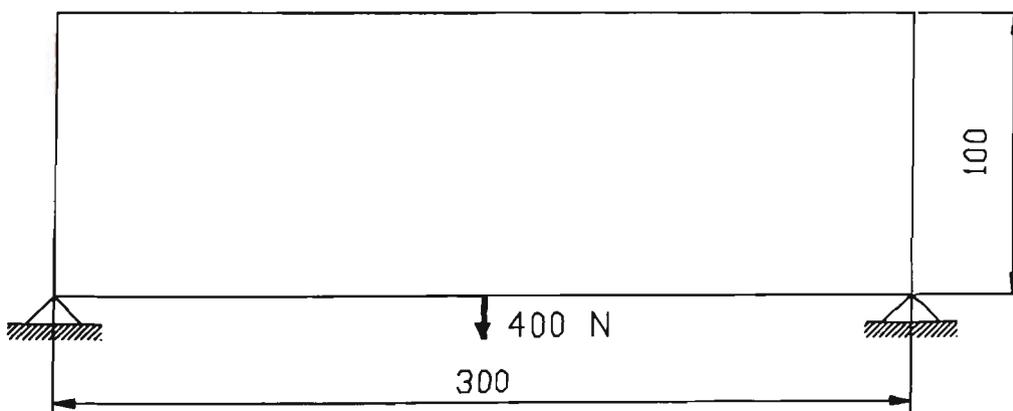
(d) Optimal topology ( $h/L=1/12$ ),  $PI_{ss} = 1.44$

**Fig. 3.9** Effects of height constraints on optimal topologies

### 3.7.3 Ranking the Performance of Structural Topologies

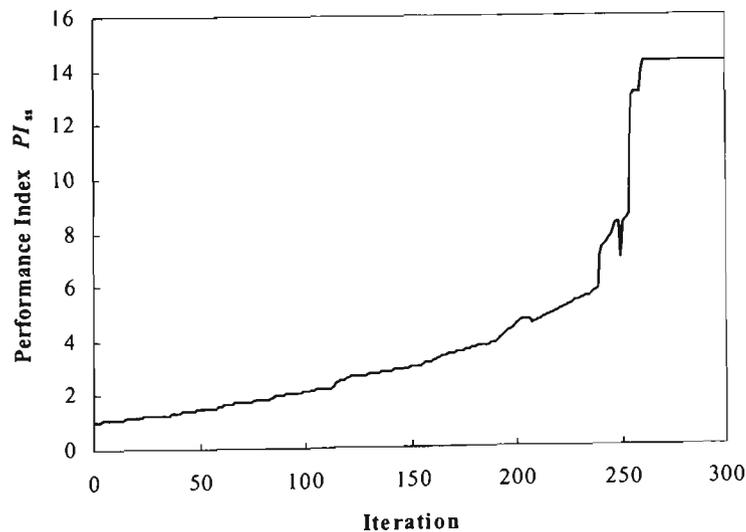
Topology optimization methods are effective tools for improving the performance of structures. However, it should be noted that the performance of optimized designs depends on the methods and criteria used. It has been found that different optimization methods usually lead to different final designs for the same design problem considered. It is difficult to evaluate the performance of different optimization algorithms because they usually involve different formulations. Little work has been done in this area in structural optimization. However, the efficiency of different structural optimization methods can be evaluated by comparing the results produced by them. The performance index developed herein is used to compare the performance of structural topologies generated by different optimization methods.

A transverse beam of homogeneous material with fixed supports shown in Fig. 3.10 is optimized by using the PBO method. The design domain is discretized into a  $90 \times 30$  mesh using four-node plane stress elements. A concentrated load of 400 N is applied to the center of the bottom. The Young's modulus  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and the thickness of elements  $t = 2$  mm are used.  $R_0 = 1\%$  and  $R_i = 1\%$  are employed in optimization process

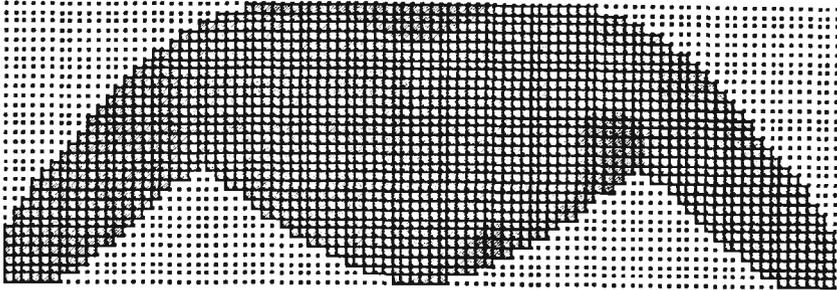


**Fig. 3.10** Design domain of the beam with fixed supports

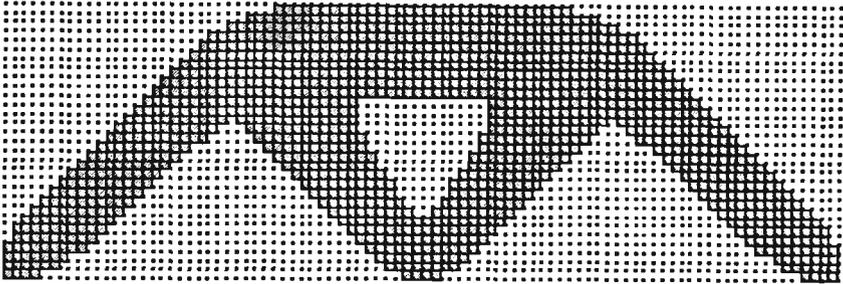
The performance index history of this beam is presented in Fig. 3.11. The maximum performance index is 14.32. The topology optimization history is shown in Fig. 3.12, where the optimal topology is uniformly stressed, as indicated by the performance index history. Fig. 3.12(d) shows the final design proposal given by Mattheck (1998) using the Soft Kill Option (SKO) approach. This proposal is regenerated by using the same mesh as used in the PBO. A linear finite element analysis is undertaken to analyze the design proposal. The performance index of the proposal calculated using Eq. (3.14) is 1.92, which is much less than that obtained by the PBO method. It can be observed that this final proposal obtained by Mattheck is very similar to the topology shown in Fig. 3.12(b), which is far from the optimum for the minimum weight design subject to the strength constraint. Therefore, it can be concluded that the proposed performance index is a useful tool for ranking the performance of structural topologies.



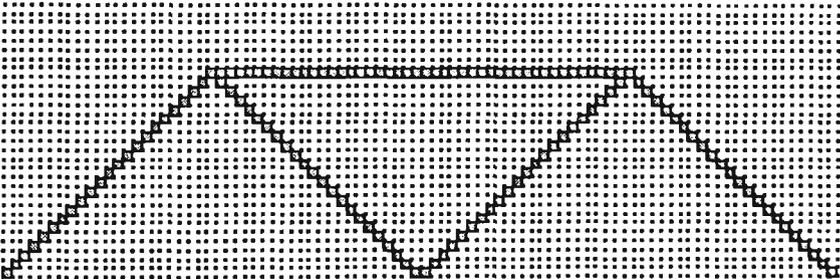
**Fig. 3.11** Performance index history of the beam with fixed supports



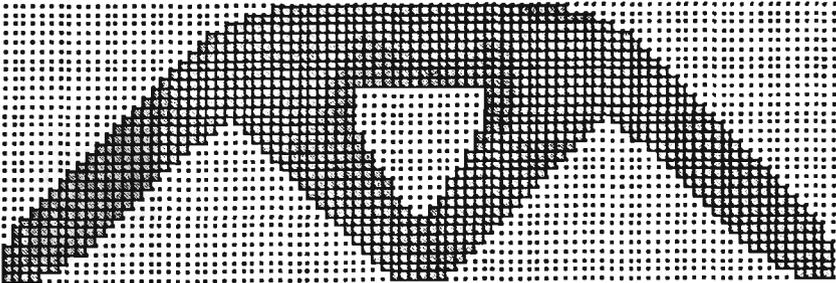
(a) Topology at iteration 50



(b) Topology at iteration 95



(c) Optimal topology



(d) Design proposal (Mattheck 1998)

Fig. 3.12 Topology optimization history of the beam with fixed supports

### 3.8 CONCLUDING REMARKS

This chapter has presented a Performance-Based Optimization (PBO) method for fully stressed topology design at minimum weight. The maximum allowable stress constraint is considered in the formulation. A performance index has been developed for evaluating the efficiency of structural topologies by using the scaling design procedure. The performance index is incorporated in optimization algorithms to monitor the optimization process and as the termination criterion. The performance objective of minimizing the weight of the structure with stress constraint is achieved by gradually eliminating lowly stressed elements from a continuum design domain until the performance index is maximized.

It has been shown that the proposed PBO method can generate efficient structural topologies that are verified by analytical solutions. The performance index can indicate not only the optimum from the optimization history but also the uniformity of stress within an optimal structure. Performance-based optimality criteria developed herein can overcome problems in stress-based continuum topology optimization methods. Moreover, the performance index can be used to rank the performance of structural topologies produced by different structural optimization methods. Examples presented show that increasing the height of an initial design domain usually improves the efficiency of the final optimal design. However, it needs to point out that stress-based continuum topology optimization methods are suitable for the design of elastic structural components or structures where the strength performance is a main concern.

## Chapter 4

# OPTIMAL TOPOLOGY AND SHAPE DESIGN FOR DISPLACEMENT PERFORMANCE

### 4.1 INTRODUCTION

The PBO method for fully stressed topology designs at minimum weight has been presented in Chapter 3. In fully stressed topology design, element removal criteria are based on the effective stress levels of elements. However, it should be noted that the fully stressed design procedure might not lead to the stiffest structural topologies. Therefore, the fully stressed optimization approach is suitable for finding optimal topologies for the strength design of structures where the system performance is not a major concern. It has wide applications in aeronautical and mechanical engineering industries.

In civil engineering, structures are often designed for required serviceability performance. For example, the maximum deflection of a steel truss under service loads must be within an acceptable limit. To generate the stiffest structural topologies and shapes, optimization methods should be formulated on the basis of displacement performance criteria. In other words, displacements should be treated as constraints. Several researchers have considered topology optimization of continuum structures subject to displacement constraints. Atrek (1989) developed a program for topology and

shape optimization of continuum structures subject to displacement constraints based on the classical optimality criteria method. The inefficient material is removed from a design domain to achieve a lighter design that has a more acceptable level of structural responses compared to other feasible designs with the same material volume. Displacement constraints are used as the termination criterion in the evolutionary structural optimization method by Chu et al. (1996) and in the density function approach by Yang (1997) to determine the optimum. However, it is worth noting that constrained displacements are significantly affected by the element thickness since the stiffness matrix of a plane stress continuum structure is a linear function of the element thickness. Prescribed displacement limits can easily be satisfied by uniformly sizing the element thickness. The result satisfying displacement constraints alone may not be the global optimum in a given design space. Therefore, it is of significant importance to develop performance-based optimality criteria that can be incorporated in optimization algorithms to obtain globally optimal designs.

In this chapter, a Performance-Based Optimization (PBO) method is proposed for the optimal topology and shape design of continuum structures for displacement performance. Continuum topology and shape optimization is treated as the problem of improving the performance of a continuum design domain in terms of the efficiency of material usage in resisting deformations. Two performance indices are developed for ranking the performance of resulting designs for plane stress continuum structures and for plates in bending in the optimization process. These performance indices are also used as termination criteria in performance-based optimization algorithms. Maximizing the performance index of a design domain is proposed as performance-based optimization criteria. The proposed performance indices and performance-based optimality criteria can be incorporated in any continuum topology and shape optimization methods to obtain optimal designs. Some of the results have been presented recently by Liang et al. (1999a, 2000a, 2001a).

## 4.2 OPTIMIZATION PROBLEM FORMULATION

Continuum topology and shape optimization is to seek the optimal material distribution within a given design domain. After the finite element analysis, it is found that some parts of the initial design domain are ineffectively used. These ineffectively used portions should be removed from the design domain to achieve a better design. By gradually eliminating underutilized material from the design, a lighter design can be generated. The performance objective is to minimize the weight of a continuum structure while maintaining concerned displacements within acceptable limits, and can be expressed in mathematical forms as follows

$$\text{minimize } W = \sum_{e=1}^n w_e(t) \quad (4.1)$$

$$\text{subject to } u_j \leq u_j^* \quad j = 1, \dots, m \quad (4.2)$$

where  $W$  is the total weight of a structure,  $w_e$  is the weight of the  $e$ th element,  $t$  is the thickness of elements,  $u_j$  is the absolute value of the  $j$ th constrained displacement,  $u_j^*$  is the prescribed limit of  $u_j$ ,  $m$  is the total number of displacement constraints and  $n$  is the total number of elements in the structure. Since the thickness of a plane continuum structure has a significant effect on the structural weight and it needs to be specified by the designer in practice, it is treated as one of the design variables. However, simultaneous topology and thickness (sizing) optimization will make the optimization problem rather complicated. Moreover, in many practical design problems, the thickness of structures needs to be uniform. Therefore, only uniform sizing of the element thickness is considered in the proposed method.

### 4.3 ELEMENT REMOVAL CRITERIA BASED ON VIRTUAL STRAIN ENERGY DENSITY

In stress-based topology optimization, lowly stressed elements are systematically removed from the design domain to achieve a fully stressed topology. For structures under displacement constraints, elements with the least effect on the change in constrained displacements should be listed as candidates for elimination. The resulting structure will be the stiffest design at minimum weight with respect to the specific displacements. The main point is to find out which element should be removed from the design. This can be done by undertaking a design sensitivity analysis on constrained displacements due to element removal. From the design sensitivity analysis, element removal criteria can be established for element removal and used in performance-based optimization algorithms.

The equilibrium equation for a static structure in the finite element analysis is expressed by

$$[K]\{u\} = \{P\} \quad (4.3)$$

If the  $e$ th element is removed from a structure discretized into finite elements, the stiffness and displacements of the structure will be changed accordingly, and Eq. (4.3) can be rewritten as

$$([K] + [\Delta K])(\{u\} + \{\Delta u\}) = \{P\} \quad (4.4)$$

in which  $[\Delta K]$  is the changes of the stiffness matrix and  $\{\Delta u\}$  is the change of the nodal displacements vector. The change of the stiffness matrix is

$$[\Delta K] = [K_r] - [K] = -[k_e] \quad (4.5)$$

where  $[K_r]$  is the stiffness matrix of the resulting structure and  $[k_e]$  is the stiffness matrix of the  $e$ th element. The change of displacement vector due to element elimination can be obtained approximately from Eqs. (4.3) and (4.4) by neglecting higher order terms as

$$\{\Delta u\} = -[K]^{-1}[\Delta K]\{u\} \quad (4.6)$$

It is assumed that the constraint is imposed on a specific displacement  $u_j$ . The change of the specific displacement due to an element removal needs to be evaluated. In order to extract the required displacement component, a virtual unit load is applied at  $u_j$  and acting in the direction of the displacement component. By multiplying Eq. (4.6) with the virtual unit load vector  $\{F_j\}^T$ , in which only the component corresponding to the  $j$ th constrained displacement is equal to unity and all the others are equal to zero, the change of the constrained displacement can be obtained approximately as

$$\Delta u_j = -\{F_j\}^T [K]^{-1} [\Delta K] \{u\} = -\{u_j\}^T [\Delta K] \{u\} = \{u_{ej}\}^T [k_e] \{u_e\} \quad (4.7)$$

where  $\{u_j\}^T$  is the nodal displacement vector of the structure under the virtual unit load,  $\{u_{ej}\}^T$  is the nodal displacement vector of the  $e$ th element under the virtual unit load and  $\{u_e\}$  is the nodal displacement vector of the  $e$ th element under real loads. It is seen from Eq. (4.7) that the change in the constrained displacement due to the elimination of the  $e$ th element can be calculated approximately by the virtual strain energy of the  $e$ th element, which is denoted as

$$s_e = \{u_{ej}\}^T [k_e] \{u_e\} \quad (4.8)$$

The element virtual strain energy can be calculated at the element level after the finite element analysis. To obtain the maximum stiffness design at minimum weight, it is obvious that elements with the lowest virtual strain energy should be eliminated

systematically from the continuum design domain being optimized. In other words, elements that have the least effect on the change in constrained displacements are eliminated from the design domain to achieve the performance objective.

The lowest virtual strain energy of elements is used as the element removal criteria for continuum structures, which are discretized into equal size finite elements. In situations where a continuum structure is divided into different size elements, element weights will differ from each other. Considering two elements with the same virtual strain energy, eliminating the element with a larger weight will result in a lighter design while the changes in specific displacements are the same. Therefore, in order to obtain the most efficient design, the virtual strain energy per unit volume of an element, which is defined as the element *virtual strain energy density*, should be used as the element removal criteria. The virtual strain energy density of the  $e$ th element can be calculated by

$$\zeta_e = \frac{|s_e|}{v_e} \quad (4.9)$$

where  $|s_e|$  is the absolute value of  $s_e$ . If the material density is not uniformly distributed in a continuum design domain, the weight of an element ( $w_e$ ) should be used in Eq. (4.9) for calculating the element virtual strain energy density.

For a structure subject to multiple displacement constraints under multiple load cases, the virtual strain energy density of the  $e$ th element can be evaluated by using the weighting average approach as

$$\zeta_e^m = \sum_{l=1}^p \sum_{j=1}^m \beta_j \zeta_e \quad (4.10)$$

in which the weighting parameter  $\beta_j$  is defined as  $u_j^l / u_j^{l*}$ , which is the ratio of the  $j$ th constrained displacement to the prescribed limit under the  $l$ th load case, and  $p$  is the

total number of load cases. It is noted that the absolute values of displacements are used in Eq. (4.10). If the constrained displacement is far from its prescribed limit, it will be less critical in the optimization process.

Since the virtual strain energy density of elements are approximately evaluated by neglecting higher order terms in the sensitivity analysis, only a small number of elements with the lowest virtual strain energy density are allowed to be removed from a structure at each iteration in order to obtain a sound optimal design. *The element removal ratio ( $R$ ) for each iteration is defined by the ratio of the number of elements to be removed to the total number of elements in the initial design domain and is kept constant during the whole optimization process.* The accuracy of the solution is obviously improved by using a smaller element removal ratio but the computational cost will considerably be increased. The element removal ratio used in the proposed performance-based optimization method is similar to the step size employed in conventional optimality criteria methods (Morris 1982; Rozvany 1989).

## 4.4 PERFORMANCE-BASED OPTIMALITY CRITERIA

### 4.4.1 General

By removing a small number of elements with the lowest virtual strain energy density from a discretized continuum structure, a more uniform distribution of element virtual strain energy density in the resulting structure can be achieved. The uniformity of element virtual strain energy density in skeletal structures has been used as an optimality criterion, which can be derived on the basis of the Kuhn-Tucker condition (Morris 1982; Rozvany 1989). In conventional optimality criteria (OC) methods (Rozvany 1989), the optimality criterion is derived for the dominant type of constraint imposed on a structure, and used to develop a recurrence relation. This recurrence relation is then used to update design variables so that the initial design is moved towards an optimum, which satisfies the optimality criterion. However, the uniformity of element virtual

strain energy density in a continuum structure may not be achieved in some cases even if constrained displacements are active. *This means that a minimum-weight optimal design is not necessarily a design in which the distribution of the element virtual strain energy density is absolutely uniform.* This is especially true for practical design problems. Therefore, the uniformity of element virtual strain energy density cannot be used as a termination condition in continuum topology optimization algorithms for determining optimal designs. To obtain globally optimal designs, new type *performance-based optimality criteria* (PBOC) must be developed and incorporated into continuum topology optimization methods. Performance-based optimality criteria that form the core of the PBO method for displacement constraints are derived in this section for plane stress and bending plate optimization problems.

#### 4.4.2 PBOC for Plane Stress Problem

Topology and shape optimization based on the element removal concept is a design problem of improving the performance of a continuum design domain in terms of the efficiency of material usage in resisting deformations. To obtain an optimum for the minimum-weight design, the performance of the resulting topology at each iteration must be evaluated by using the performance index, which can be derived on the basis of the scaling design procedure. Since the stiffness matrix of a plane stress continuum structure is a linear function of the thickness of elements, the element thickness can be uniformly scaled to keep the critical constraint at the constraint surface (Kirsch 1982; Morris 1982). As seen from Eq. (3.9), in order to satisfy the displacement constraints imposed on a structure, the actual design needs to be scaled by the following scaling factor

$$\varphi = \frac{u_j}{u_j^*} \quad (4.11)$$

where  $u_j$  is the absolute value of the  $j$ th constrained displacement that is the most critical in the design. By scaling the initial design with respect to the most critical constrained displacement, the scaled weight of the initial design domain can be represented by

$$W_0^s = \left( \frac{u_{0j}}{u_j^*} \right) W_0 \quad (4.12)$$

in which  $W_0$  is the actual weight of the initial design domain and  $u_{0j}$  is the absolute value of the most critical constrained displacement in the initial design under applied loads. In a similar manner, the current design can be scaled to satisfy the prescribed displacement limit at each iteration in the optimization process. The scaled weight of the current design at the  $i$ th iteration is expressed by

$$W_i^s = \left( \frac{u_{ij}}{u_j^*} \right) W_i \quad (4.13)$$

where  $u_{ij}$  is the absolute value of the most critical constrained displacement in the current design at the  $i$ th iteration under the applied loads and  $W_i$  is the actual weight of the current design at the  $i$ th iteration.

The performance index of the resulting design at the  $i$ th iteration is proposed as

$$PI_{ds} = \frac{W_0^s}{W_i^s} = \frac{(u_{0j} / u_j^*) W_0}{(u_{ij} / u_j^*) W_i} = \frac{u_{0j} W_0}{u_{ij} W_i} \quad (4.14)$$

It can be seen from Eq. (4.14) that the performance index is a dimensionless number that indicates the performance of a structural topology or shape in terms of the material efficiency in resisting deformations. It depends on the topology or shape but not the scale of a structure. The performance index of the initial design is equal to unity. The

performance of a structural topology is gradually improved when elements with the lowest virtual strain energy density are systematically removed from the design domain. The displacement limit ( $u_j^*$ ) is consequently eliminated from Eq. (4.14). This indicates that an optimal topology for the minimum-weight design of a continuum structure is unique for any value of prescribed displacement limits. The performance objective can be achieved by maximizing the performance index in the optimization process. Therefore, *the performance-based optimality criterion for plane stress continuum structures subject to displacement constraints* is proposed as

$$\text{maximize } PI_{ds} = \frac{u_{0j} W_0}{u_{ij} W_i} \quad (4.15)$$

It is noted that uniformly changing the element thickness does not affect the topology of a plane stress continuum structure and the performance index, but significantly influences the weight of the structure and the constrained displacements. As a result of this, the thickness of elements is not changed in the finite element analysis at each iteration. Displacement limits are usually set to large values in order to obtain the optimum design, which can then be sized to satisfy actual displacement limits.

#### 4.4.3 PBOC for Plate Bending Problem

Plate structures are commonly used as structural systems in engineering practice. Research work on the optimal topology and shape design and reinforcement of plates has been undertaken by many researchers (Olhoff, 1975; Lurie and Cherkaev 1976; Cheng and Olhoff 1982). Their results show that there are an infinite number of fine ribs in the optimal reinforcement of plates. Atrek (1989) used a material removal procedure to generate optimal topologies of bending plates subject to displacement constraints. The homogenization-based optimization method has been applied to the topology optimization of bending plates (Tenek and Hagiwara 1993).

The performance-based optimization method can also be applied to plates in bending. In the proposed method, the optimal design is to seek the stiffest topology of a bending plate at minimum weight. Unlike plane stress problems, the bending stiffness of a plate is not a linear function of the plate thickness. As a result of this, the performance index proposed for plane stress structures cannot be used for plates in bending. However, the scaling design concept used in deriving performance indices for plane stress structures can be adopted for bending plates if an appropriate-scaling factor is found.

To obtain the best topology for the design of a plate in bending, the thickness of the plate is treated as one of design variables. The plate thickness is uniformly scaled to satisfy displacement constraints. By scaling the design, the scaled thickness of the plate is represented by

$$t^s = \varphi t \quad (4.16)$$

in which  $\varphi$  is the scaling factor which is the same for all elements. The material elastic constants of an element are written in matrix form as

$$[D_e] = \frac{t^3}{12} \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 \\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \quad (4.17)$$

Eq. (4.17) can be denoted as

$$[D_e] = t^3 [C] \quad (4.18)$$

where  $E$  is the Young's modulus and  $\nu$  is the Poisson's ratio. The material elastic constants of an element can be expressed in term of the scaled design variable as

$$[D_e] = \left( \frac{t^s}{\varphi} \right)^3 [C] = \frac{1}{\varphi^3} [D_e^s] \quad (4.19)$$

in which  $[D_e^s]$  is the scaled material elastic constant matrix of an element. The equilibrium equation for a plate can be expressed in the finite element analysis as

$$\frac{1}{\varphi^3} [K^s] \{u\} = \{P\} \quad (4.20)$$

where  $[K^s]$  is the stiffness matrix of the scaled plate, which is calculated by using the scaled design variable  $t^s$ ,  $\{u\}$  is the actual nodal displacement vector and  $\{P\}$  is the nodal load vector. Using the scaled design variables, the equilibrium equation for the scaled plate is denoted as

$$[K^s] \{u^s\} = \{P\} \quad (4.21)$$

From Eqs. (4.20) and (4.21), the scaled displacement vector can be obtained as

$$\{u^s\} = \frac{1}{\varphi^3} \{u\} \quad (4.22)$$

It can be seen from Eqs. (4.16) and (4.22) that when the thickness of the plate is reduced by a factor  $\varphi$ , the deflections will increase with a factor of  $1/\varphi^3$ . In order to satisfy the displacement limit, the actual design needs to be scaled by

$$\varphi = \left( \frac{u_j}{u_j^*} \right)^{1/3} \quad (4.23)$$

By using the above scaling factor, the design can be scaled to keep the most constrained displacement active after each iteration in the optimization process. Consequently, the

best possible constrained design can be obtained from the history of the reduction in the weight of a plate. The performance index of a bending plate at the  $i$ th iteration is defined as the ratio of the scaled weight of the initial design to the scaled weight of the current design at the  $i$ th iteration, and is proposed as

$$PI_{dp} = \frac{W_0^s}{W_i^s} = \frac{(u_{0j}/u_j^*)^{1/3} W_0}{(u_{ij}/u_j^*)^{1/3} W_i} = \left( \frac{u_{0j}}{u_{ij}} \right)^{1/3} \frac{W_0}{W_i} \quad (4.24)$$

It can be seen from Eq. (4.24) that topology performance does not depend on the prescribed displacement limits. However, it is noteworthy that large values must be given to displacement limits in order to obtain the global optimum from the optimization history. The scaling design concept allows for the structural response (displacement) to be entered into the performance index formulas. In other words, the performance index is a measure of structural responses and the reduction in the weight of plates in an optimization process, and thus quantifies the performance of a bending plate. Therefore, *the performance-based optimality criterion for bending plates subject to displacement constraints* can be proposed as

$$\text{maximize } PI_{dp} = \left( \frac{u_{0j}}{u_{ij}} \right)^{1/3} \frac{W_0}{W_i} \quad (4.25)$$

The performance-based optimality criterion can be achieved by gradually removing elements with the lowest virtual strain energy density from the discretized plate.

In the optimality criteria method, a Lagrangian function is usually constructed for the objective function and constraints to obtain the optimality condition. A recurrence formula derived from this optimality condition is used to modify design variables in order to generate the next structural layout. The optimality condition provides information on how to modify the design but it does not indicate which topology generated in the optimization process is the optimum in the given design space. In contrast, in the PBO method proposed herein, the structure is modified by removing  $R$

(%) elements with the lowest virtual strain energy density from the design at each iteration. Maximizing the performance indices is proposed as performance-based optimality criteria. The main advantage of the PBO method is that performance-based optimality criteria can be used to monitor the optimization process and to identify the optimum from the optimization history.

## 4.5 PERFORMANCE OPTIMIZATION PROCEDURE

The performance optimization process for plane stress problems is basically the same as that for plates in bending, except that different performance indices are used in the optimization algorithms to evaluate the performance of resulting designs. The performance optimization process includes modeling of the initial design, finite element analysis, performance evaluation, element removal, checking model connectivity and maintaining symmetry of resulting design under symmetrical conditions. A flowchart is given in Fig. 4.1 to illustrate the performance-based optimization procedure for topology and shape design of continuum structures subject to displacement constraints. The optimization procedure is also explained as follows:

- (1) Model the initial design domain with applied loads, material properties and boundary conditions. Elements around the applied loads are usually treated as the non-design region and are not removed in the optimization process.
- (2) Carry out a linear elastic finite element analysis on the structure.
- (3) Evaluate the performance of resulting topology or shape using Eq. (4.14) for plane stress structures and Eq. (4.24) for plates in bending. In the evaluation of topology or shape performance, the most critical constrained displacement is used.
- (4) Calculate the virtual strain energy density of elements ( $\zeta_e^m$ ) for each loading case.

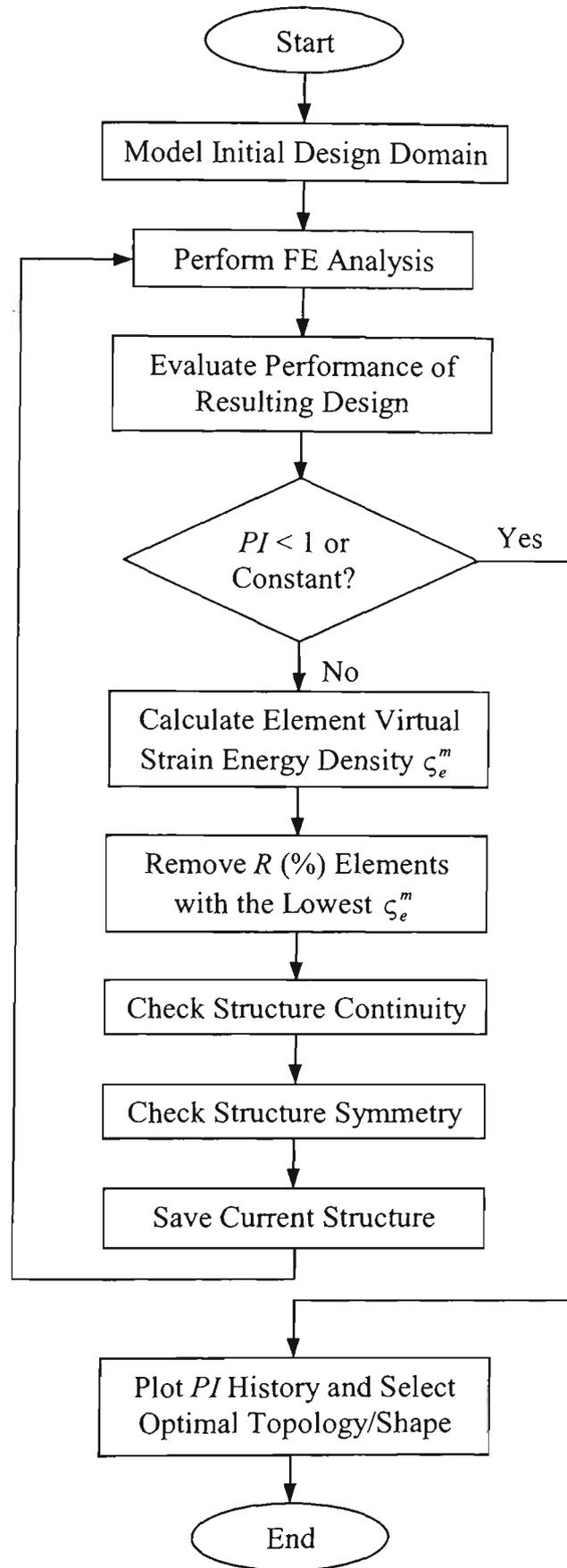


Fig. 4.1 Flowchart of PBO procedure for optimal design for displacement performance

- (5) Eliminate  $R$  (%) elements with the lowest virtual strain energy density ( $\zeta_e^m$ ) from the design domain. In topology optimization, underutilized elements can be removed from the interior of the design as well as the boundary. However, in shape optimization, elements can only be eliminated from the boundary of a structure.
- (6) Check topology/shape connectivity. It is assumed that two elements are connected together if they have at least one common edge. Any element that is not connected with other elements is considered as a singular element, and is removed from the model.
- (7) Maintain the symmetry of resulting structure under initially symmetrical geometry and loading conditions. It may be necessary to remove extra elements from the structure to maintain the symmetry.
- (8) Save information for current structure.
- (9) Repeat step (2) to (8) until the performance index is kept constant in later iterations or less than unity.
- (10) Plot the performance index history and select the optimal topology or shape.

It is noteworthy that the optimal design produced by the PBO method depends on the element removal ratio ( $R$ ), which is similar to the step size used in the conventional optimality criteria method (Rozvany 1989). The smooth convergence of the performance index to the maximum value may not be guaranteed in the optimization process. Moreover, the structure can always be modified by eliminating elements from the structure since the distribution of the element virtual strain energy density is seldom uniform in a continuum design domain. Therefore, the prescribed tolerance for the relative change in the performance index cannot be used as a termination criterion in the optimization procedure. If the performance index is less than unity, the performance of the corresponding topology is less than the initial design domain. The iterative

performance optimization process can be terminated when the performance index is less than unity. This can ensure that the optimal topology that corresponds to the maximum performance index is included in the optimization history. It is also possible that the performance index is kept constant in later iterations. In such situations, the optimization process can be terminated. In order to continue the optimization process, displacement limits must be set to large values that allow for the performance index to be less than unity. Actual displacement limits are easily satisfied by changing the thickness of the optimal design obtained.

## **4.6 EXAMPLES: PLANE STRESS STRUCTURES**

### **4.6.1 General**

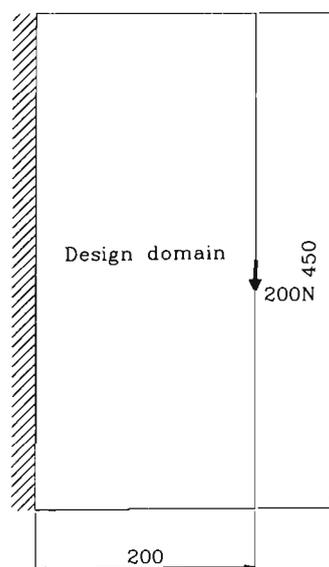
Several examples are provided in this section to demonstrate the effectiveness and validity of the PBO method for optimal topology design of plane stress continuum structures for displacement performance. The PBO method is verified by the well-known analytical solution of a two-bar frame structure and the Michell structure, which are presented in Sections 4.6.2 and 4.6.3. The effect of geometry constraints imposed on initial design domains on the optimal designs is also illustrated in Section 4.6.3. Multiple displacement constraints are treated in Section 4.6.4. The performance evaluation of structural topologies generated by different structural optimization methods is presented in Section 4.6.5. Finally, in Section 4.6.6, topology and shape optimization of plates in bending is considered.

### **4.6.2 Verification of the PBO Method**

The PBO method for optimal topology design of continuum structures for displacement performance is examined by solving a simple optimization problem whose optimal solution may be obtained by the analytical method. The optimization problem of a two-

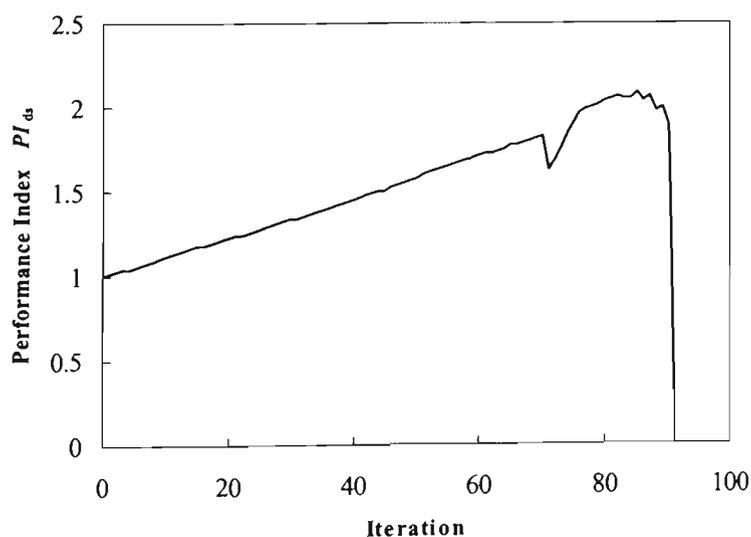
bar frame structure shown in Fig. 3.2 is treated here for displacement performance. If the frame structure is assumed to be a truss for the minimum-weight design, its optimal height  $H$  is obtained as  $H = 2L$  using the analytical method (Rozvany 1976). This problem was considered by Suzuki and Kikuchi (1991) using the homogenization-based optimization method for the mean compliance minimization.

A continuum design domain that is slightly larger than the two-bar frame structure is used to derive the optimal topology of the structure as shown in Fig. 4.2. A displacement constraint is imposed at the loaded point in the vertical direction, and its limit is set to a large value in order to ensure the optimum is included in the optimization process. The design domain is divided into a  $32 \times 72$  mesh using four-node plane stress elements. The left edge of the design domain is fixed. A concentrated load of 200 N is applied to the centre of the free end. The values of Young's modulus  $E = 200$  GPa, the Poisson's ratio  $\nu = 0.3$  and the thickness of elements  $t = 1$  mm are used in the finite element analysis. Plane stress conditions are assumed. The element removal ratio  $R = 1\%$  is adopted in the optimization process.

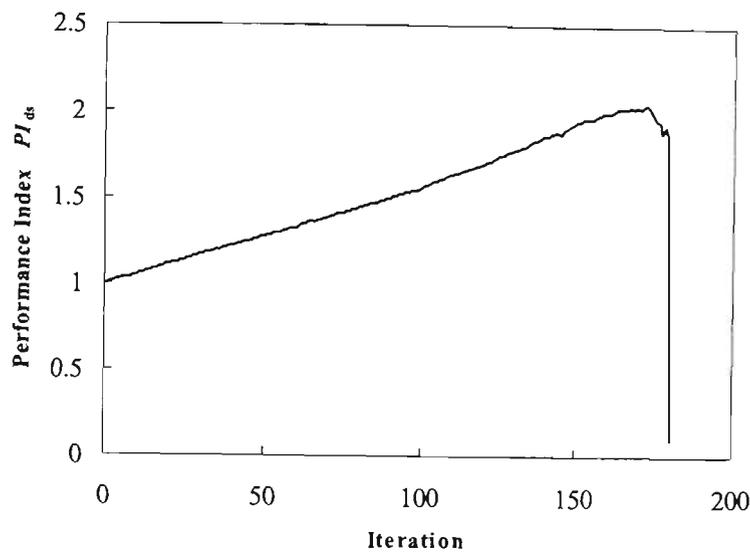


**Fig. 4.2** Design domain of two-bar frame structure

Figure 4.3 shows the performance index history of the two-bar frame structure. While elements with the lowest virtual strain energy density are eliminated from the design domain, the performance index gradually increases from unity to the maximum value of 2.08. It is seen from Fig. 4.3 that the performance index may jump in the optimization process. This is because the element removal ratio of 1% used is still high. A smoother solution may be obtained by using a smaller removal ratio such as  $R = 0.5\%$  for this two-bar frame structure as shown in Fig. 4.4, but the computational time is approximately double of that using a removal ratio of 1%. The maximum performance index of the optimal design obtained using  $R = 0.5\%$  is 2.05. This indicates that the element removal ratios do not have a significant effect on the performance of optimal topologies produced by the PBO method. Moreover, it can be observed from Fig. 4.5 that optimal topologies obtained using different element removal ratios are almost the same. Therefore, it is concluded that the  $R = 1\%$  can be used in the design of practical engineering optimization problems with sufficient accuracy and efficiency.



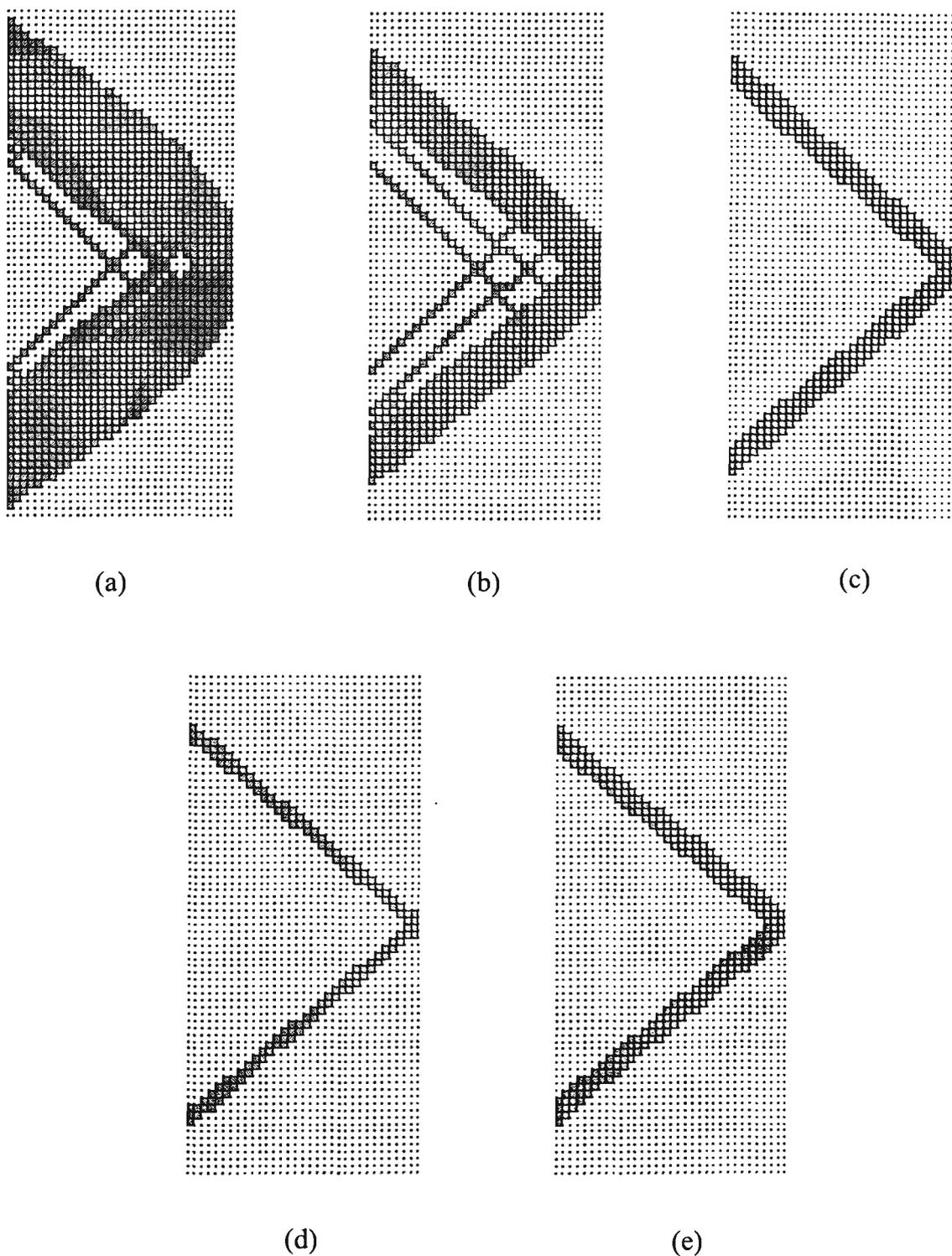
**Fig. 4.3** Performance index history of the two-bar frame structure ( $R = 1\%$ )



**Fig. 4.4** Performance index history of the two-bar frame structure ( $R = 0.5\%$ )

The topology optimization history is presented in Fig. 4.5. The optimal topology evolves to a two-bar truss-like structure whose optimal height is two times of its span (Rozvany 1976). This proves that the PBO method can generate reliable optimal topologies for the design of continuum structures.

As discussed previously, the performance of an optimal topology does not depend on the actual values of displacement limits. This is illustrated in Table 4.1 where a comparison of material volumes required for the initial design and four topologies shown in Fig. 4.5 (a) to (d) for various displacement limits is made. It can be seen from the table that the volumes of the optimal topologies are always less than those of other four topologies for each displacement limit. It also shows that the maximum performance indices are the same for different displacement limits.



**Fig. 4.5** Topology optimization history of two-bar frame structure: (a) topology at iteration 50; (b) topology at iteration 70; (c) optimal topology ( $R = 1\%$ ); (d) topology at iteration 90; (e) optimal topology ( $R = 0.5\%$ ).

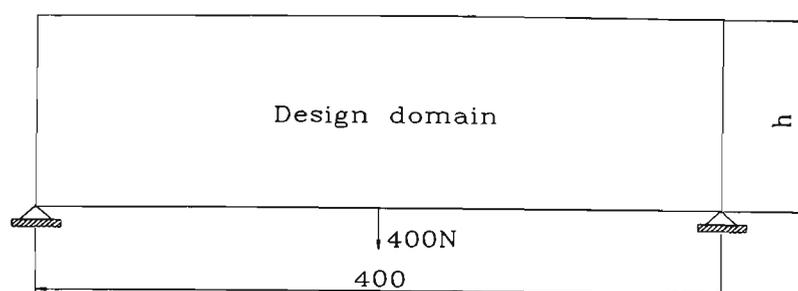
**Table 4.1** Performance of topologies with various displacement limits

| $u_j^*$<br>(mm) | $V_0^s$<br>(mm <sup>3</sup> ) | $V_{50}^s$<br>(mm <sup>3</sup> ) | $V_{70}^s$<br>(mm <sup>3</sup> ) | $V_{\text{optimal}}^s$<br>(mm <sup>3</sup> ) | $V_{90}^s$<br>(mm <sup>3</sup> ) | $PI_{\text{ds.max}}$ |
|-----------------|-------------------------------|----------------------------------|----------------------------------|----------------------------------------------|----------------------------------|----------------------|
| 0.5             | 740                           | 468                              | 405                              | 355                                          | 390                              | 2.08                 |
| 0.75            | 493                           | 312                              | 270                              | 237                                          | 260                              | 2.08                 |
| 1.0             | 370                           | 234                              | 203                              | 178                                          | 195                              | 2.08                 |

### 4.6.3 Effect of Geometry Constraints

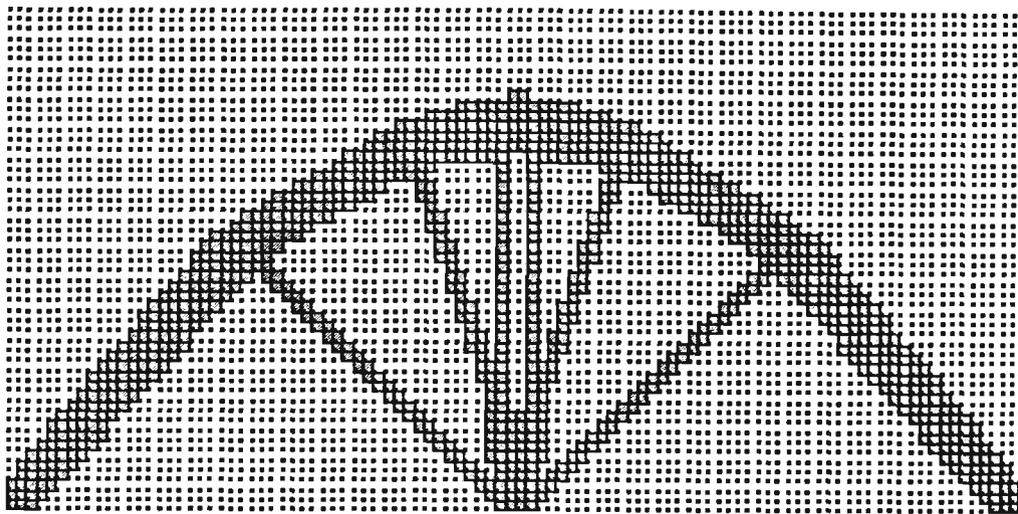
The design spaces in practical engineering design problems are often limited and have a significant effect on the performance of final optimal designs. This example is to investigate the effects of height constraints imposed on the initial design domains of Michell type structures on the performance of optimal topologies. In addition, the optimal topology obtained by the PBO method is compared with the classical solution given by Michell (1904).

The design domain for Michell type structures with fixed supports subject to various height constraints is shown in Fig. 4.6. In case (a), the design domain with  $h/L = 1/2$  is divided into  $100 \times 50$  mesh using plane stress elements. In case (b), the design domain with  $h/L = 1/4$  is divided into  $100 \times 25$  mesh. In case (c), the design domain with  $h/L = 1/8$  is divided into  $100 \times 13$  mesh. In case (d), a  $100 \times 9$  mesh is used for the structure with  $h/L = 1/12$ . The Young's modulus  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and thickness of elements  $t = 2$  mm are used for all cases. A point load  $P = 400$  N is applied to the structure. A displacement constraint imposed at the loaded point in the vertical direction is set to large value in the analysis. The  $R = 1\%$  is used in the optimization process.

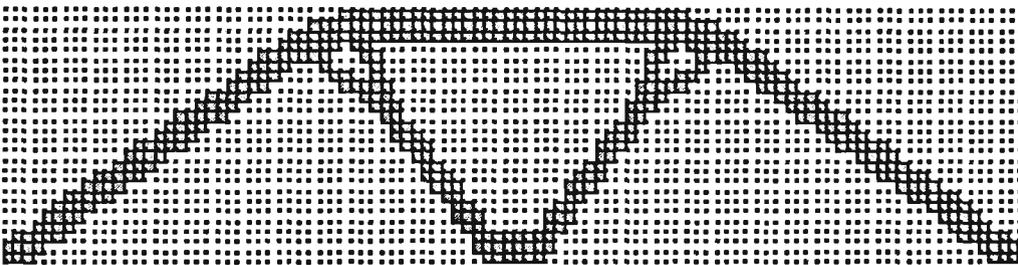


**Fig. 4.6** Design domain of Michell type structures with height constraints

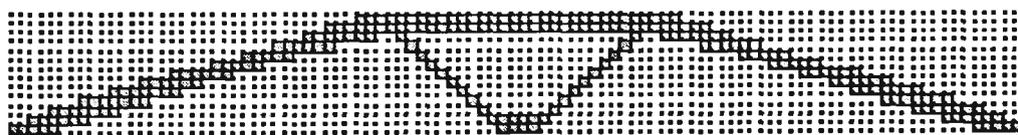
Optimal topologies selected by the performance index Eq. (4.14) from the optimization history of continuum structures with various height constraints are shown in Fig. 4.7. The case (a) is the well-known Michell type structure with fixed supports. The optimal topology produced by the PBO method as shown in Fig. 4.7(a) agrees extremely well with the analytical solution derived by the Australian inventor A.G.M. Michell (1904). Maximum performance indices for cases (a) to (d) are 1.89, 1.67, 1.73 and 1.58. It is noted that the performance index may not increase with the increase in height constraints imposed on initial design domains. This is because the performance index for each structure is defined by the scaled weights of the initial design and the current design at each iteration. Therefore, the performance of optimal topologies for different structures such as structures with different heights cannot be ranked by comparing their performance indices. The suitable method is to compare their scaled weights or volumes with respect to the same displacement limit such as 0.5 mm imposed at the loaded point. A comparison of material volumes required for each optimal design while satisfying the same displacement performance level is given in Table 4.2. It is seen from the Table 4.2 that the material volume of the optimal design increases with the decrease in the height constraints. Therefore, the performance of the optimal topology of a structure is improved when increasing the height of its initial design domain.



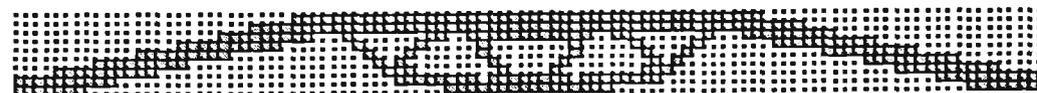
(a) Optimal topology ( $h/L=1/2$ ),  $PI_{ds} = 1.89$



(b) Optimal topology ( $h/L=1/4$ ),  $PI_{ds} = 1.67$



(c) Optimal topology ( $h/L=1/8$ ),  $PI_{ds} = 1.73$



(d) Optimal topology ( $h/L=1/12$ ),  $PI_{ds} = 1.58$

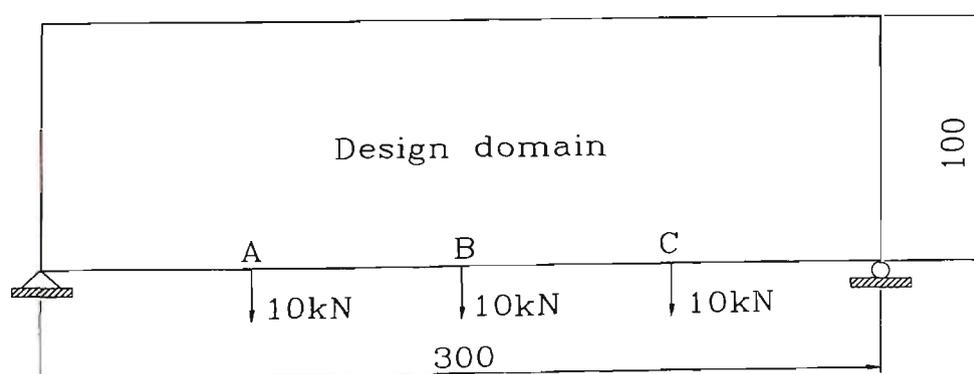
**Fig. 4.7** Effect of height constraints on optimal topologies

**Table 4.2** Effects of height constraints on the material volumes of optimal topologies

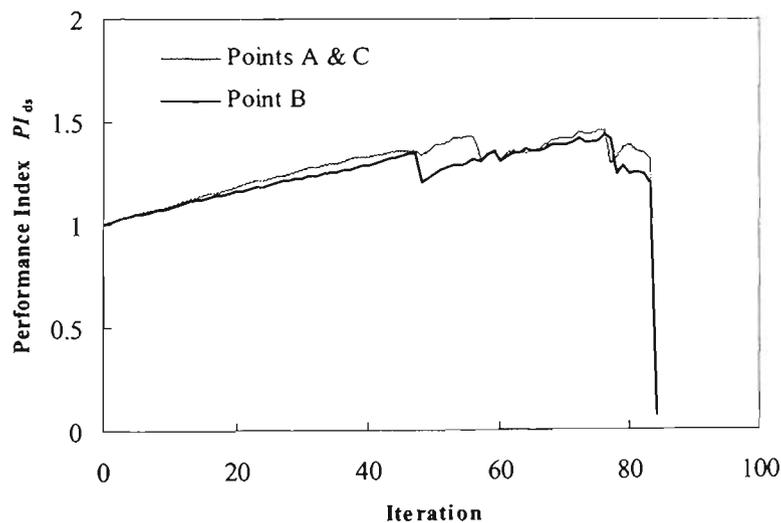
| Height<br>$h/L$ | $u_{optj}$<br>(mm) | $u_j^*$<br>(mm) | $V_{opt}$<br>(mm <sup>3</sup> ) | $V_{opt}^s$<br>(mm <sup>3</sup> ) |
|-----------------|--------------------|-----------------|---------------------------------|-----------------------------------|
| 1/2             | 0.023              | 0.5             | 28800                           | 1342                              |
| 1/4             | 0.036              | 0.5             | 20864                           | 1481                              |
| 1/8             | 0.244              | 0.5             | 7138                            | 3483                              |
| 1/12            | 0.428              | 0.5             | 9125                            | 7811                              |

#### 4.6.4 Multiple Displacement Constraints

This example is to demonstrate the effectiveness of the PBO method in dealing with continuum topology design problems subject to multiple displacement constraints. Fig. 4.8 illustrates the design domain of a simply supported beam under three concentrated loads of 10 kN, each, applied at points A, B and C. This beam are subjected to multiple displacement constraints of the same limit imposed at three loaded points. The design domain is discretized into a  $96 \times 32$  mesh using four-node plane stress elements. The Young's modulus  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and the thickness of elements  $t = 5$  mm are assumed. The  $R = 1\%$  is used in the optimization process.

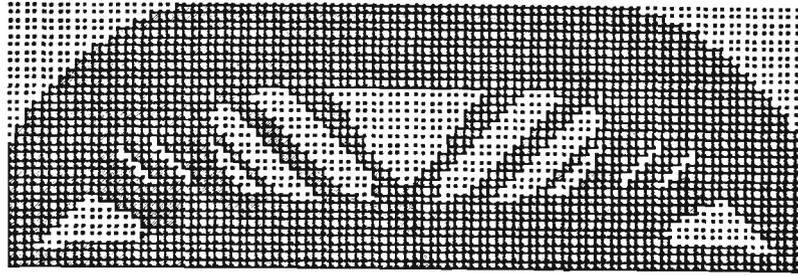
**Fig. 4.8** Design domain of structure with multiple displacement constraints

The performance index history of the simply supported beam is shown in Fig. 4.9. Due to symmetry, the displacements at points A and C are the same. The performance index curves shown in Fig. 4.9 are obtained by using Eq. (4.14) based on the constrained displacements at points A and B in the optimization process. The maximum performance index calculated for point A is 1.46 whilst it is 1.43 for point B. The optimal topology that corresponds to the maximum performance index occurs at iteration 76 for both displacement constraints. It is obvious that the material volume required for the optimal design is governed by the critical displacement constraint imposed at point B.

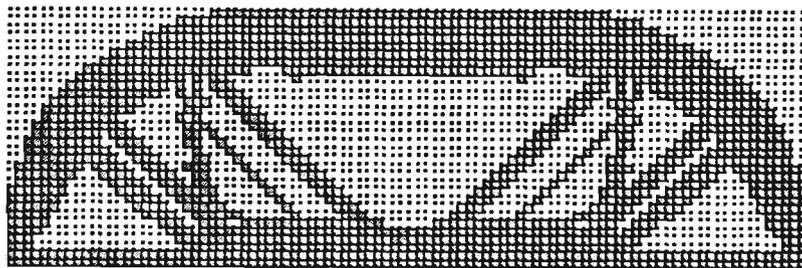


**Fig. 4.9** Performance index history of structure with multiple displacement constraints

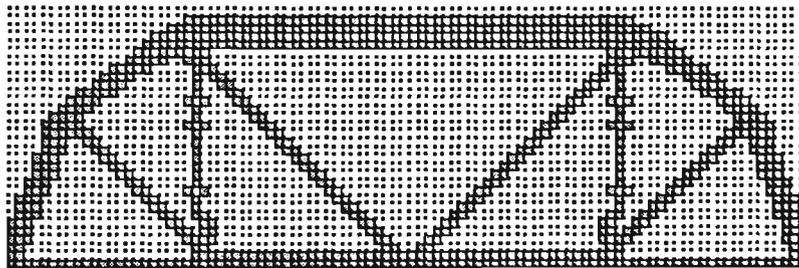
The topology optimization history of the structure subject to multiple displacement constraints is shown in Fig. 4.10. The optimal topology presented in Fig. 4.10(c) indicates a truss-like structure. It is observed that the in-plane size of members in the truss is approximately proportional to the force carried by that member since the virtual strain energy density of elements within the optimal topology is approximately uniform.



(a) Topology at iteration 30



(b) Topology at iteration 50



(c) Optimum at iteration 76



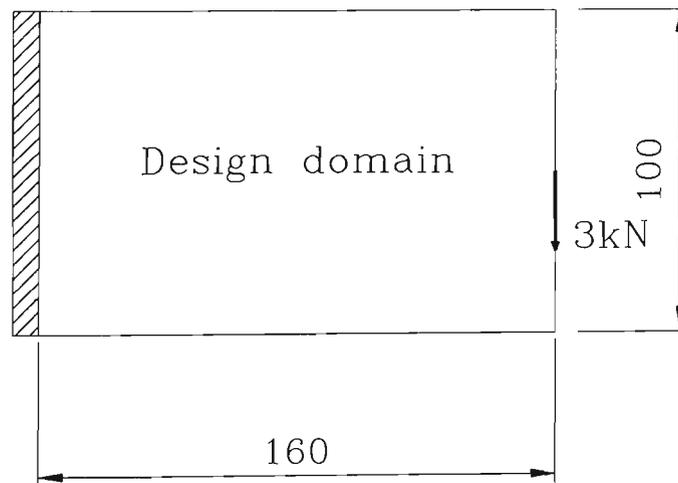
(d) Topology at iteration 83

**Fig. 4.10** Topology optimization history of structure with multiple displacement

### 4.6.5 Ranking the Performance of Structural Topologies

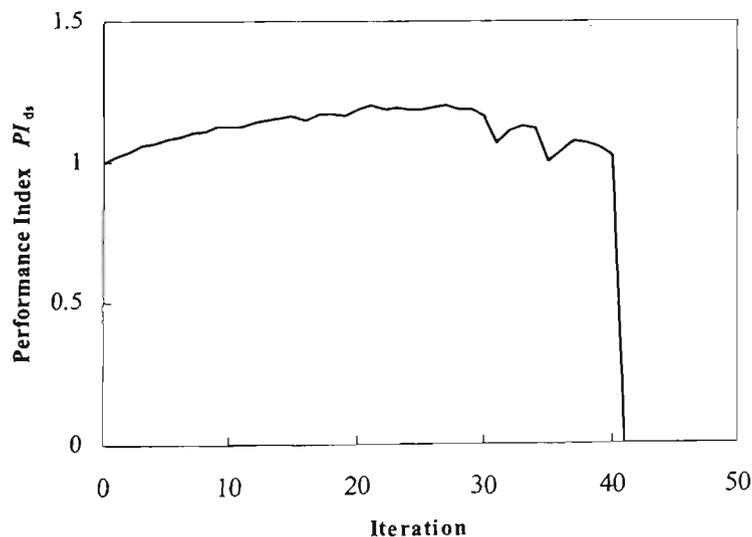
For a same design problem, different optimization methods usually lead to different final designs. In order to select the best topology for the design of continuum structures, the proposed performance index is used to rank the performance of structural topologies produced by different continuum topology optimization methods.

A short cantilever beam subject to a displacement constraint imposed at the loaded point in the vertical direction as shown in Fig. 4.11 is optimized by using the PBO method. The design domain is divided into  $32 \times 20$  plane stress elements. A concentrated load of 3 kN is applied to the center of the free end. The Young's modulus  $E = 207$  GPa, Poisson's ratio  $\nu = 0.3$  and the thickness of the structure  $t = 1$  mm are adopted in the analysis. The element removal ratio  $R = 2\%$  is used in this problem.

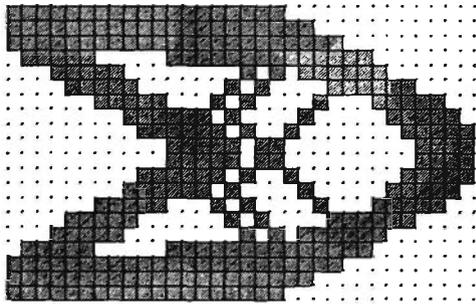


**Fig. 4.11** Design domain of short cantilever beam

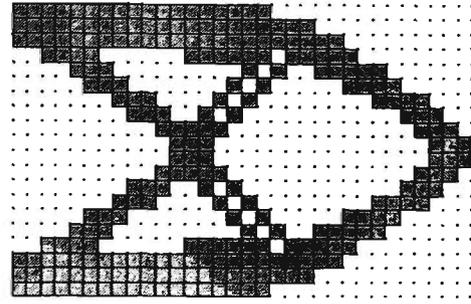
The performance index history of this short cantilever beam is shown in Fig. 4.12. For the initial design without any hole, the performance index is equal to unity whilst the maximum performance index of 1.20 occurs at iteration 27. The optimal topology obtained by the PBO method is presented in Fig. 4.13(a). The topology shown in Fig. 13(b) is given by Chu et al. (1996). The performance index corresponding to Fig. 13(b) is 1.11. The topology given by Zhao et al (1998) is presented in Fig. 13(c), where the model is regenerated. The performance index of the topology shown in Fig. 13(c) is obtained as 1.18. The performance index of the topology given by Suzuki and Kikuchi (1991) using the homogenization-based optimization method as shown in Fig. 13(d) is found to be 1.04. This indicates that the performance of the optimal topology obtained by the PBO method is higher than those presented by other researchers. It also demonstrates that the PBO method developed in this study is a reliable and efficient topology optimization technique.



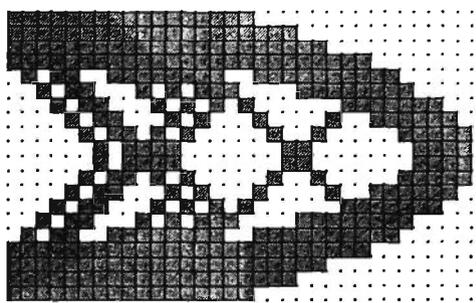
**Fig. 4.12** Performance index history of short cantilever beam



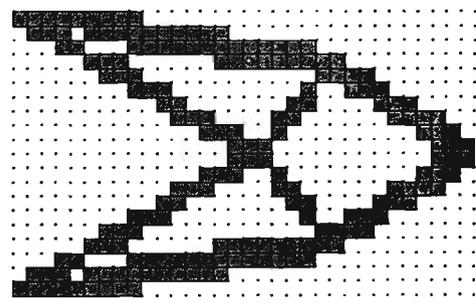
(a) Topology by PBO method  
( $PI_{ds} = 1.20$ )



(b) Topology by Chu et al. (1996)  
( $PI_{ds} = 1.11$ )



(c) Topology by Zhao et al. (1998)  
( $PI_{ds} = 1.18$ )



(d) Topology by Suzuki and Kikuchi (1991)  
( $PI_{ds} = 1.04$ )

**Fig. 4.13** Ranking the Performance of structural topologies produced by different continuum topology optimization methods

## 4.7 EXAMPLES: PLATES IN BENDING

### 4.7.1 General

It is traditionally believed that continuum topology optimization can offer more material savings than continuum shape optimization. However, this statement may hold true only for plane stress structures, and it may not be the case for optimal design of plates in bending. Little work has been undertaken so far to compare the performance of optimal designs for bending plates, which are generated by using topology and shape optimization methods. Therefore, the PBO method is used to investigate the effects of topology and shape optimization techniques, loading and boundary conditions on the optimal designs of plates in bending in this section.

### 4.7.2 Clamped Plate under Concentrated Loading

The design domain of a clamped square plate under a concentrated load of 500 N applied to the centre of the plate is shown in Fig. 4.14. A single displacement constraint is imposed at the loaded point in the vertical direction. The design domain is divided into a  $50 \times 50$  mesh using four-node plate elements. The Young's modulus  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and the thickness of the plate  $t = 5$  mm are used in the finite element model. Four elements around the loaded point are frozen so that this region is not removed during the optimization process. The element removal ratio  $R = 1\%$  is adopted in the performance optimization process.

Fig. 4.15 shows the performance index histories for topology and shape optimization of clamped plate in bending. It can be seen that performance indices are gradually increased while inefficient elements are eliminated from the plate in the optimization process. It is interesting of that performance indices for topology and shape optimization are almost identical up to iteration 59. However, shape optimization provides a slightly higher performance index than topology optimization. The maximum performance

index of the topology is 2.09 whilst the maximum performance index of the shape is 2.13. After the performance is maximized, further element removal will destroy the structure as indicated by performance index histories shown in Fig. 4.15.

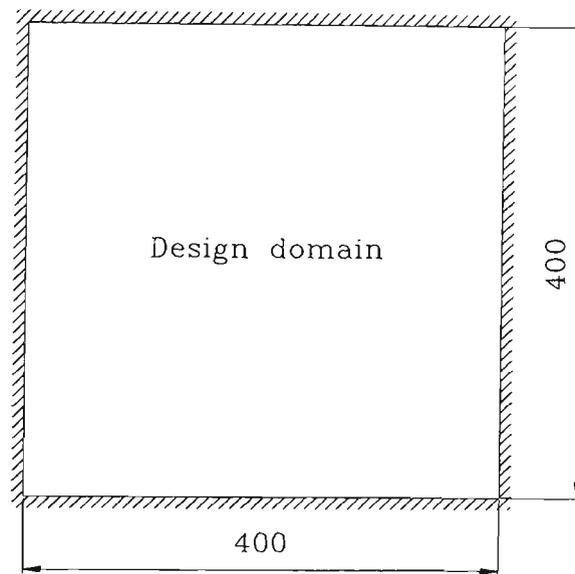


Fig. 4.14 Design domain of clamped plate under concentrated loading

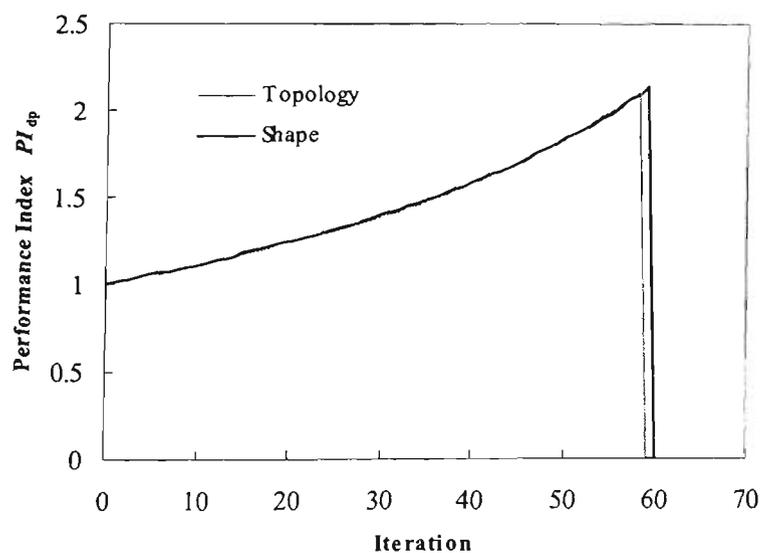


Fig. 4.15 Performance index histories of clamped plate under concentrated loading

The histories of topology and shape optimization for the plate are shown in Fig. 4.16 and Fig. 4.17. It is noted that cavities in the interior of the plate are created by topology optimization whilst no holes in the interior of the plate are generated by shape optimization. Based on the consideration of manufacture and structural performance, shape optimization technique should be used to optimize plates in bending.

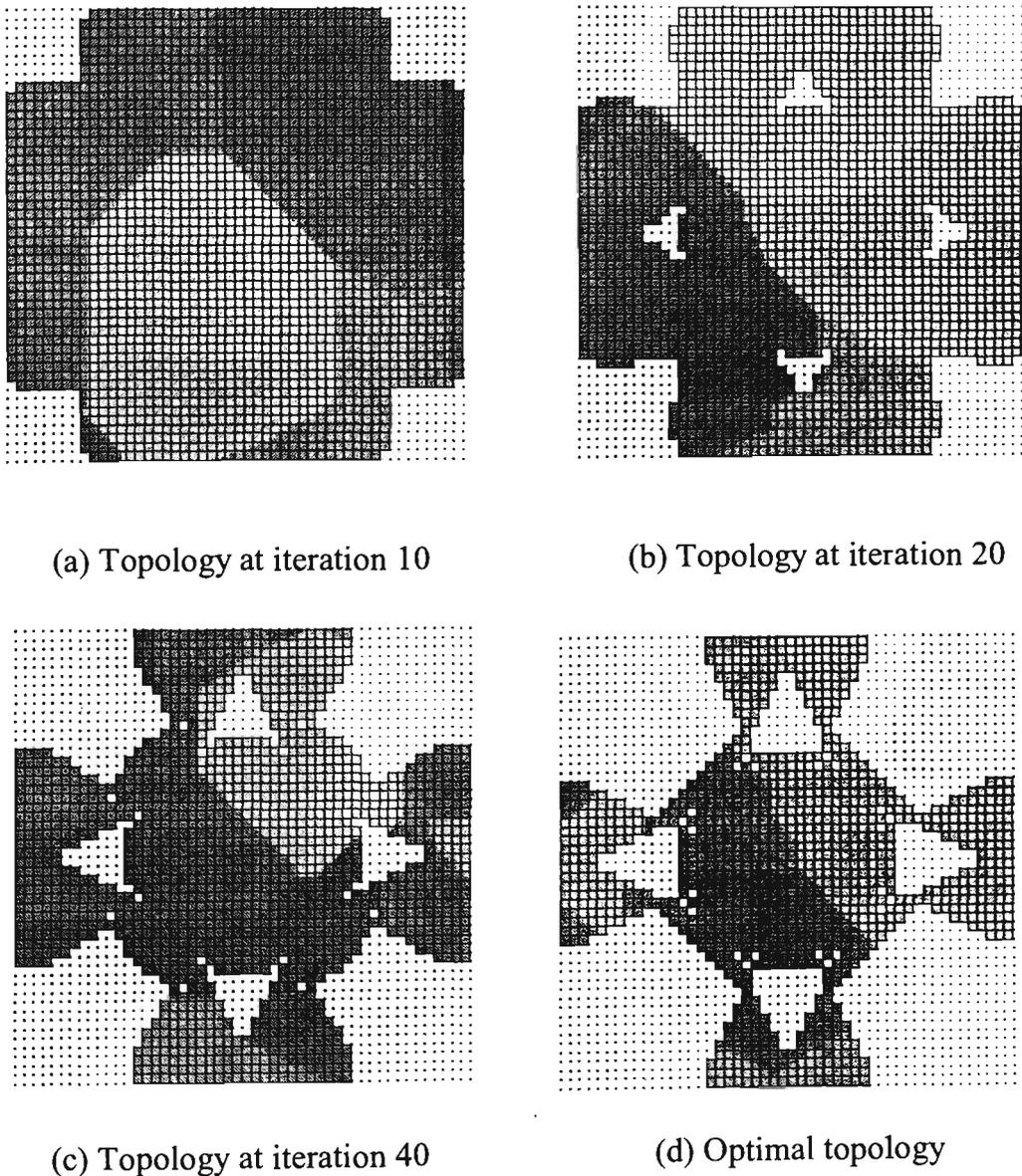
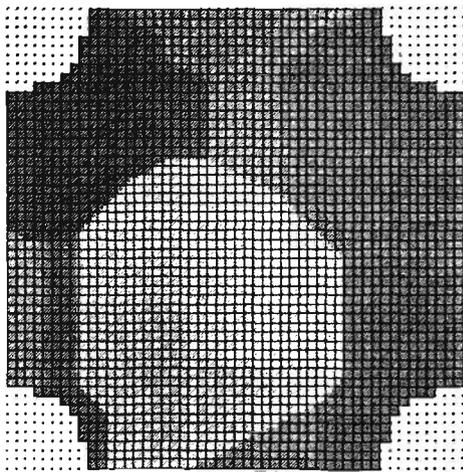
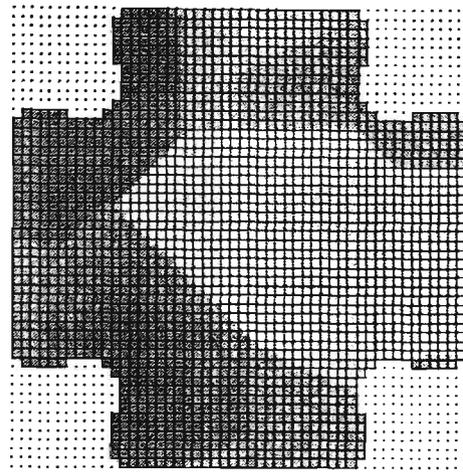


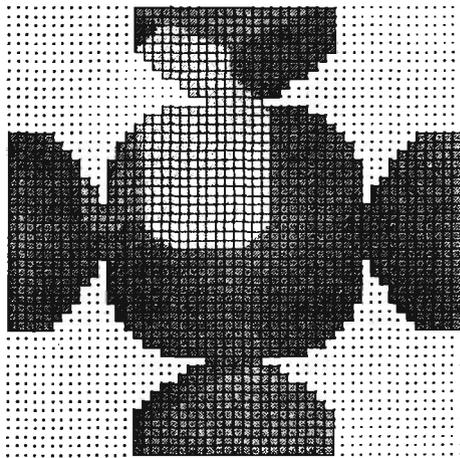
Fig. 4.16 Topology optimization history of clamped plate under concentrated loading



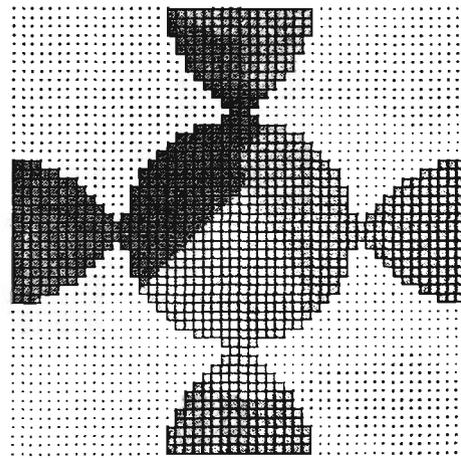
(a) Topology at iteration 10



(b) Topology at iteration 20



(c) Topology at iteration 40



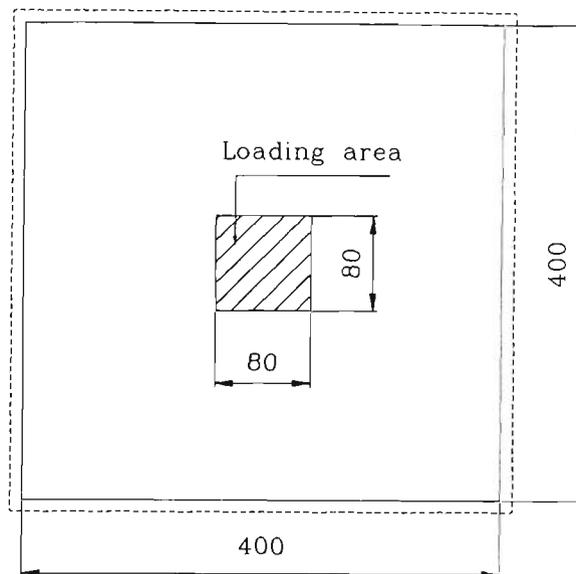
(b) Optimal topology

**Fig. 4.17** Shape optimization history of clamped plate under concentrated loading

### 4.7.3 Simply Supported Plate under Area Loading

Figure 4.18 shows the design domain of a simply supported plate under a local area pressure of 0.1 MPa normal to the surface of the plate. A single displacement constraint is imposed at the centre of the plate. The mesh and material properties used are the same

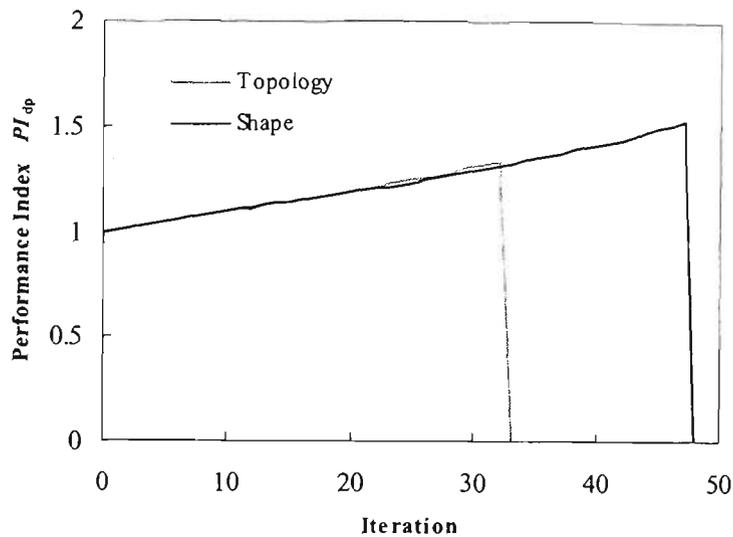
as the previous example. The loading region is frozen. The element removal ratio  $R = 1\%$  is used.



**Fig. 4.18** Design domain of simply supported plate under area loading

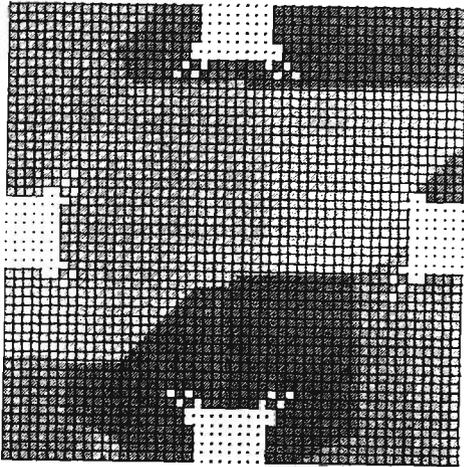
The performance index histories for topology and shape optimization are demonstrated in Fig. 4.19. It is seen that performance indices at each iteration are almost identical up to iteration 32 for the plate using topology and shape optimization techniques. However, the maximum performance index of the optimal shape is 1.53 while the maximum performance index of the optimal topology is only 1.34. This illustrates that shape optimization for plates in bending can generate higher performance optimal designs than topology optimization.

The histories of topology and shape optimization are presented in Figs. 4.20 and 4.21. It is observed that hinge lines are formed between the corners and the central region of the

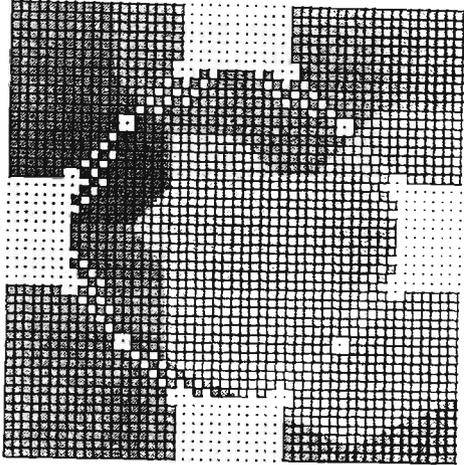


**Fig. 4.19** Performance index histories of simply supported plate under area loading

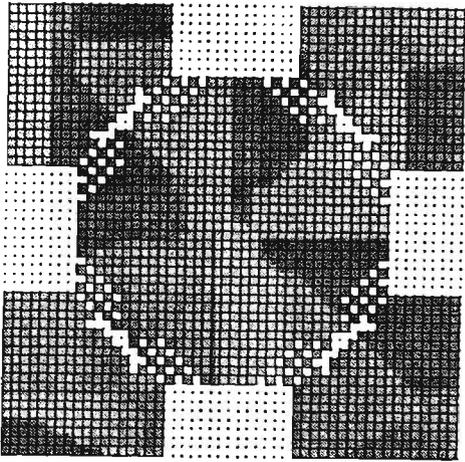
optimal topology using the topology optimization technique. However, no hinge lines are observed in the optimal shape because elements are only eliminated from the boundaries of the plate in shape optimization. Moreover, it is seen that checkerboard pattern appear in resulting topologies whilst no checkerboard pattern is present in shapes obtained. It is difficult to manufacture structures with checkerboard pattern. Although the checkerboard pattern can be eliminated by incorporating an intuitive smoothing scheme into the PBO algorithms, the computational cost will be penalized. From the manufacturing, computational cost and structural performance points of view, it is suggested that shape optimization technique should be used to optimize plates in bending. The effects of boundary conditions on optimal topologies and shapes of bending plates can be seen from the example presented in Section 4.7.2 and this example.



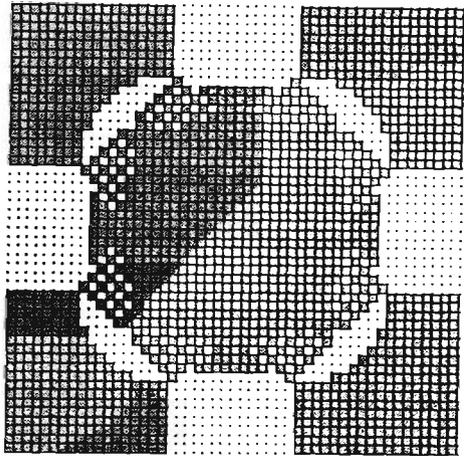
(a) Topology at iteration 10



(b) Topology at iteration 20

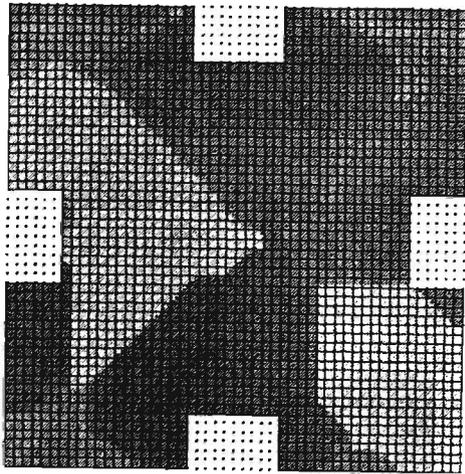


(c) Topology at iteration 40

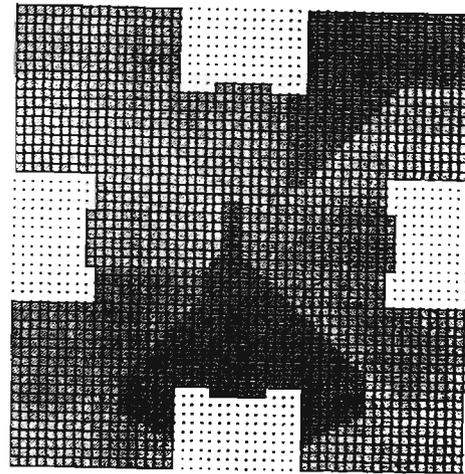


(d) Optimal Topology

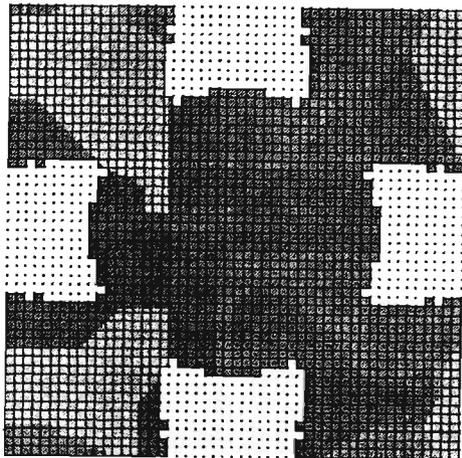
Fig. 4.20 Topology optimization history of simply supported plate under area loading



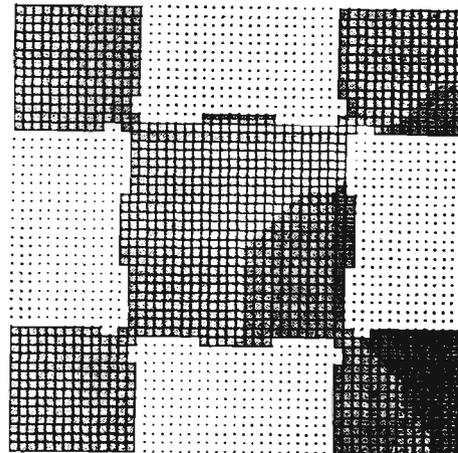
(a) Shape at iteration 10



(b) Shape at iteration 20



(c) Shape at iteration 25



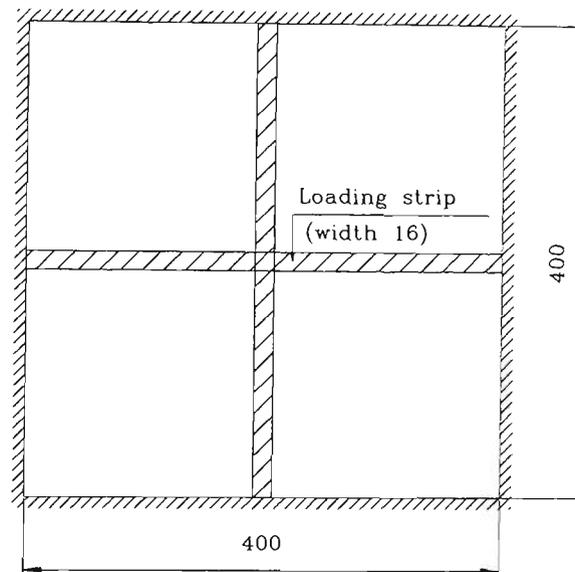
(d) Optimal shape

Fig. 4.21 Shape optimization history of simply supported plate under area loading

#### 4.7.4 Clamped Plate under Strip Loading

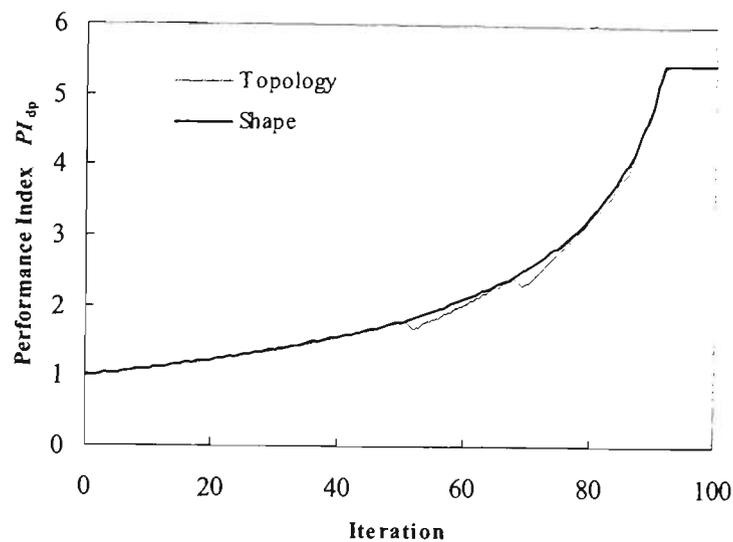
The design domain of a clamped square plate under strip pressures of 0.1 MPa is illustrated in Fig. 4.22. A single displacement constraint imposed at the centre of the plate is considered. The mesh and material properties are the same as used in previous

example. Elements in the two loaded strips are frozen. The element removal ratio  $R = 1\%$  is employed in this problem. This plate is optimized using the PBO method for topology and shape optimization.



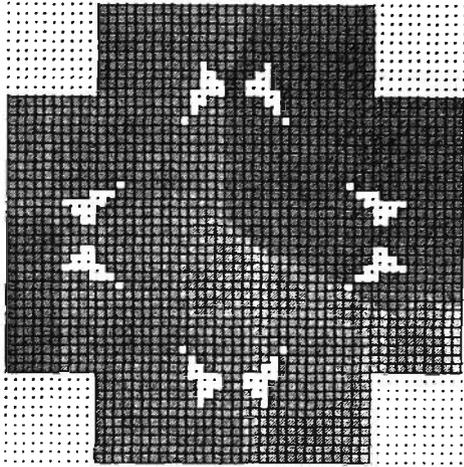
**Fig. 4.22** Design domain of clamped plate under strip loading

Figure 4.23 shows performance index histories for the topology and shape optimization of the clamped plate under a strip loading. It can be observed that the performance index curve obtained using the shape optimization method is smoother than the one generated using the topology optimization scheme. This may be the effect of holes, which are created in the interior of the plate in topology optimization process. However, it is shown that these two optimization methods provide optimal designs with the same maximum performance index of 5.44, which is constant in later iterations. This is because loaded strips are frozen so that no elements can be removed from loading strips after eliminating all of the un-frozen elements from the design.

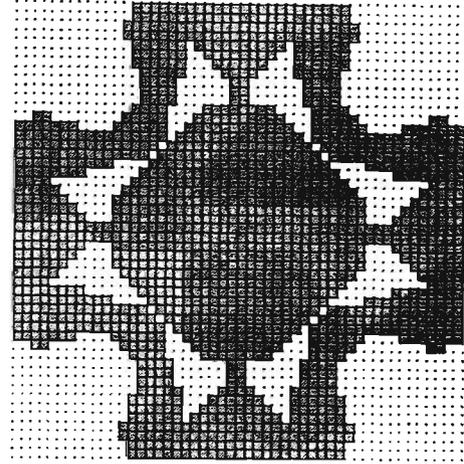


**Fig. 4.23** Performance index histories of clamped plate under strip loading

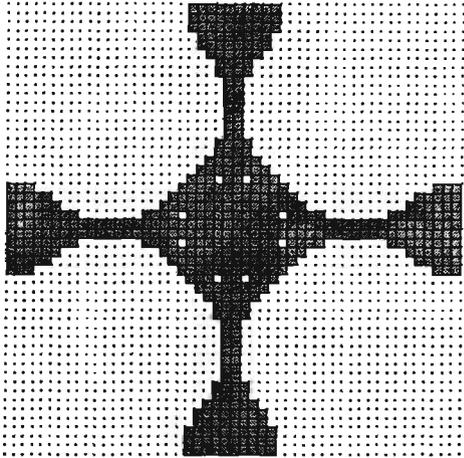
The topology and shape optimization histories of the plate under the strip pressure are presented in Figs. 4.24 and 4.25. It is seen from these figures that although the results generated at the same iteration in the performance optimization process are different using different optimization techniques, final optimal designs are the same for this plate under strip loading. The optimal shape suggests that the most efficient design can be achieved by using beam to support the strip loading.



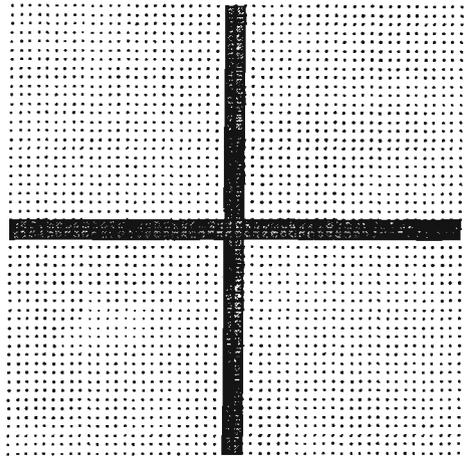
(a) Topology at iteration 20



(b) Topology at iteration 40

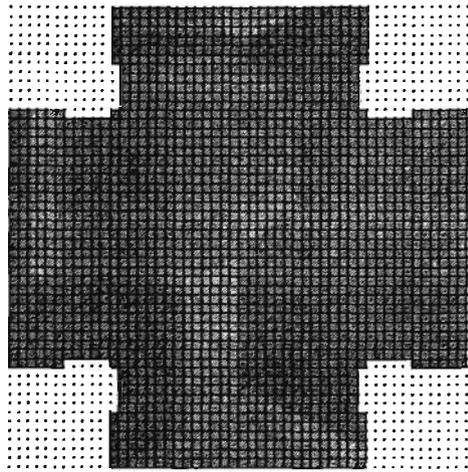


(a) Topology at iteration 80

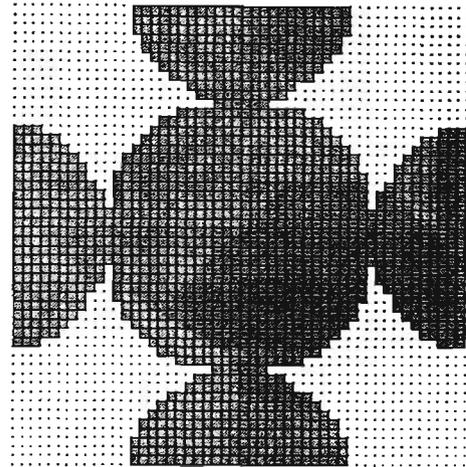


(b) Optimal topology

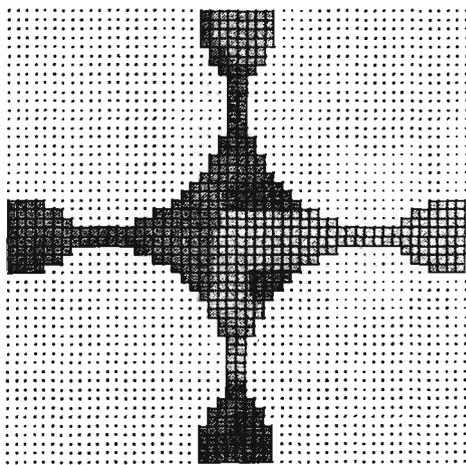
**Fig. 4.24** Topology optimization history of clamped plate under strip loading



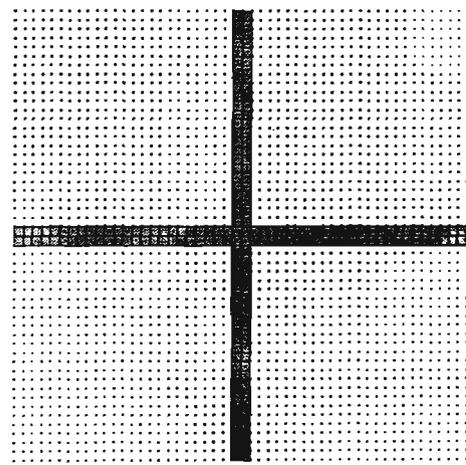
(a) Shape at iteration 20



(b) Shape at iteration 40



(c) Shape at iteration 80



(d) Optimal shape

Fig. 4.25 Shape optimization history of clamped plate under strip loading

## 4.8 CONCLUDING REMARKS

In this chapter, the performance-based optimization (PBO) method has been developed for the optimal topology and shape design of continuum structures subject to

displacement constraints. The PBO method is formulated on the basis of displacement performance criteria, which allows for elements with the lowest virtual strain energy density to be gradually eliminated from a continuum design domain to achieve a lightweight design with minimum deformations. In the proposed method, continuum topology and shape optimization is treated as the problem of improving the performance of a continuum design domain in terms of the material usage in effectively resisting deformations. Maximizing the performance indices in the design space is proposed as performance-based optimization criteria. The proposed performance indices are used to monitor the optimization process and as a termination criterion in performance optimization algorithms.

It has been demonstrated that the PBO method can effectively generate optimal topologies and shapes, which have been verified by analytical solutions. It is shown that a smoother solution can be achieved by using a smaller element removal ratio in the performance optimization process but at the expense of a higher computational cost. The results indicate that increasing the height of an initial design domain usually improves the efficiency of the final optimal design. It has been shown that the shape optimization technique provides higher performance optimal designs than the topology optimization method does. From the manufacturing and efficient points of view, the shape optimization technique should be used to optimize plates in bending.

Performance-based optimality criteria developed herein can be incorporated in any existing continuum topology optimization methods to guarantee success in obtaining the global optimal designs with reasonable effort. Furthermore, the proposed performance indices can be used to rank the performance of structural topologies and shapes produced by different structural optimization methods and the efficiency of continuum topology optimization methods.

## Chapter 5

# OPTIMAL TOPOLOGY DESIGN FOR OVERALL STIFFNESS PERFORMANCE

### 5.1 INTRODUCTION

The Performance-Based Optimization (PBO) method for topology and shape designs of continuum structures with displacement constraints has been presented in Chapter 4. Element removal criteria are based on the virtual strain energy density of elements. In order to calculate the virtual strain energy density of elements, the structure has to be analyzed under virtual unit loads, which are applied to loaded points. The optimal design produced by using the PBO method for displacement performance is in favor of specific displacements. This means that the optimal design is a maximum stiffness design at minimum-weight with respect to specific displacements. For a structure under only a few point loads, the PBO method for structures with displacement constraints is efficient in generating minimum-weight designs for stiffness. However, for a continuum structure under many concentrated loads and multiple loading conditions, many virtual unit loads have to be used in order to calculate the virtual strain energy density of elements for elimination. This will considerably increase the computational cost. In addition, the weighting average approach used to evaluate the virtual strain energy density of elements may not be efficient for structures under multiple displacement

constraints and loading conditions. To overcome these limitations, a more general approach that considers the overall stiffness performance needs to be developed.

Structural topology optimization involves a large number of design variables and is the most computationally expensive design task. To simplify the design optimization problem, the objective for stiffness design is usually to minimize the mean compliance of a structure. It is noted that minimizing the mean compliance of a structure is equivalent to maximizing its overall stiffness. The homogenization-based design concept has been extensively adopted in continuum topology optimization for compliance minimization problems (Bendsøe and Kikuchi 1988; Suzuki and Kikuchi 1991; Diaz and Bendsøe 1992; Tenek and Hagiwara 1993; Youn and Park 1997; Krog and Olhoff 1999). Other optimization approaches for the stiffness design of continuum structures have also been developed recently (Gea 1996; Swan and Kosaka 1997; Yang 1997).

The difficulty involved in continuum topology optimization is to incorporate an appropriate termination criterion in optimization algorithms to obtain the global optimum. The prescribed material volume has commonly been used in optimization approaches as the termination criterion (Bendsøe and Kikuchi 1988; Suzuki and Kikuchi 1991; Diaz and Bendsøe 1992; Tenek and Hagiwara 1993; Youn and Park 1997; Krog and Olhoff 1999; Gea 1996; Swan and Kosaka 1997). Based on this termination criterion, using a different percentage of material volume as the constraint leads to different designs. The mean compliance has been used as the termination criterion that is the only condition for determining the final design (Yang 1997). The results satisfying these termination criteria mentioned might not be the global optimum in the given design space. Therefore, there is a strong need to develop performance-based termination criteria that can be used in continuum topology optimization methods to identify globally optimal designs for overall stiffness performance.

In this chapter, the Performance-Based Optimization (PBO) method is developed for optimal topology design of continuum structures for overall stiffness performance. In

the proposed method, the weight of a structure is used as the objective function and the constraint is imposed on the mean compliance of the structure. Continuum topology optimization is treated as the problem of improving the performance of a continuum design domain in terms of the efficiency of material usage and overall stiffness. Two stiffness performance indices are developed for evaluating the performance of resulting topologies for plane stress continuum structures and for plates in bending in the optimization process. These performance indices are also used as termination criteria in performance-based optimization algorithms. The maximization of performance indices is proposed as performance-based optimization criteria. Part of the performance-based optimization theory has been addressed by Liang et al. (2000b) and Liang and Steven (2001a)

## 5.2 TOPOLOGY DESIGN PROBLEM FORMULATION

The formulation of a design optimization problem is of significant importance to the success of an optimization method. A poor formulation can lead to poor results or to prohibitive development cost. Moreover, an improper problem formulation may lead to meaningless results that cannot be used in practical design.

For the maximum stiffness topology design, the mean compliance of a structure has commonly been used as the objective function, and the constraint is imposed on a somewhat arbitrarily specified material volume (Bendsøe and Kikuchi 1988; Suzuki and Kikuchi 1991; Diaz and Bendsøe 1992; Tenek and Hagiwara 1993; Youn and Park 1997; Krog and Olhoff 1999; Gea 1996; Swan and Kosaka 1997). However, it is noteworthy that the designer usually does not know what percentage of the material volume is the minimum for supporting applied loads in advance. Optimization methods based on such a problem formulation certainly lead to a trial-and-error design process if the designer really wants to find the minimum-weight design. The optimal material usage shall be sought by using optimization methods rather than specified by the designer. Therefore, realistic optimization approaches for stiffness design are to treat

the weight of a structure as the objective function and behavior quantities such as the mean compliance or displacements as constraints since limitations on behavior quantities are specified in national or international design codes.

In the proposed method, the weight of a structure is used as the objective function and the mean compliance is treated as the constraint. In other words, the performance objective is to minimize the weight of a continuum design domain while maintaining its overall stiffness within an acceptable limit. The performance objective can be expressed in mathematical forms as follows:

$$\text{minimize } W = \sum_{e=1}^n w_e(t) \quad (5.1)$$

$$\text{subject to } C \leq C^* \quad (5.2)$$

in which  $W$  is the total weight of a structure,  $w_e$  is the weight of the  $e$ th element,  $t$  is the thickness of elements,  $C$  is absolute value of the mean compliance of the structure,  $C^*$  is the prescribed limit of  $C$ , and  $n$  is the total number of elements in the structure. The mean compliance of a structure is usually used as an inverse measure of the overall stiffness of a structure. Since the thickness of a continuum structure, which is specified by the designer in practice, has a significant effect on the weight of the structure it is treated as one of design variables. However, only uniform sizing of the element thickness is considered in the proposed method owing to its practical engineering applications.

### 5.3 ELEMENT REMOVAL CRITERIA BASED ON STRAIN ENERGY DENSITY

Element removal criteria have been derived for element elimination in the PBO method for continuum structures with stress and displacement constraints in Chapters 3 and 4. To obtain fully stressed topology designs, lowly stressed elements are gradually removed from a discretized continuum design domain. In contrast, elements with the lowest virtual strain energy density are gradually deleted from a design domain in order to generate an optimal topology with the least deformations at specific locations at minimum weight. However, these element removal criteria cannot be used to optimize continuum structures under an overall stiffness constraint. As a result of this, new element removal criteria need to be derived for topology designs when considering the system performance. As presented in Chapter 4, the criteria for element removal can be developed on the basis of the design sensitivity analysis of constraints with respect to design variables. In the PBO method, the design sensitivity analysis is to study the effect of element removal on the changes of the mean compliance of a continuum structure. Approximate concepts are employed in the design sensitivity analysis (Kirsch 1993).

The equilibrium equation for a linearly elastic structure in the finite element formulation can be expressed by

$$[K]\{u\} = \{P\} \quad (5.3)$$

The element removal concept is used in the proposed topology optimization method. When a structure is modified by eliminating the  $e$ th element from a continuum design domain modeled by finite elements, the stiffness matrix and displacement vector of the structure will be changed accordingly. Eq. (5.3) can be rewritten as

$$([K] + [\Delta K])(\{u\} + \{\Delta u\}) = \{P\} \quad (5.4)$$

in which  $[\Delta K]$  is the changes of the stiffness matrix and  $\{\Delta u\}$  is the change of nodal displacements vector. It is noted that the loads applied to the structure are unchanged. Since only the  $e$ th element is removed from the structure, the change of the stiffness matrix can be derived as

$$[\Delta K] = [K_r] - [K] = -[k_e] \quad (5.5)$$

where  $[K_r]$  is the stiffness matrix of the resulting structure and  $[k_e]$  is the stiffness matrix of the  $e$ th element. The change of displacement vector due to element elimination can approximately be obtained from Eqs. (5.3) and (5.4) by neglecting higher order terms as

$$\{\Delta u\} = -[K]^{-1}[\Delta K]\{u\} \quad (5.6)$$

The strain energy or mean compliance of a structure can be expressed by

$$C = \frac{1}{2}\{P\}^T\{u\} \quad (5.7)$$

The change of the strain energy of a structure due to the removal of the  $e$ th element can approximately be determined by

$$\Delta C = \frac{1}{2}\{P\}^T\{\Delta u\} = -\frac{1}{2}\{P\}^T[K]^{-1}[\Delta K]\{u\} = -\frac{1}{2}\{u\}^T[\Delta K]\{u\} = \frac{1}{2}\{u_e\}^T[k_e]\{u_e\} \quad (5.8)$$

in which  $\{u_e\}$  is the displacement vector of the  $e$ th element. Eq. (5.8) shows that the change of the strain energy of a structure due to the removal of the  $e$ th element can be approximately evaluated by the strain energy of the  $e$ th element. Therefore, the element strain energy can be used as a measure of element contribution to the overall stiffness of a structure, and is denoted as

$$c_e = \frac{1}{2} \{u_e\}^T [k_e] \{u_e\} \quad (5.9)$$

To achieve the performance objective, it is obvious that a small number of elements with the lowest strain energy should be systematically removed from a structure. For continuum structures modeled with different size finite elements, the element strain energy per unit volume (mass), which is defined as the *strain energy density* of elements, should be calculated for element removal. The strain energy density of the  $e$ th element is calculated by

$$\zeta_e = \frac{|c_e|}{v_e} \quad (5.10)$$

in which  $v_e$  is the volume of the  $e$ th element. If the material density and element thickness are uniformly distributed in a design domain, either the volume or the weight ( $w_e$ ) of an element can be used to calculate the element strain energy density in Eq. (5.10). However, if the material density is varied in the continuum design domain, the element weight ( $w_e$ ) shall be used in Eq. (5.10).

For structures subject to multiple loading conditions, a logical AND scheme can be used in optimization algorithms to take account of the effects of different loading conditions on optimal designs. In the logical AND scheme, an element is eliminated from the design domain only if its strain energy density is the lowest for all loading conditions. The element removal criteria can be expressed as follows:

$$\text{Load case 1: } \zeta_e^1 \in \{Q^1\} \quad (5.11a)$$

$$\text{Load case } q: \zeta_e^q \in \{Q^q\} \quad (5.11b)$$

in which  $\zeta_e^1, \dots, \zeta_e^q$  are the strain energy density of the  $e$ th element under load case 1,  $\dots, q$ ;  $\{Q^1\}, \dots, \{Q^q\}$  are the vectors of the lowest strain energy density of elements

under load case 1, ...,  $q$ ; and  $q$  is the total number of loading cases. A loop is used to count elements with the lowest strain energy density until they made up the specified amount that is the element removal ratio times the total number of elements in the initial design domain. The element removal ratio ( $R$ ) for each iteration is defined as the ratio of the number of elements to be removed to the total number of elements in the initial design domain. The element removal ratio is not changed in the whole optimization process.

## 5.4 PERFORMANCE-BASED OPTIMALITY CRITERIA

### 5.4.1 General

By gradually eliminating elements with the lowest strain energy density from a design domain, the distribution of element strain energy density will consequently become more and more uniform within the resulting structure. The uniform strain energy density has been used as an optimality condition in continuum topology optimization approaches, and can be derived by using Kuhn-Tucker conditions (Ramm et al. 1994). However, the uniform condition of element strain energy density in a continuum design domain may not be achieved even if the overall stiffness constraint is violated. *This means that a minimum-weight design with acceptable overall stiffness performance is not necessarily a design where the distribution of element strain energy density is absolutely uniform.* Therefore, as stated in the fully stressed design and design for displacement performance, the uniformity of element strain energy density cannot be incorporated in continuum topology optimization methods as a termination condition for identifying the optimal topology. Performance-based optimality criteria for plane stress and plates in bending problems are developed in this section.

### 5.4.2 PBOC for Plane Stress Problem

In design problems with element thickness or cross-sectional design variables, an infeasible design in the optimization process can be converted into a feasible one by the scaling procedure. Due to its simplicity and efficiency, this scaling procedure has been used in conventional Optimality Criteria (OC) methods (Morris 1982; Kirsch 1993) for truss layout optimization. The scaling design concept has been utilized to develop performance indices in previous chapters for evaluating the performance of structural topologies and shapes with stress and displacement constraints. The scaling design procedure is also employed to derive a performance index for structures with overall stiffness constraint in this section.

For plane stress continuum structures, the stiffness matrix of a structure is a linear function of the element thickness. Therefore, for structures with the mean compliance constraint, the element thickness can be uniformly scaled to keep the mean compliance constraint active in the optimization process. By scaling the initial design with a factor of  $C_0 / C^*$ , the scaled weight of the initial design can be expressed by

$$W_0^s = \left( \frac{C_0}{C^*} \right) W_0 \quad (5.12)$$

in which  $W_0$  is the actual weight of the initial design domain and  $C_0$  is the absolute value of the strain energy of the initial design under applied loads. Similarly, by scaling the current design with respect to the mean compliance limit, the scaled weight of the current design at the  $i$ th iteration can be determined by

$$W_i^s = \left( \frac{C_i}{C^*} \right) W_i \quad (5.13)$$

where  $C_i$  is the absolute value of the strain energy of the current structure under applied loads at the  $i$ th iteration and  $W_i$  is the actual weight of the current structure at the  $i$ th iteration.

The performance of the resulting topology at the  $i$ th iteration can be quantified by the performance index, which is proposed as

$$PI_{cs} = \frac{W_0^s}{W_i^s} = \frac{(C_0 / C^*)W_0}{(C_i / C^*)W_i} = \frac{C_0 W_0}{C_i W_i} \quad (5.14)$$

To obtain the optimum, the *performance-based optimality criterion* (PBOC) for plane stress structures with overall stiffness constraint can be proposed as

$$\text{maximize } PI_{cs} = \frac{C_0 W_0}{C_i W_i} \quad (5.15)$$

*This PBOC means that the optimal topology of a continuum structure under applied loads is found when its associated strain energy and material consumption are a minimum.* The optimal topology obtained represents an efficient load-carrying mechanism within the design domain. It can be seen that the format of the performance index for plane stress structures with mean compliance constraint is the same as that for plane stress structures with displacement constraints.

It is traditionally believed that the optimization problems of minimizing the mean compliance for a somewhat arbitrarily chosen material volume constraint and minimizing the weight of a structure for a given mean compliance constraint result in equivalent solutions. However, it can be seen from Eq. (5.14) that this statement holds true only when the chosen material volume is the minimum value, which maximizes the performance index in Eq. (5.14). In practice, the minimum material usage is difficult to be determined a priori by the design engineer. Therefore, the applications of structural

optimization methods formulated with an objective of minimizing the mean compliance for a somewhat arbitrarily chosen material volume are limited.

It can be observed from Eq. (5.14) that the optimal topology does not depend on the mean compliance limit, which is usually set to a large value in an optimization process. Since the performance index is a dimensionless number, the uniform scaling of element thickness does not affect its values. In other words, the element thickness of an initial design domain can be assumed and needs not to be changed in the finite element analysis and optimization process. The performance index can be employed in optimization algorithms to monitor the optimization history so that the optimum can be identified. After obtaining the optimum, the actual mean compliance constraint can be satisfied by changing the element thickness. The mean compliance constraint can be interpreted as displacement constraints. Maximum displacements that are the most critical can be checked after obtaining the optimal topology.

For a structure subjected to multiple loading cases, the performance index of the structure at each iteration can be calculated by using the strain energy of the structure under the most critical loading case in the optimization process.

### 5.4.3 PBOC for Plate Bending Problem

The performance index and performance-based optimality criterion for topology and shape optimization of bending plates subject to displacement constraints have been deduced by using the scaling procedure in Chapter 4. When the thickness of a bending plate is uniformly scaled, the displacement vector of the scaled plate as given by Eq. (4.22) is represented by

$$\{u^s\} = \frac{1}{\phi^3} \{u\} \quad (5.16)$$

It can be seen from Eq. (5.7) that the mean compliance of a bending plate is proportional to the displacement vector. When the thickness of a bending plate is reduced by a factor  $\varphi$ , the mean compliance will increase with a factor of  $1/\varphi^3$ . In order to satisfy the mean compliance constraint, the plate needs to be scaled by

$$\varphi = \left( \frac{C}{C^*} \right)^{1/3} \quad (5.17)$$

By using the scaling procedure, the performance index of a bending plate at the  $i$ th iteration can be derived as follows

$$PI_{cp} = \left( \frac{C_0}{C_i} \right)^{1/3} \frac{W_0}{W_i} \quad (5.18)$$

It can be seen from Eq. (5.18) that the performance index formula is composed of the structural response (the mean compliance) and the weight of the structure. In other words, the performance index is a measure of structural responses and the reduction in the weight of the plate in the optimization process, and thus quantifies the performance of a bending plate. To obtain the optimum, *the performance-based optimality criterion for plates in bending with an overall stiffness constraint* can be proposed as

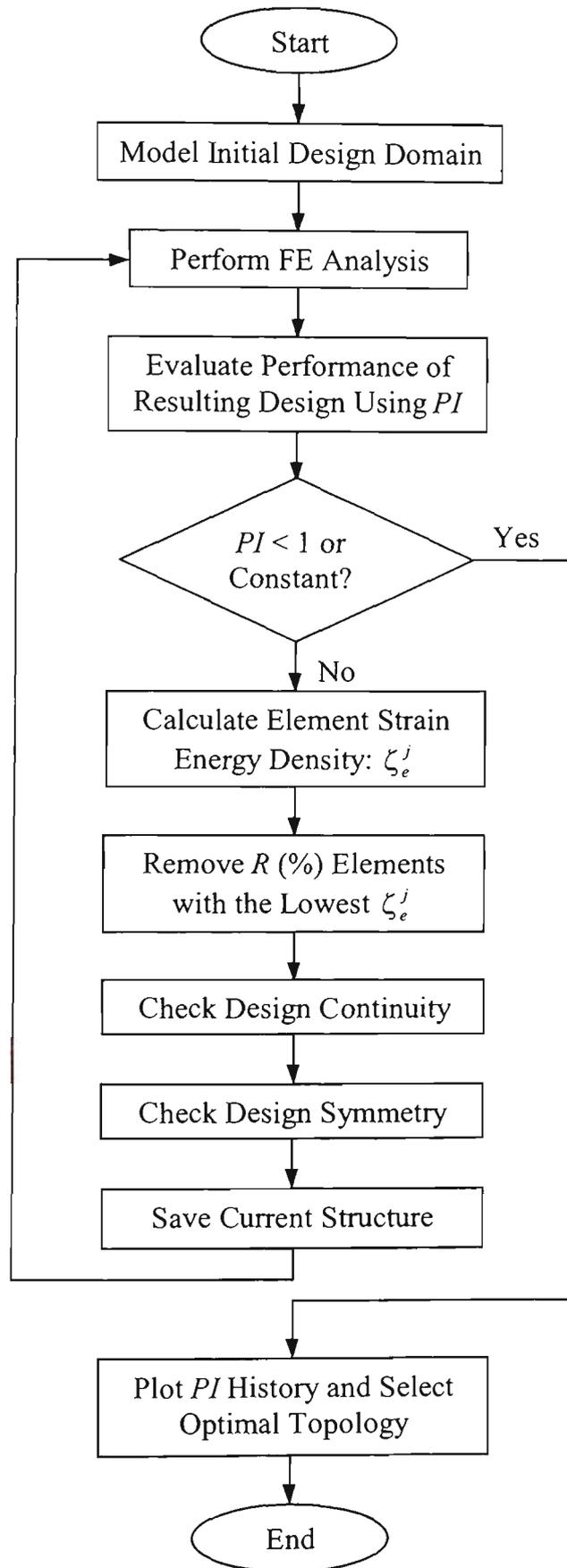
$$\text{maximize } PI_{cp} = \left( \frac{C_0}{C_i} \right)^{1/3} \frac{W_0}{W_i} \quad (5.19)$$

For a bending plate under multiple load cases, the strain energy of the plate under the most critical load case should be used in the calculation of the performance index.

## 5.5 PERFORMANCE OPTIMIZATION PROCEDURE

The performance-based optimization method proposed herein utilizes the finite element method as the modeling and analytical tool. Based on the information obtained from the results of the finite element analysis (FEA), underutilized elements can be identified. The performance of a structural topology can then be improved by gradually eliminating these underutilized elements from the structure. The process of FEA and performance improvement is repeated until the termination criterion is satisfied. The main steps of the performance-based optimization procedure are illustrated in the flowchart given in Fig. 5.1. The optimization procedure is also explained as follows:

- (1) Model the initial design domain with fine finite elements. Applied loads, material properties and support conditions are specified. Non-design regions that are not removed in the optimization process are defined by specifying the number of their material properties to a different number from that of design regions.
- (2) Perform a linear elastic finite element analysis on the structure.
- (3) Evaluate the performance of resulting topology using Eq. (5.14) for plane stress structures and Eq. (5.18) for plates in bending.
- (4) Calculate the strain energy density of elements ( $\zeta_e^j$ ) under each loading case.
- (5) Remove  $R$  (%) elements with the lowest strain energy density ( $\zeta_e^j$ ) from the design domain.
- (6) Check continuity of the resulting structure. The continuity constraint affects resulting topologies in the optimization process. Implementing the continuity scheme in optimization algorithms will prevent the discontinuity of the design domain from occurring. It is assumed that two elements are connected together if



**Fig. 5.1** Flowchart of performance optimization procedure for topology design with overall stiffness constraint

they have at least one common edge. Any element that is not connected with other elements is considered as a singular element, and is removed from the model.

- (7) Check the symmetry of the resulting structure in the optimization process. The symmetrical conditions of the initial design domain are specified before performing the analysis and optimization. Numerical errors may occur in the calculation of strain energy density for elements, in which approximate concepts are adopted. This may result in an unsymmetrical structure even if the initial structure has a symmetrical geometry, loading and support condition. A scheme for checking the symmetry of resulting structures is incorporated in the optimization algorithm. Extra elements are removed from the structure to maintain the symmetry of resulting structures under an initially symmetrical condition.
- (8) Save current structure. The data for structures generated in the optimization process is saved to files so that the optimization history can be kept track.
- (9) Repeat step (2) to (8) until the performance index is kept constant in later iterations or less than unity.
- (10) Plot the performance index history and select the optimum. The optimal topology that corresponds to the maximum performance index can be identified from the plot of the performance index history.

## 5.6 EXAMPLES

Benchmark examples are provided in this section to demonstrate the effectiveness and validity of the PBO method proposed for optimal topology design of continuum structures with overall stiffness constraint. Firstly, the PBO method is verified by the well-known analytical solution of the Michell type structure. In Section 5.6.2, the PBO technique is used to find the best layout of a bridge structure. Topology optimization of a structure under multiple load cases is treated in Section 5.6.3. The characteristics of

the performance index for plates in bending are investigated in Section 5.6.4. Finally, the effects of mesh discretization on optimal designs are investigated in Section 5.6.5.

### 5.6.1 Verification of the PBO Method

The Michell truss shown in Fig. 5.2 is known as an optimal solution, which was obtained by using the analytical method (Michell 1904). This example is to show that whether the PBO method proposed for topology design problems considering the system performance criteria can reproduce the Michell truss. A continuum structure shown in Fig. 5.3 is used as the initial structure for deriving the Michell truss (Bendsøe, Diaz and Kikuchi 1993). The initial structure is discretized into  $110 \times 80$  four-node finite elements. The circular non-design domain constructed approximately on the basis of rectangular elements is treated as the fixed support in which no deformations are allowed. A tip load is applied to the centre of the free end as illustrated in Fig. 5.3. The Young's modulus of material  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$ , and thickness of elements  $t = 5$  mm are assumed in the analysis. The element removal ratio  $R = 2\%$  is employed in the optimization process.

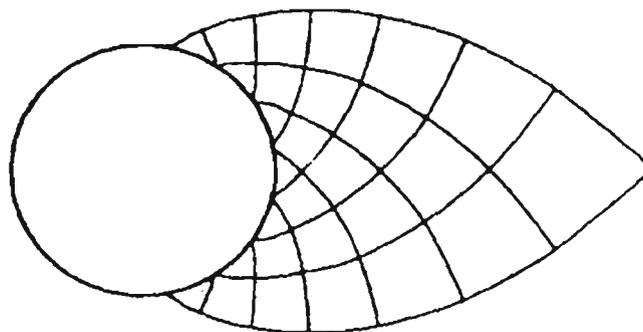
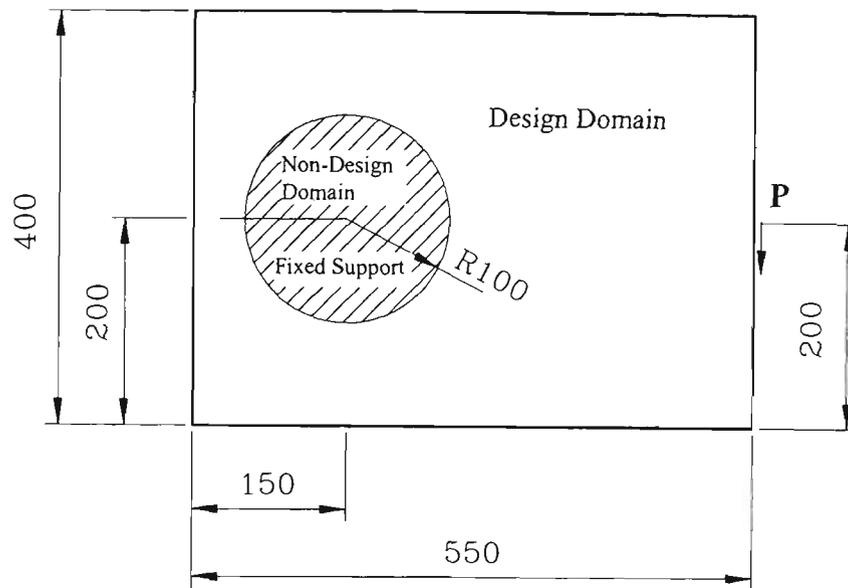


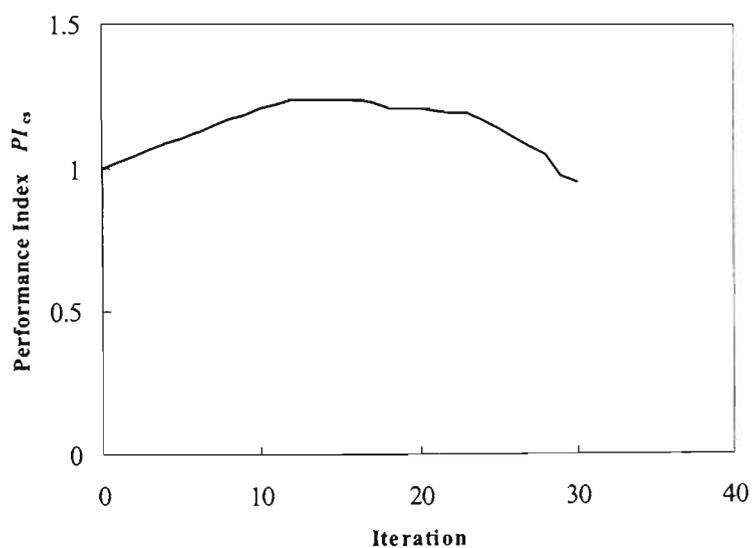
Fig. 5.2 Michell truss



**Fig. 5. 3** Design domain of the Michell type structure

The performance index history of the Michell type structure is presented in Fig. 5.4. It is seen that the performance of resulting topologies in the optimization process is gradually improved by the elimination of elements with the lowest strain energy density from the design domain. The maximum performance index is 1.33, which occurs at iteration 14. The optimal topology corresponding to the maximum performance index is shown in Fig. 5.5 based on the performance-based optimality criteria. It is observed that the optimal topology obtained is a continuum rather than a truss-like Michell structure. It is noted that continuum topology optimization may or may not result in truss-like structures, and is a more general approach than the truss topology optimization method.

In order to generate a truss-like structure, the optimization process is continued. The resulting topologies at iterations 17, 20, 23 and 25 are shown in Figs. 5.6 to 5.9. It can be seen from these figures that when more and more elements are removed from the design domain, the resulting topology is gradually evolved towards a truss-like structure.



**Fig. 5.4** Performance index history of the Michell type structure

If the resulting structure is to be designed as a truss, the topology shown in Fig. 5.8 agrees extremely well with the Michell truss given in Fig. 5.2 and solutions produced by the homogenization-based optimization method (Suzuki and Kikuchi 1991; Bendsøe, Diaz and Kikuchi 1993).

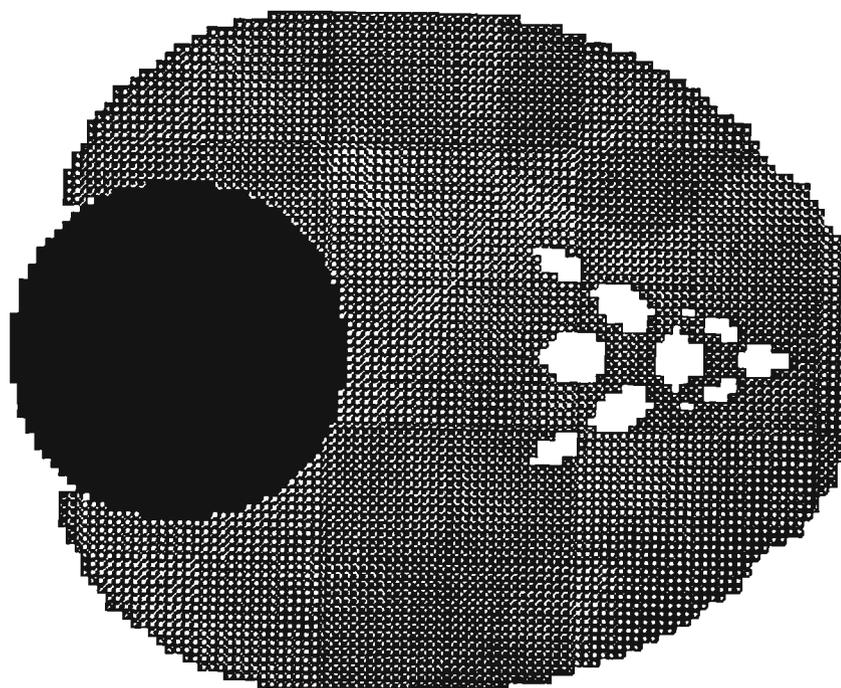


Fig. 5.5 Optimal topology at iteration 14,  $PI_{cs} = 1.23$

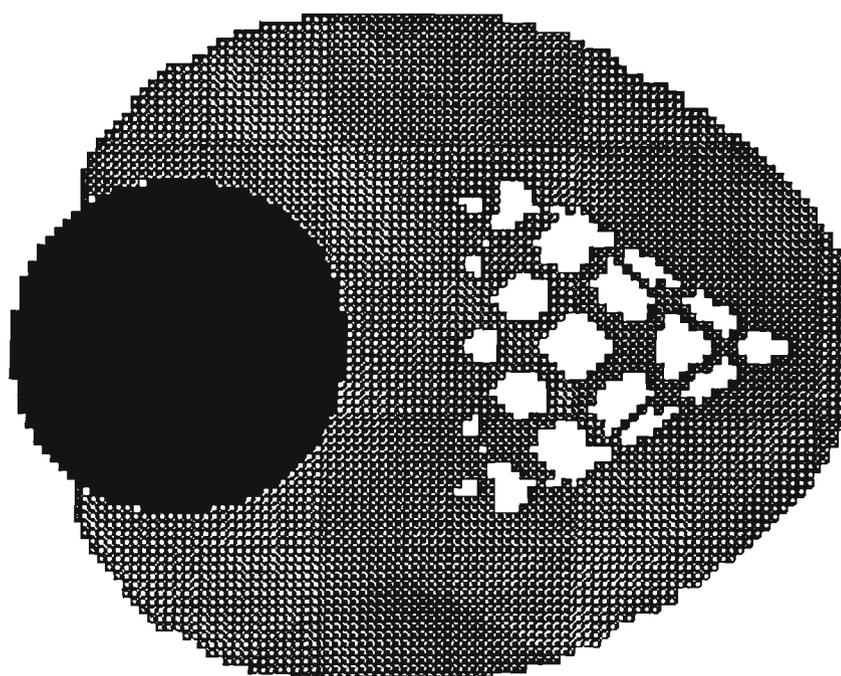


Fig. 5.6 Topology at iteration 17,  $PI_{cs} = 1.22$

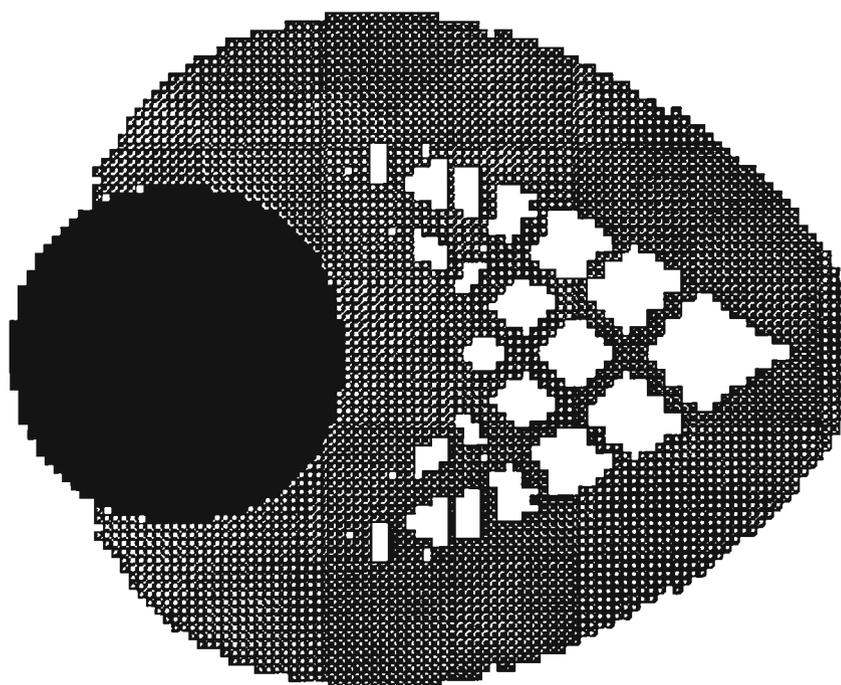


Fig. 5.7 Topology at iteration 20,  $PI_{cs} = 1.21$

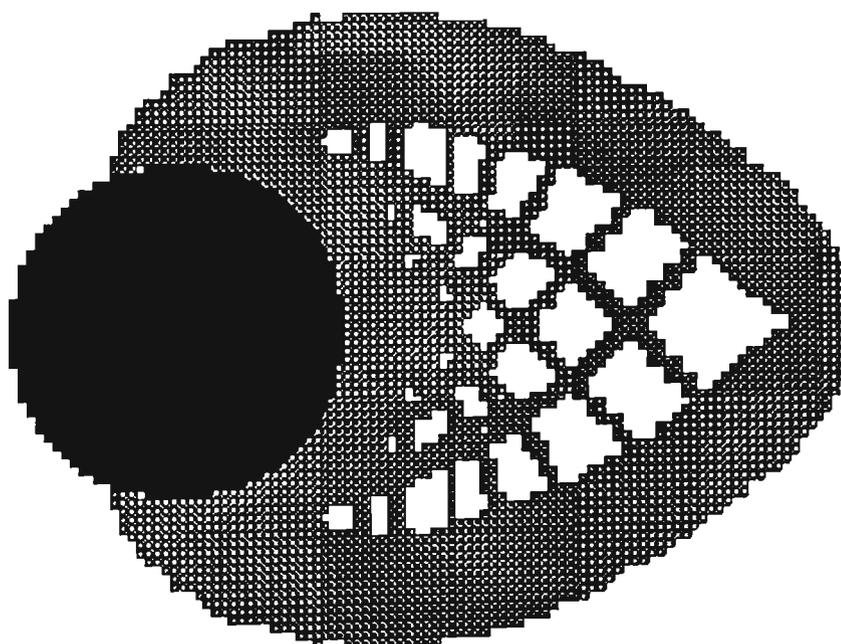


Fig. 5.8 Topology at iteration 23,  $PI_{cs} = 1.19$

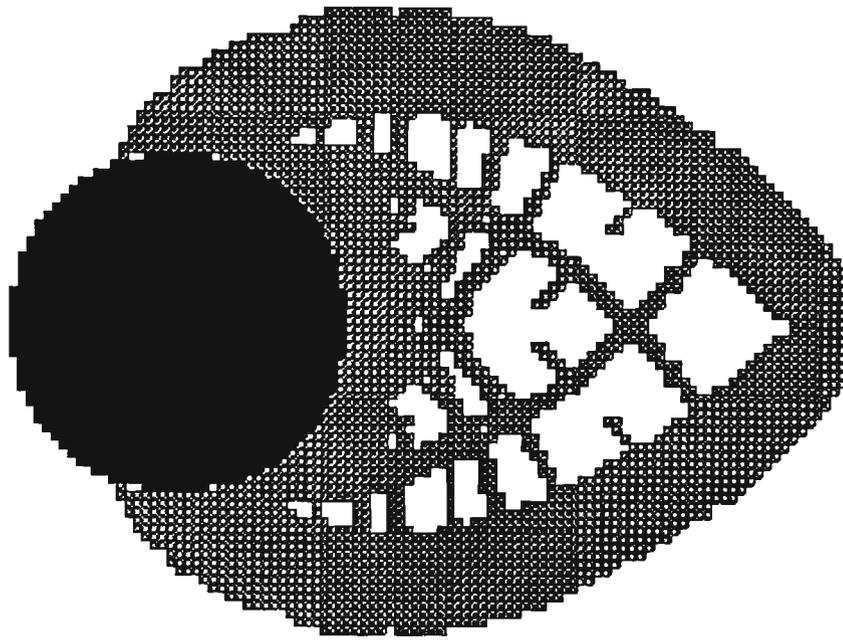
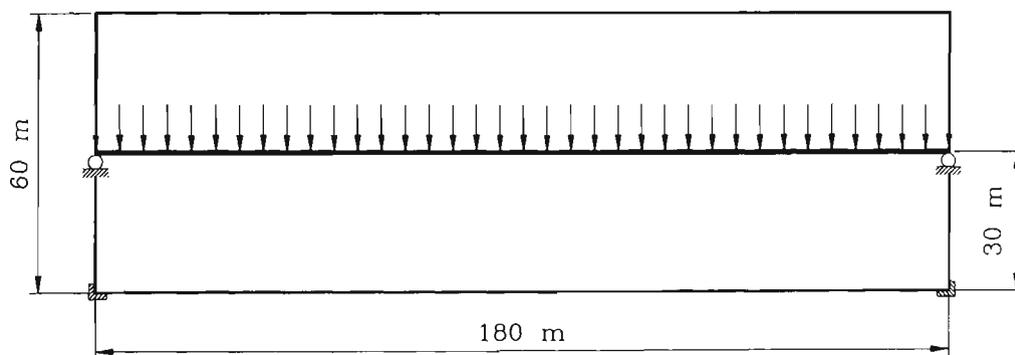


Fig. 5.9 Topology at iteration 25,  $PI_{cs} = 1.14$

It is observed from Fig. 5.4 that the performance of resulting topology at iterations from 14 to 23 decreases only slightly. This means that the material volume that is needed to construct these structures is almost the same while maintaining their overall stiffness performance within the same acceptable level. In other words, the structure can be designed by selecting one of these topologies shown in Figs. 5.5 to 5.8. After iteration 23, the interior of the topology obtained is broken up as shown in Fig. 5.9. As a result of this, the performance of the resulting topologies decreases considerably as indicated by the performance index history. Continuum topology optimization is actually the selection of the best configurations for the design of continuum structures. The performance index is an extremely useful tool for assisting the selection of the best topology in structural design when considering the structural performance, aesthetic and construction constraints. The importance of the system performance should be ranked the first since it relates to the safety of the design.

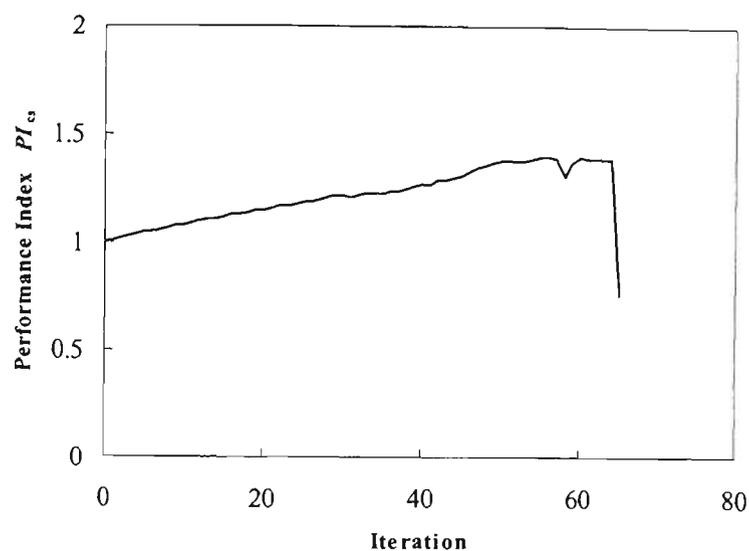
### 5.6.2 Layout Design of Bridge Structures

In this example, the PBO method is used to produce the best layout of a bridge structure under uniformly distributed traffic loading at the conceptual design stage. The design domain and support conditions of a bridge structure are illustrated in Fig. 5.10, where the bottom supports are fixed. The continuum design domain is discretized into  $90 \times 30$  four-node finite elements. The two rows of elements below the loading level are treated as the non-design domain, which represents the bridge deck. The uniformly distributed loading is modeled by applying a 500 kN point load per node. The Young's modulus of material  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and the thickness of elements  $t = 300$  mm are used in the analysis. The  $R = 1\%$  is adopted in the optimization process.



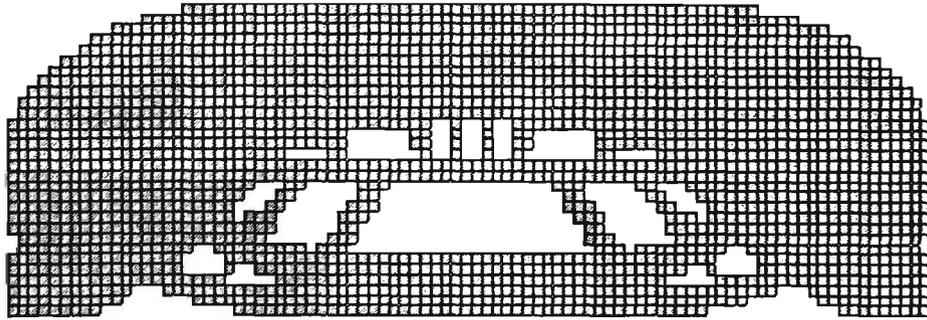
**Fig. 5.10** Design domain of a bridge structure

Fig. 5.11 shows the performance index history of the bridge structure optimized by the PBO method. When underutilized elements are eliminated from the design domain, the strain energy of the bridge increases. The increase of strain energy and the reduction in the weight of the bridge are characterized by the performance index. The maximum performance index is 1.40, which corresponds to the iteration 56 as shown in Fig. 5.11. After iteration 64, the performance index drops sharply and this indicates that the load-carrying mechanism is destroyed by further element elimination. Therefore, the topologies obtained after iteration 64 are not recommended as the final design proposal.

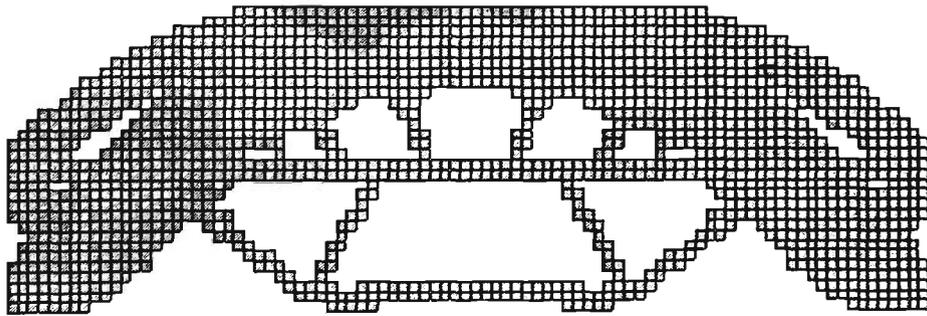


**Fig. 5.11** Performance index history of the bridge structure

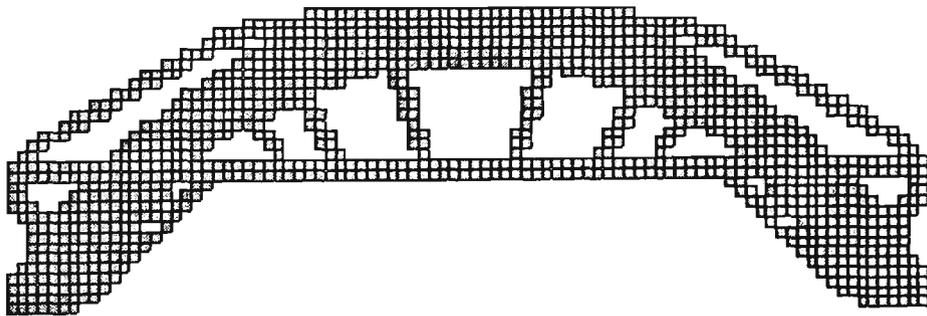
The topology optimization history of the bridge structure is presented in Fig. 5.12. It is observed that the part below the bridge deck is systematically removed since they have no contributions to the structural efficiency. The optimal topology obtained is shown in Fig. 5.12(c). This optimum design indicates a well-known *tie-arch bridge* structural system that has commonly been used in bridge engineering. In the design of bridge structures, the designer usually needs to consider various important aspects, such as structural performance, economy, aesthetic and constructability. To select a bridge layout that is not only beautiful but also has a high structural performance, the performance index plays an important role. It is seen from Fig. 5.12 that the performance of the topology obtained at iteration 64 is almost the same as that of the optimum. However, the layout shown in Fig. 5.12 (d) is more beautiful than the optimum. Therefore, it is suggested that the topology shown in Fig. 5.12 (d) shall be used as the final design proposal for the bridge layout. The arch can be constructed by using either concrete or steel trusses, which were the construction form used in the Sydney Harbor Bridge.



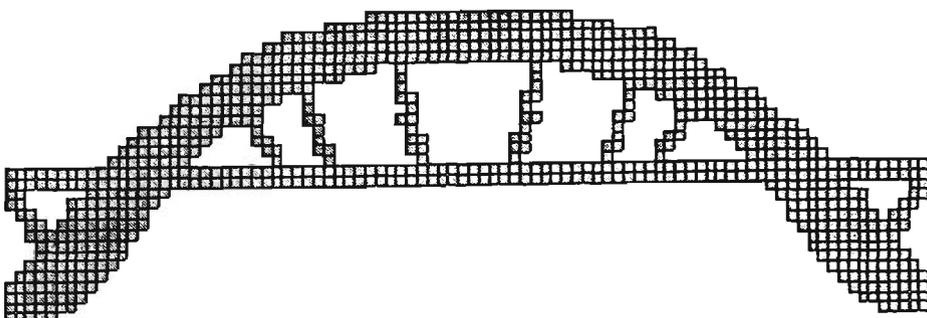
(a) Topology at iteration 20,  $PI_{cs} = 1.15$



(b) Topology at iteration 40,  $PI_{cs} = 1.26$



(c) Optimal topology at iteration 56,  $PI_{cs} = 1.40$

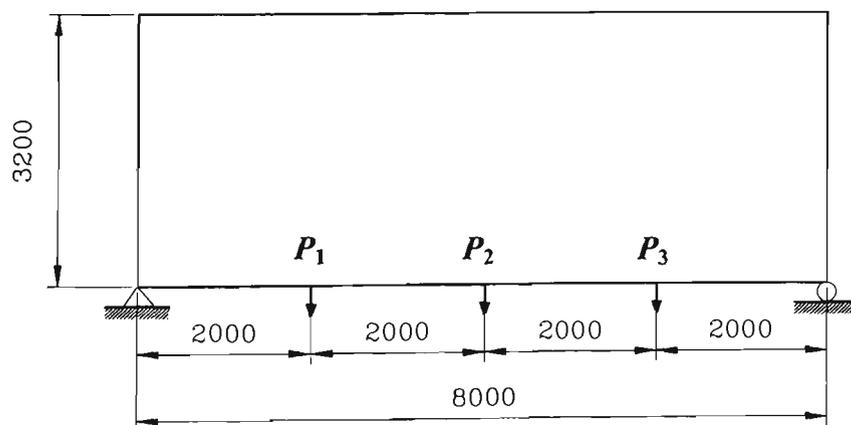


(d) Final design proposal at iteration 64,  $PI_{cs} = 1.39$

**Fig. 5.12** Layout optimization of a tie-arch bridge structure

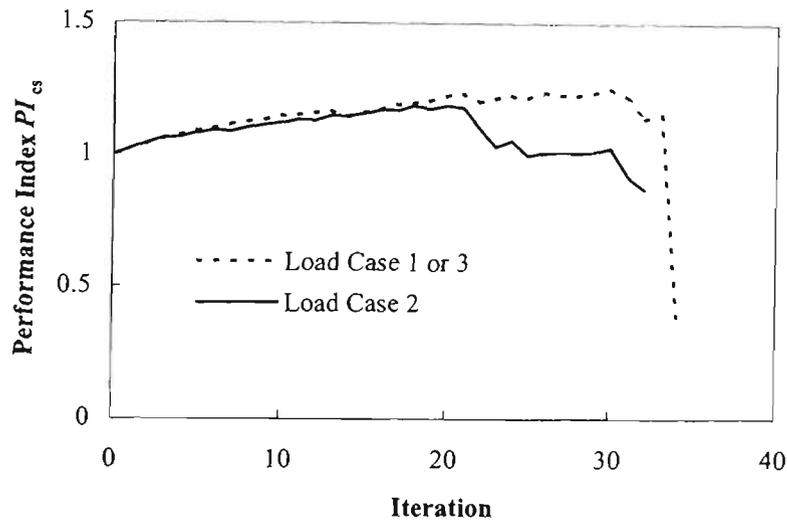
### 5.6.3 Multiple Loading Conditions

This example is to investigate the effect of multiple loading conditions on the optimal topology. For multiple loading cases, the performance index at each iteration is determined by the strain energy of the resulting structure under the most critical loading case, as discussed in Section 5.4.2. To verify this statement, a simply supported structure under three loading cases is considered herein as shown in Fig. 5.13. The concentrated load  $P_1 = P_2 = P_3 = 100$  kN is applied to the structure at a different time. Since in loading case 2 the load  $P_2$  results in the maximum deflection, it is the most critical loading case. The design domain is modeled using  $80 \times 32$  four-node plane stress elements. The Young's modulus  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and element thickness  $t = 10$  mm are used in the analysis. The element removal ratio  $R = 2\%$  is employed in the optimization process.



**Fig. 5.13** Design domain of the structure under multiple loading conditions

In order to compare the performance of resulting topologies under different loading cases, the performance indices for loading cases 1 and 2 are computed using Eq. (5.14). It is noted that the loading conditions 1 and 3 are symmetrical about the vertical axis of the structure. The calculated performance indices are shown in Fig. 5.14. It is observed that at the same iteration in the optimization process, the performance index of the resulting topology under loading case 2 is lower than that under loading case 1 or 3. This is because the absolute value of the strain energy of the resulting structure under loading case 2 is higher than that of the same structure under loading case 1 or 3. If the maximum deflection of the structure under three loading cases must be within an acceptable limit, the weight of the final design will be determined by the loading case 2.

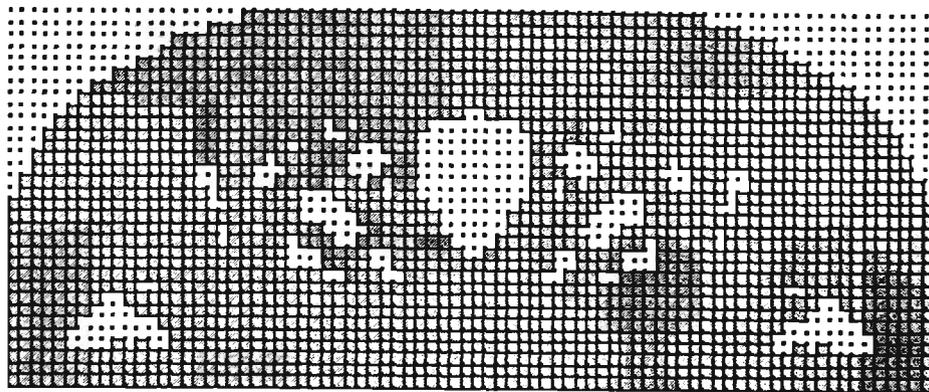


**Fig. 5.14** Performance index history of the structure under multiple load cases

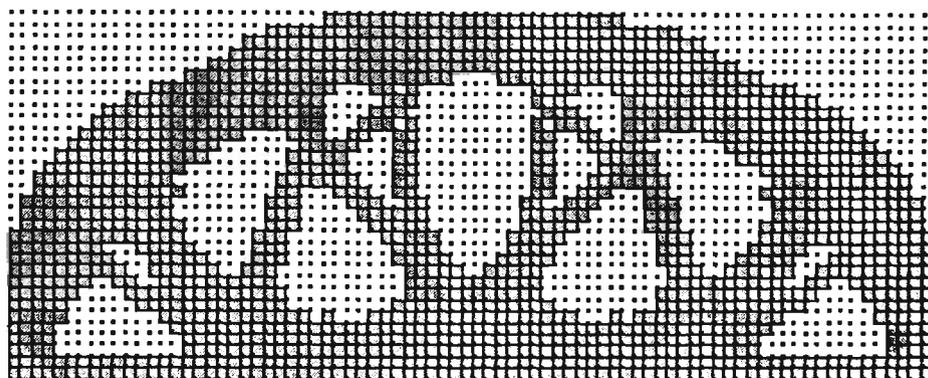
The topologies obtained at iterations 10, 20 and 26 are shown in Fig. 5.15. The optimal topology presented in Fig. 5.15(b) indicates a symmetrical continuum structure with holes inserted. It is seen that the loads are transmitted from the bottom part of the

structure to the arch, and then form the arch to the supports. However, the members that transmit the loads are intersected under multiple loading cases as shown in Fig. 5.15(b). Since underutilized elements for three loading cases were systematically removed from the design domain, the optimal topology obtained is the best possible design in the sense of the load combination.

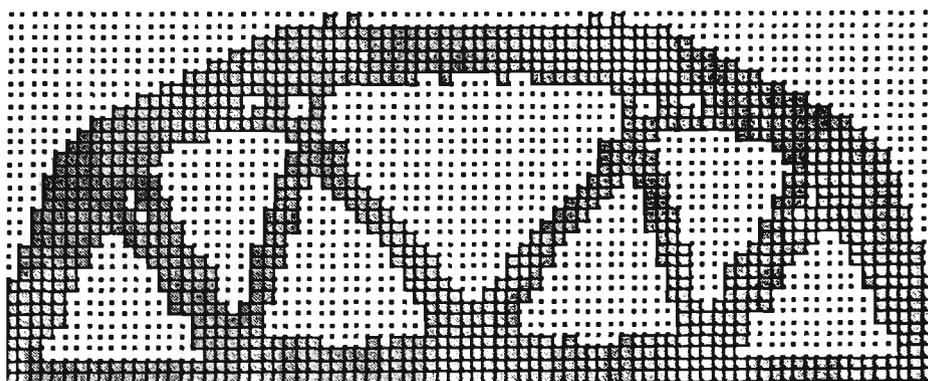
To maximize the utilization of material for the design of the optimal topology shown in Fig. 5.15(b), it has to be designed as a continuum structure. However, a continuum structure with many holes inserted usually complicates the construction process and consequently increases the construction cost. In civil engineering, such structures are usually designed as trusses, which are easily constructed using standard members. The performance index of the structure shown in Fig. 5.15(c) under loading case 2 is 1.01 while it is 1.24 under loading case 1 or 3. Although its performance is not much better than that of the initial design domain, the topology shown in Fig. 5.15(c) clearly indicates a stable truss structure in which its members are connected together by following the principle of triangle. If this structure is to be designed as a truss, its interpretation is illustrated in Fig. 5.16. This truss can be constructed by using standard steel sections.



(a) Topology at iteration 10,  $PI_{cs} = 1.11$

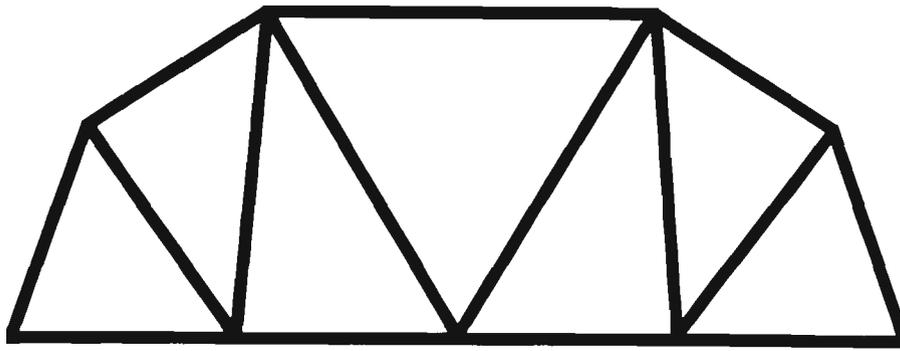


(b) Optimum at iteration 20,  $PI_{cs} = 1.18$



(c) Topology at iteration 26,  $PI_{cs} = 1.01$

**Fig. 5.15** Topology optimization history of the structure under multiple loading cases



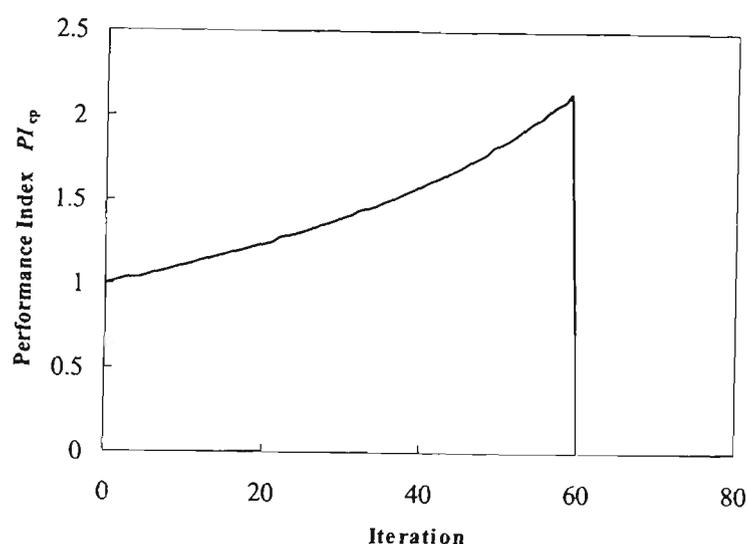
**Fig. 5.16** Truss interpretation

### 5.6.4 Plate in Bending

For a structure under a single point load, both of the PBO methods for structures with displacement constraints and overall stiffness constraint are equally efficient and shall produce a same solution for the same design problem. This example is to investigate whether the PBO methods with different formulations can produce the same optimal topology for a bending plate. The clamped plate under a concentrated load applied at its centre presented in Section 4.7.2 is optimized using the PBO method for overall stiffness performance. The design and optimization parameters given in Section 4.7.2 are adopted, except that the overall stiffness constraint in stead of a displacement constraint is considered herein.

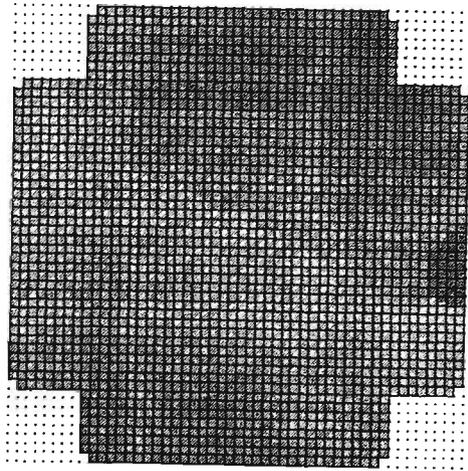
Fig. 5.17 shows the history of the performance index calculated using Eq. (5.18). The performance index curve shown in Fig. 5.17 is similar to that presented in Fig. 4.15. This is because for a structure under a single point load, the performance index calculated using Eq. (4.24) based on the displacement under the point load is the same as that calculated using Eq. (5.18) based on the mean compliance of the structure. The maximum performance index by the PBO method for structures with displacement

constraints is 2.09 whilst it is 2.13 by the PBO method for structures with overall stiffness constraint. It is shown that the PBO method based on the overall stiffness performance criteria provides higher performance topologies than that based on displacement performance criteria.

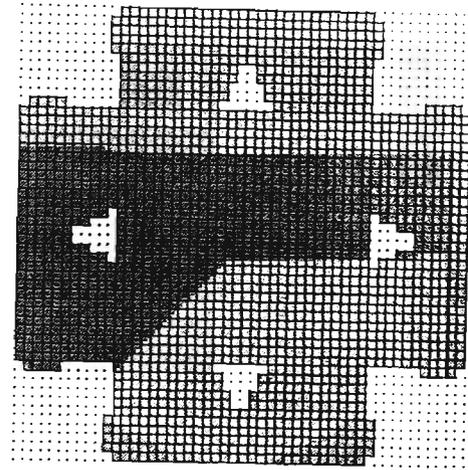


**Fig. 5.17** Performance index history of clamed plate in bending

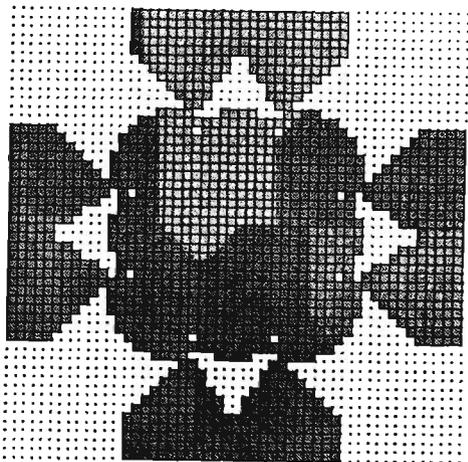
The topology optimization history of the plate in bending is presented in Fig. 5.18. The topology obtained at iteration 10 as shown in Fig. 5.18(a) is identical to that presented in Fig. 4.16(a). The slight differences between topologies obtained by the two PBO methods are observed at iteration 20, 40 and the optimum. These differences may be caused by the element removal criteria adopted in the optimization algorithms since the virtual strain energy density and strain energy density are approximately calculated for element elimination in both methods.



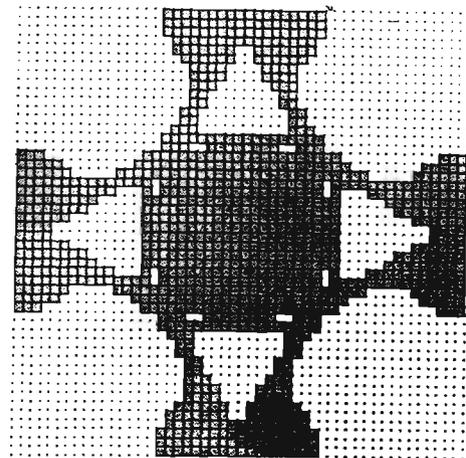
(a) Topology at iteration 10,  $PI_{cp} = 1.11$



(b) Topology at iteration 20,  $PI_{cp} = 1.24$



(c) Topology at iteration 40,  $PI_{cp} = 1.59$



(d) Optimal at iteration 59,  $PI_{cp} = 2.13$

**Fig. 5.18** Topology optimization history of bending plate under overall stiffness constraint

### 5.6.5 Effects of Finite Element Meshes

This example is to investigate the effects of finite element meshes on the optimal topologies of continuum structures optimized by the performance-based optimization method while other conditions are fixed. The Michell type structure with a simply supported condition is used as the test example.

Fig. 5.19 shows the design domain for the simply supported Michell type structure under a concentrated load of  $P = 100$  kN. The design domain is divided into three different meshes, such as  $70 \times 35$ ,  $100 \times 50$  and  $120 \times 60$ , using four-node plane stress elements. The Young's modulus  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and the thickness of all elements  $t = 10$  mm are specified. Plane stress conditions are assumed in the finite element modeling. To eliminate the effects of the element removal ratio on the solutions, the element removal ratio  $R = 1\%$  is used for all cases. The structure is analyzed and optimized by using the PBO method for three different finite element discretizations.

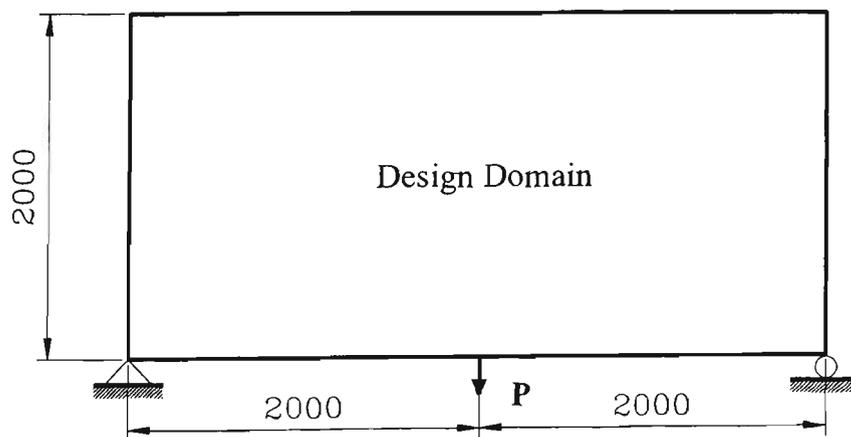
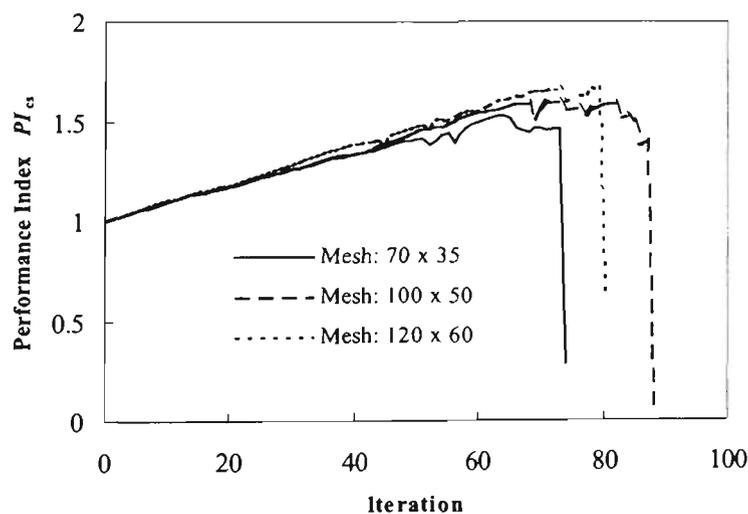


Fig. 5.19 Design domain of Michell type structure

Fig. 5.20 shows the performance index histories for the Michell type structure that is modeled using three different meshes. It is observed from Fig. 5.20 that the discrepancies of the performance index value between different mesh increase with the increases in the iteration numbers. The finer the mesh used to model the structure, the higher the performance of the optimal topology obtained. The maximum performance indices of optimized structures for meshes  $70 \times 35$ ,  $100 \times 50$  and  $120 \times 60$  are 1.53, 1.60 and 1.67. It is seen that performance indices reach the peak values at different iterations for the structure optimized using different meshes for the same element removal ratio. After reaching the peak, performance indices decrease and finally drop very sharply. It is also observed that more iteration is usually needed for a structure modeled using a finer mesh to obtain the optimum.



**Fig. 5.20** Effects of finite element meshes on performance indices

Optimal topologies obtained using three different finite element meshes are presented in Figs. 5.21 to 5.23. The optimal topology generated using  $70 \times 35$  finite elements indicates a truss-like structure as shown in Fig. 5.21. By inspection, it is seen that the in-plane member size of the truss is approximately proportional to the axial force carried

by that member. In other words, the distribution of element strain energy density within the optimal topology is approximately uniform since elements with the lowest strain energy density are systematically eliminated from the design domain. Fig. 5.22 shows the optimal topology obtained by using a  $100 \times 50$  mesh. It is observed from Figs. 5.21 and 5.22 that the topologies of these two optimal designs are almost identical. However, the optimal design with a finer mesh discretization exhibits a truss-like structure in which the in-plane member size is smaller than that with a coarse one.

The optimal solution obtained by Michell (1904) using the analytical method indicates a pin-jointed truss with finite members. The strain field in the Michell truss was assumed to be linearized. The compression and tension members are subjected to the same allowable stress. It should be noted that the Michell truss is theoretical optima, which is not necessarily a practical design. It has been shown that a more accurate solution can be achieved by using a finer mesh in the finite element analysis. To see whether the optimal design can approach the Michell truss, a  $120 \times 60$  mesh is used to divide the design domain. The optimal topology obtained using  $120 \times 60$  elements is presented in Fig. 5.23. It is seen that a more Michell truss-like structure is obtained by using the finer elements in continuum topology optimization. If elements are refined infinitely, the optimal structure will be towards the Michell truss.

This investigation shows that the optimal structure converges to the theoretical optimum as sufficiently fine elements are used. Even coarse mesh can produce a rough idea of the optimal structure. At the conceptual design stage, if the PBO method is used to find the primary layout of an optimal structure, a coarse mesh can be used to solve the optimization problem. After obtaining the optimal topology, shape optimization techniques can be employed to further improve its performance.

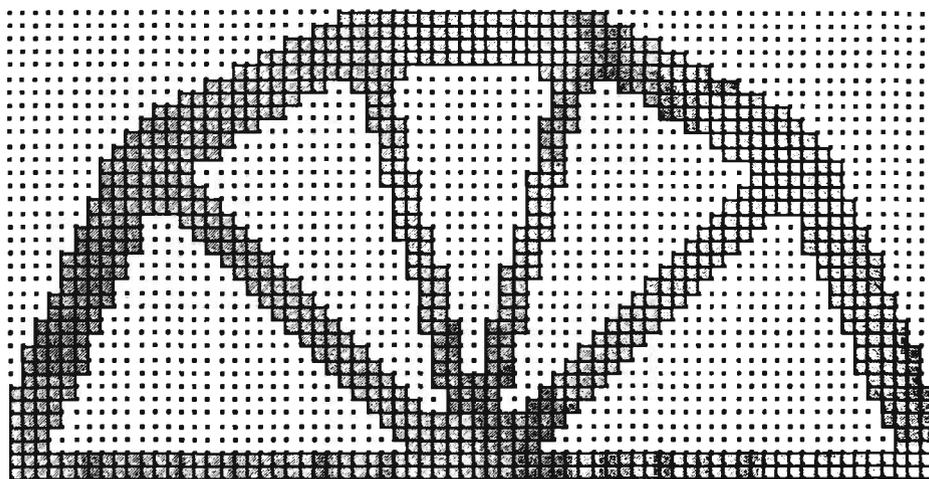


Fig. 5.21 Optimal topology obtained using a  $70 \times 35$  mesh,  $PI_{cs} = 1.53$

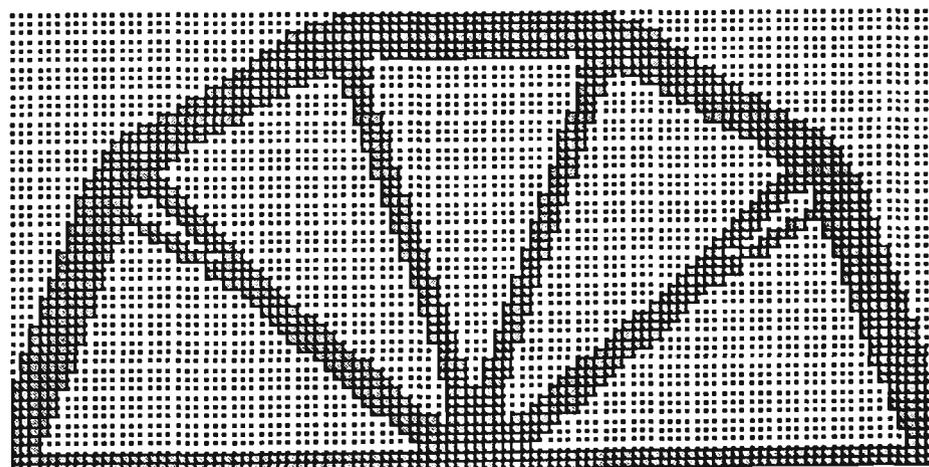


Fig. 5.22 Optimal topology obtained using a  $100 \times 50$  mesh,  $PI_{cs} = 1.60$

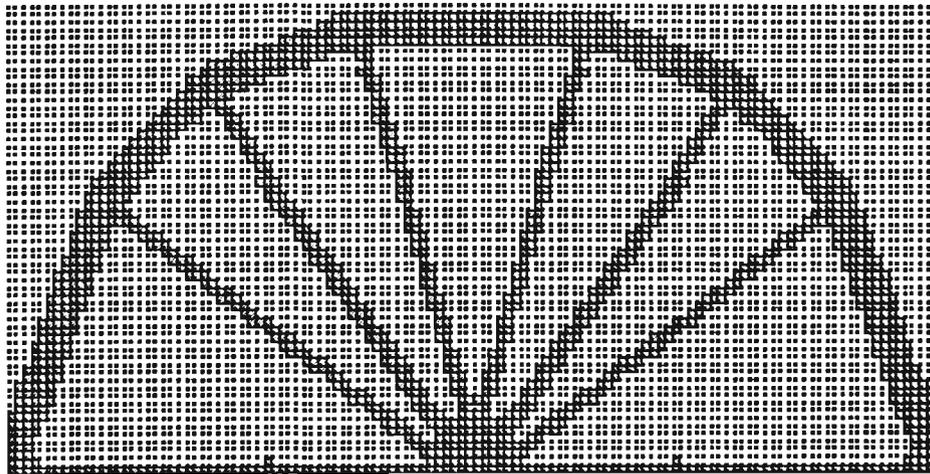


Fig. 5.23 Optimal topology obtained using a  $120 \times 60$  mesh,  $PI_{cs} = 1.67$

## 5.7 CONCLUDING REMARKS

A Performance-Based Optimization (PBO) method formulated on the basis of system performance criteria for optimal topology design of continuum structures has been developed in this chapter. In the proposed approach, element removal criteria are based on the strain energy density of elements. By gradually deleting elements with the lowest strain energy density from a continuum design domain, the optimal topology of the structure with maximum stiffness at minimum weight can be generated. Two stiffness performance indices have been derived using the scaling design concept for evaluating the topology performance of plane stress structures and of plates in bending. Performance indices are used in optimization algorithms as the termination criteria in the PBO method. Maximization of performance indices is proposed as performance-based optimality criteria. Benchmark examples have been provided to demonstrate the effectiveness and validity of the PBO method for topology design for stiffness.

Based on the study presented in this chapter, the following conclusions are drawn.

- (1) The PBO method is a realistic structural optimization approach, in which the performance objective is to minimize the weight of a structure while maintaining its mean compliance within an acceptable performance level.
- (2) The sensitivity analysis shows that the element strain energy density is a measure of the element contribution to the overall stiffness performance of a structure, and thus can be used as element removal criteria in continuum topology optimization. The maximum stiffness topology design at minimum weight can be obtained by removing elements with the lowest strain energy density from a structure.
- (3) The proposed performance indices are dimensionless and can be used to evaluate the performance of structural topologies in terms of the mass and overall stiffness efficiency.
- (4) Stiffness performance indices are extremely useful tools in continuum topology optimization approaches. They can be used to monitor the optimization process and as termination criteria.
- (5) Performance-based optimality criteria can be incorporated in any stiffness-based continuum topology optimization methods to obtain the global optimum. For example, they can be incorporated in the homogenization-based optimization method (Bendsøe and Kikuchi 1988; Bendsøe 1995), and the density function approach (Yang and Chuang 1994).
- (6) Continuum topology optimization is a generalized shape optimization approach. It has been shown that continuum topology optimization methods may or may not result in truss-like structures, they are in this sense more general than the truss topology optimization based on the ground structure approach.
- (7) The optimal topology produced by continuum topology optimization methods is not necessarily the one that meets the construction requirements. As a result of

this, performance indices can be used to assist the selection of the best design in terms of the structural performance, aesthetics and constructability.

- (8) For a structure under multiple loading conditions, a logical AND scheme is employed in the optimization method to take account of the effects of multiple load cases on the optimal design. The performance index at each iteration is determined by using the strain energy of the resulting structure under the most critical loading case.
- (9) It has been shown that the finer the mesh used to model a structure, the higher the performance of the optimal topology obtained.
- (10) More iteration is usually needed for a structure modeled using a finer mesh to obtain the optimum.

Chapters 3, 4 and 5 concern with the theoretical development and verification of the Performance-Based Optimization (PBO) method for layout design of continuum structures for strength, displacement and overall stiffness performance. Examples presented in these chapters are mainly used to verify the proposed methods from theoretical aspects rather than practical application. However, in order to develop continuum topology optimization methods into practical design tools for practicing design engineers, the PBO method for structures with displacement and overall stiffness constraints are extended to solve real world civil engineering problems in Chapters 6 and 7. In Chapter 6, the PBO method is proposed as an automated design tool for generating optimal strut-and-tie models in structural concrete. In Chapter 7, the PBO method formulated on the basis of system performance criteria is developed for conceptual layout design of bracing systems for multistory steel building frameworks.

## Chapter 6

# OPTIMIZATION OF STRUT-AND-TIE MODELS IN STRUCTURAL CONCRETE

### 6.1 INTRODUCTION

#### 6.1.1 General

The truss model is considered as the basic tool in the design and detailing of reinforced concrete beams under shear and torsion. However, standard truss models can only be used to design regions of a concrete structure where the Bernoulli hypothesis of plane strain distribution is assumed valid. At regions where the strain distribution is significantly nonlinear such as point loads, corbels, deep beams, beam-column connections, and openings, the truss model theory is not applicable. *The strut-and-tie model, which is a generalization of the truss analogy method for beams, is therefore used to design the disturbed regions of structural concrete, which includes reinforced and prestressed concrete structures.* The strut-and-tie model approach is justified by the fact that loads applied to a structural concrete member are transmitted through a set of compressive stress fields that are distributed and interconnected by tensile ties. The flow of compressive stresses is idealized using compression members called struts, and the tension is carried by tension ties. Tensile ties can be reinforcing bars or prestressed tendons or concrete tensile stress fields.

The strut-and-tie model developed is employed to investigate the equilibrium between the loads, the reactions and the internal forces in the concrete struts and in the reinforcement (Marti 1985). The actual load carried by the strut-and-tie model is treated as a lower bound ultimate load for the structural concrete member based on the lower bound theorem of plasticity. This simple approach provides a clear understanding of the behavior of structural concrete. The designer can easily idealize the flow of forces in a structural concrete member by strut-and-tie modeling. Moreover, the influence of shear and moment are accounted for simultaneously and directly in one model. Furthermore, it offers a unified, intelligible, rational and safe design framework for structural concrete under combined load effects (ASCE-ACI Committee 445 on Shear and Torsion 1998).

### **6.1.2 Historical Development of Strut-and-Tie Model Approach**

The truss model was originally developed by Ritter (1899) for the analysis and design of reinforced concrete beams under shear in 1899. Ritter found that a reinforced concrete beam after cracking due to diagonal tension stresses can be idealized as a parallel chord truss with compressive diagonals inclined at  $45^\circ$  with respect to the longitudinal axis of the beam. Later, Mörsch (1920, 1922) extended truss models to the design of reinforced concrete members under torsion. The truss analogy method was refined and expanded by Kupfer (1964), and Leonhardt (1965). The truss model with diagonals having a variable angle of inclination was considered as a viable model for design of reinforced and prestressed concrete beams under shear and torsion (Kupfer 1964; Lampert and Thürlimann 1971; Thürlimann et al. 1983). Collins and Mitchell (1980) proposed the truss model approach considering deformations for the design of reinforced and prestressed concrete.

Marti (1985) developed the strut-and-tie model approach which considers the consistent equilibrium and ultimate strength for reinforced concrete beams. The determination of the ultimate strength of a strut-and-tie model is based on the lower-bound plastic theory. Struts, ties, nodes, fans and arches were proposed as basis tools for the design and

detailing of reinforced concrete beams. Schlaich et al. (1987, 1991) extended the truss model theory for beams to a consistent strut-and-tie model approach for the design and detailing of structural concrete that includes reinforced and prestressed concrete structures. This consistent design approach allows any part of a concrete structure to be designed using strut-and-tie systems. The concept of B- and D-regions was introduced by Schlaich et al. The distribution of strains in B-regions (where B stands for beam or Bernoulli) is linear whereas the distribution of strains in D-regions (where D stands for discontinuity or disturbance) is nonlinear. More often, a concrete structure can be divided into B and D regions. The B-regions are designed on the basis of standard truss models. However, specific strut-and-tie models have to be developed for D-regions where standard truss models are not applicable.

The modified truss model approach with variable angle of inclination diagonals and a concrete contribution has been proposed for design of reinforced and prestressed concrete beams by Ramirez and Breen (1991). Ramirez (1994) gave some guidelines for the strut-and-tie design of pretensioned concrete members. Experimental and analytical study on the use of strut-and-tie models for the design of post-tensioned anchorage zones has been conducted by Sanders and Breen (1997). The strut-and-tie model approach and related theories for the design of structural concrete were summarized in the state-of-the-art report by the ASCE-ACI Committee 445 on Shear and Torsion (1998). Modern concrete model codes and standards rely on the strut-and-tie model approach as the basis for the design and detailing of reinforced and prestressed concrete structures (CEB 1978; AS 3600 1994).

### **6.1.3 Conventional Methods for Developing Strut-and-Tie Models**

The elastic stress distribution and load path methods can be used to develop strut-and-tie models in structural concrete, as suggested by Schlaich et al. (1987). In the elastic stress distribution approach, the strut-and-tie model is constructed by orientating struts and ties to the mean direction of principal stress trajectories, which are obtained by performing a

linear elastic finite element analysis on an uncracked homogenization concrete member. However, due to the uncracked assumption of concrete in the elastic finite element analysis, the strut-and-tie model obtained by this approach may differ from the actual load transfer mechanism at ultimate limit states, as shown by Schlaich and Schäfer (1991). It is often required to adjust the strut-and-tie model obtained on the basis of the elastic stress analysis in order to represent the real behavior of cracked structural concrete. In the load path method, it is firstly required to determine all loads and reactions acting on the D region for the outer equilibrium. The load paths are then traced by following the centre of gravity of the corresponding stress diagrams. The principle to be followed is that loads transmit through the shortest paths in nature. After plotting all load paths in the direction of applied loads, further struts and ties must be added for transverse equilibrium acting between nodes. However, it is difficult to find the correct models in members with complex loading and geometry conditions using these conventional methods, which usually involve a trial-and-error process. It is also a difficult design task for the designer to select correct strut-and-tie models from many possible equilibrium configurations for complex design situations by using traditional drawing board methods.

Because of the limitations of conventional methods for developing strut-and-tie models, Marti (1985) called for the development of computer-based design aids for strut-and-tie modeling of structural concrete. Attempts to develop computer programs with graphical input and output routines for strut-and-tie modeling have been made by several researchers. Kumar (1978) applied the truss topology optimization theory to finding the load transfer mechanism in reinforced concrete deep beams. A continuum concrete structure with specified geometry and loading conditions was modeled by using a highly indeterminate truss (ground structure). The best truss used as a basis for the design of reinforced concrete beams is the one with the maximum stiffness. In other words, the minimum strain energy principle was used as the optimal design criterion for trusses. The linear programming technique was employed to solve the truss topology optimization problem. Kumar's study provides important insight into the practical application of truss optimization theory to the specific field of structural concrete. The

truss topology optimization technique has also been used by Biondini et al. (1998) to find optimal strut-and-tie models in structural concrete members based on the ground structure approach. These methods offer an automatic search for strut-and-tie models in reinforced concrete members in an iterative process. However, since the ground structure grid has a significant effect on the optimal topology of the structure (Dorn et al. 1964), the chosen ground structure may not adequately simulate the nature of a continuum concrete structure. In other words, the load paths are restricted to the chosen ground structure grid.

Computer graphics have been used as a design aid for the strut-and-tie modeling of structural concrete by Alshegeir and Ramirez (1992), and by Mish (1994). In these computer graphical methods, finite element analysis packages are used in the construction of a strut-and-tie model. After a continuum finite-element model of the structural concrete member is created and analyzed, the load paths in the structure may be visualized by locating the maximum principal stresses and the direction of these principal stresses. The equivalent force resultant in the member is obtained by the summation of stresses. These computer graphical methods are useful design aids for developing strut-and-tie models in structural concrete. It might be necessary to incorporate stiffness-based optimization procedures into computer graphical methods to develop strut-and-tie models for complex design situations.

More recently, Yun (2000) has presented a so-called nonlinear strut-and-tie model approach for structural concrete. In this approach, the principal stresses and their trajectories are firstly evaluated by performing a nonlinear finite element analysis on the plain concrete member. The strut-and-tie model is then sketched by the designer according to the principal stress trajectories. A linear analysis on the strut-and-tie model obtained is then carried out in order to determine the cross-sectional areas of struts and ties. If the strut-and-tie model does not satisfy the geometric compatibility condition, a new strut-and-tie model must be selected, and the above process is repeated until the condition is satisfied. At this stage, the strut-and-tie model obtained is analyzed nonlinearly to verify the bearing capacities of nodal zones. The limitations of this

approach are that the success of this approach depends on the designer's ability to sketch an appropriate model for members with complex loading and geometry conditions. Although the nonlinear analysis is used for developing strut-and-tie models in this approach, it does not have much meaning to the real behavior of reinforced concrete structures. This is because the nonlinear analysis is performed on a plain concrete member rather than a reinforced concrete member.

### 6.1.4 Research Significance

The shear design of structural concrete members is a complex problem that has not been solved fully although extensive work has been carried out on shear in structural concrete. Empirical equations adopted in current concrete model codes lead to complex design procedures for shear and generally yield shear strength predictions, which deviate considerably from experimental results. *The load transfer mechanism of a structural concrete member is not the function of a single variable and it depends on the geometry, loading and support conditions of the member. Design procedures based on test results, rules of thumb, guess work and past experience have been one of the main reasons for poor structural performance and even failure of concrete structures.* Experiments should be used to verify a theory but not to derive it. A consistent theory associated with reliable design approaches will significantly improve the performance of concrete structures.

It has been recognized that the simple strut-and-tie model theory provides a better understanding of the behavior of structural concrete under bending, shear and torsion. *The strut-and-tie model is primarily used to represent the actual load transfer mechanism in a structural concrete member under the ultimate loading condition.* As discussed previously, however, conventional methods are not efficient in developing strut-and-tie models of structural concrete since they involve a trial-and-error process, which largely depends on the designer's intuition and previous experience. It is time consuming and difficult for the designer to find correct strut-and-tie models for

members with complex loading and geometry conditions by using trial-and-error methods. Therefore, the development of an efficient and reliable technique for strut-and-tie modeling of structural concrete will be of significant practical importance.

Continuum topology and shape optimization has received considerable development in the past two decades. Several textbooks have been published (Bendsøe 1995; Seireg and Rodriguez 1997; Xie and Steven 1997; Mattheck 1998; Hassani and Hinton 1999). However, these continuum topology optimization methods focus mainly on mathematical aspects rather than engineering applications. Examples used to test these methods are very simple, and far from practical relevance. Moreover, no performance-based optimality criteria are used in these methods to obtain the global optimum. *The gap between the progress of continuum topology optimization theory and its application to practice of civil engineering does not appear to have been reduced during the last two decades*, as addressed by Cohn and Dinovitzer (1994). The work presented in this chapter plays a significant role in reducing the gap between the theoretical development of continuum topology optimization and its practical applications to civil engineering. It is also a significant contribution to the field of structural concrete.

### **6.1.5 Scope and Objective**

In this chapter, the Performance-Based Optimization (PBO) method formulated on the basis of displacement and overall stiffness performance criteria is extended and proposed for automatically developing optimal strut-and-tie models in reinforced and prestressed concrete structures. In the proposed approaches, developing strut-and-tie models in structural concrete is transformed to the topology optimization problem of continuum structures. The optimal strut-and-tie model in a structural concrete member is generated by gradually removing inefficient elements from the continuum concrete member. An integrated design optimization procedure is proposed for strut-and-tie

design. Some of the findings have been reported by Liang et al. (1999c, 1999d, 2000c, 2001b) and Liang and Steven (2001b).

The rest of this chapter is organized as follows. In Section 6.2, the strut-and-tie model optimization problem is formulated. This is followed by the presentation of the limit analysis and finite element modeling for strut-and-tie models in Section 6.3. In Section 6.4, optimization criteria for strut-and-tie models are described. The design optimization procedure of strut-and-tie models is presented in Section 6.5. Section 6.6 provides design criteria for dimensioning struts, ties and nodes. The PBO method is used to develop strut-and-tie models in reinforced concrete members in Section 6.7. The generation of optimal strut-and-tie models in prestressed concrete beams is treated in Section 6.8. The optimization of strut-and-tie models in low-rise shearwalls is presented in Section 6.9. Finally, Section 6.10 gives significant conclusions on the present work.

## 6.2 STRUT-AND-TIE MODEL OPTIMIZATION PROBLEM

Since the concrete permits only limited plastic deformations, the strut-and-tie model in a structural concrete member has to be selected so that the structural system has the least deformations. Based on the principle of the minimum strain energy for the linear elastic behavior of strut-and-tie models after cracking, Schlaich et al. (1987) proposed the following equation for assisting the selection of strut-and-tie models as

$$\sum_{i=1}^N F_i l_i \varepsilon_{mi} = \text{minimum} \quad (6.1)$$

where  $F_i$  is the force in the  $i$ th strut or tie,  $l_i$  is the length of the  $i$ th member and  $\varepsilon_{mi}$  is the mean strain of the  $i$ th member and  $N$  is the total number of members in a strut-and-tie system. This equation is helpful for selecting a better strut-and-tie model from several possible ones. However, it will be cumbersome to find the optimal strut-and-tie

model that represents the load transfer mechanism using this method, since there are a large number of possibilities for the equilibrium configurations of a complex structure.

Strut-and-tie models are used to idealize the load transfer mechanism in a cracked structural concrete member at ultimate limit states. The design task is mainly to identify the load transfer mechanism in a structural concrete member and reinforce the member such that this load path will safely transfer applied loads to the supports. Obviously, some regions of a structural concrete member are not as effective in carrying loads as others. By eliminating these underutilized portions from a structural concrete member, the actual load transfer mechanism in the structure can be found. The PBO method has the capacity to find the underutilized portions of a structure and remove them from the structure to improve its performance. Therefore, *developing an appropriate strut-and-tie model in a structural concrete member can be transformed to the topology optimization problem of continuum structures.*

In nature, loads are transmitted by the principle of minimum strain energy (Kumar 1978). This means that *strut-and-tie systems in structural concrete should be developed on the basis of system performance criteria (stiffness) rather than component performance criteria (strength).* Component performance criteria can be satisfied by dimensioning the component. Based on these design criteria, the PBO methods for structures with displacement constraints and overall stiffness constraint are modified and tailored for the development of strut-and-tie models for the design and detailing of structural concrete. The performance objective of strut-and-tie model optimization is to minimize the weight of a concrete structure while maintaining its stiffness within an acceptable performance level. The performance objective can be expressed in mathematical forms as follows:

$$\text{minimize } W = \sum_{e=1}^n w_e(t) \quad (6.2a)$$

$$\text{subject to } u_j \leq u_j^* \quad j = 1, 2, \dots, m \quad (6.2b)$$

$$\text{or } C \leq C^* \quad (6.2c)$$

If the PBO method for structures with displacement constraints is used to develop strut-and-tie models in structural concrete, sufficient displacement constraints have to be specified in order to generate the stiffest strut-and-tie models. In practice, the width of a structural concrete member is designed as uniform. As a result of this, only uniform sizing of the element thickness (or the width of the member) is considered in the optimization process. By means of systematically removing elements from the discretized concrete member, actual load paths within the structural concrete member can be gradually characterized by remaining elements.

### 6.3 LIMIT ANALYSIS AND FINITE ELEMENT MODELING

Topology optimization theory is rarely applied to the special and important field of reinforced and prestressed concrete due to the difficulty in modeling the nonlinear behavior of structural concrete for optimization. Concrete has a considerable strength in compression, but a very low strength in tension. Moreover, reinforced concrete is a composite material. The nonlinear behavior of reinforced concrete is characterized by the cracking of concrete and the yielding of steel reinforcement. The behavior can be well approximated by the uncracked linear, cracked linear and limit analysis (Marti 1999), as shown in Fig. 6.1.

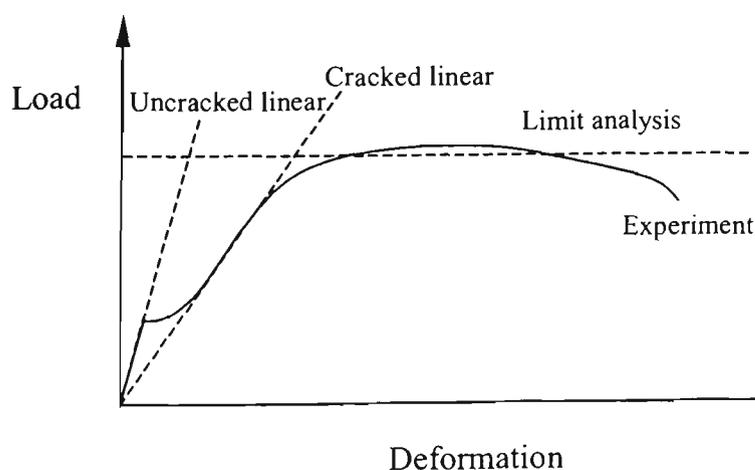


Fig. 6.1 Load-deformation curves of reinforced concrete

The shrinkage and temperature have significant effects on the load-deformation response of reinforced concrete. However, neither the stresses induced by them nor associated deformations can be determined accurately in practice. If a sufficiently ductile behavior is ensured, the ultimate strength of a structural concrete member is not affected by the loading history including the effects of shrinkage and temperature (Marti 1999). Therefore, the ultimate strength based on the limit analysis will be reliable if a structural concrete member is designed with adequate ductility and detailing.

The limit analysis is well documented in the IABSE state-of-the-art report (1979) and in the book by Nielsen (1984). The limit analysis can be divided into lower-bound and upper-bound methods. Lower-bound methods require the designer to design a concrete structure by strengthening its load transfer mechanism. They are particularly suitable for designing new structures. On the other hand, upper-bound methods allow for quick checks for ultimate strength, dimensions and details of existing structures. They are suitable for the performance evaluation of existing structures. For example, the nonlinear finite element analysis developed for reinforced concrete is an upper-bound method, which is suitable for the evaluation of load-deformation response, but rarely used in designing new structures in practice. *Strut-and-tie models correspond to the lower-bound limit analysis, and can indicate the necessary amount, the correct location and the required detailing of the steel reinforcement.* Moreover, they allow for checking of critical zones of concrete. Therefore, the strut-and-tie model approach is a rational method for the design and detailing of structural concrete.

After extensive cracking of concrete, loads applied to a reinforced concrete member are mainly carried by concrete struts and steel reinforcement. The failure of a reinforced concrete member cannot simply be explained by that the tensile stress attains the tensile strength of concrete, rather it is due to the breakdown of the load transfer mechanism, such as the yielding of steel reinforcement in ductile structural concrete members (ASCE-ACI Committee 445 on Shear and Torsion 1998). Before designing a structural concrete member, the locations of tensile ties and the amounts of steel reinforcement are not known in advance. Actually, it is the designer's task to identify an appropriate strut-

and-tie model in a structural concrete member in order to reinforce it. As a result of this, *the stiffness of reinforcing steel and the nonlinear behavior of reinforced concrete cannot be taken into account in the finite element model for developing strut-and-tie systems.*

Only two-dimensional models are considered here. Plain concrete members are treated as homogenization continuum structures, and modeled using plane stress elements in the present study. The linear elastic behavior of cracked concrete structures is assumed in the proposed method, as suggested by Schlaich et al. (1987). Since tensile ties in the strut-and-tie model obtained will be reinforced with steel reinforcement in a reinforced concrete structure, the effect of cracking due to stresses attaining the tensile strength of concrete is not considered. However, the progressive cracking of a concrete member is characterized by gradually removing concrete from the member, which is fully cracked at the optimum. The proposed method is to find a strut-and-tie system as stiff as possible. The strength of struts, ties and nodes can be treated when dimensioning the strut-and-tie model obtained.

*It is proposed here to develop strut-and-tie models in structural concrete based on the linear elastic theory of cracked concrete for system performance criteria (stiffness) and to design the structure based on the theory of plasticity for component performance criteria (strength). It is worth noting that the load-deformation response of a concrete member in optimization process is highly nonlinear due to the changing of the topology of the structure at each iteration. Generally, only two material properties are involved in the limiting performance analysis of reinforced concrete structures. One is the effective compressive strength of concrete ( $f_{c,cal}$ ), and the other is the yield strength of the steel reinforcement ( $f_{yr}$ ).*

## 6.4 OPTIMIZATION CRITERIA FOR STRUT-AND-TIE MODELS

### 6.4.1 Element Removal Criteria

In the PBO method formulated on the basis of displacement performance criteria, the objective is to maximize the performance of an initial continuum design domain in terms of the efficiency of material usage in resisting deformations. The element removal criteria are based on the virtual strain energy density (VSED) of elements. For concrete structures, the optimal strut-and-tie model can be generated by gradually eliminating a small number of elements with the lowest virtual strain energy density from a discretized concrete structure. The virtual strain energy density of the  $e$ th element is approximately calculated by

$$\zeta_e = \frac{\{u_e\}^T [k_e] \{u_e\}}{w_e} \quad (6.3)$$

For a concrete structure under multiple displacement constraints and loading cases, the weighted average approach is used to calculate the virtual strain energy density of elements for elimination. The virtual strain energy density of the  $e$ th element for multiple displacement constraints and load cases is determined by

$$\zeta_e^m = \sum_{l=1}^p \sum_{j=1}^m \beta_j \zeta_e \quad (6.4)$$

Similarly, in the PBO method formulated on the basis of overall stiffness performance criteria, element removal criteria are based on the strain energy density (SED) of elements. For concrete structures, the optimal strut-and-tie model can be generated by gradually eliminating a small number of elements with the lowest strain energy density from a discretized concrete structure. The strain energy density of the  $e$ th element is approximately evaluated by

$$\zeta_e = \frac{|\{u_e\}^T [k_e] \{u_e\}|}{2w_e} \quad (6.5)$$

For a concrete structure under multiple load case, a logical AND scheme is employed in the calculation of element strain energy density for elimination, as described in Chapter 5. An element is deleted from the concrete structure only if its strain energy density is the lowest for all load cases.

By systematically removing elements with the lowest VSED or SED from a concrete structure, the maximum stiffness topology design at minimum weight can be obtained. It is possible to select the best one from resulting topologies in the optimization process as the strut-and-tie model for a structural concrete member. These two stiffness-based optimization methods can be used to develop strut-and-tie models in structural concrete.

### 6.4.2 Performance-Based Optimality Criteria

In order to obtain the optimal strut-and-tie model, the performance of the resulting structure at each iteration must be quantitatively evaluated by using performance indices. In performance-based design, structural responses such as stresses and displacements are used as performance indices to quantify the performance of structures (ICCMC 1999). In performance-based optimal design, the performance objective is to minimize the weight of the structure while maintaining structural responses within acceptable limits. Therefore, displacements alone are not sufficient for evaluating the performance of optimal designs. The minimum weight of a structure with acceptable structural responses is a sound measure of the performance of optimal designs for stiffness.

*It is proposed here to treat optimal topologies generated by the PBO technique as optimal strut-and-tie models in structural members. Therefore, the performance-based optimality criteria for strut-and-tie models can be expressed as follows:*

For concrete structures with displacement constraints,

$$\text{maximize } PI_{ds} = \frac{u_{0j}W_0}{u_{ij}W_i} \quad (6.6)$$

and for concrete structures with the overall stiffness constraint,

$$\text{maximize } PI_{cs} = \frac{C_0W_0}{C_iW_i} \quad (6.7)$$

The optimal topology, which satisfies the performance-based optimality criteria, adapts to the condition of the member geometry, loading and supports, and actually represents the load-carrying mechanism of a structural concrete member at ultimate limit states. *The physical meaning of the performance-based optimality criteria is that the optimal strut-and-tie model transmits loads in a way such that the associated strain energy and material consumption are a minimum.*

It is worth noting that changing the width of a concrete structure under a plane stress condition has no effect on the topology of the structure or on the performance index, but it has a significant influence on the weight of the structure and structural responses. As a result of this, it is not necessary to change the width of the concrete structure in the finite element model at each iteration. Performance indices can be employed to evaluate the performance of the resulting topology at each iteration and to identify the optimum, which can then be sized by adjusting the width of the structure in order to satisfy actual displacement or overall stiffness requirements.

## 6.5 DESIGN OPTIMIZATION PROCEDURE

The design of a concrete structure by using strut-and-tie models usually involves the estimation of an initial size, finding an appropriate strut-and-tie model and

dimensioning struts, ties and nodes. Developing an appropriate strut-and-tie model for a complex structural concrete member perhaps is the most challenging task in the design process. Afterwards, dimensioning the strut-and-tie model is straightforward according to codes of practice. The main steps of the design optimization procedure are illustrated in Fig. 6.2, and explained as follows:

- (1) Model the two-dimensional concrete structure using the finite element method. The initial size of a concrete structure should be estimated on the basis of serviceability performance criteria. Experienced engineers can usually select an appropriate size for the concrete structure. It should be pointed out that it is not necessary to use a very fine mesh to divide the concrete structure. A rough layout of a strut-and-tie model is usually adequate for designing a concrete structure, as demonstrated in Chapter 5. At this step, the loads, support conditions and openings, if any, are specified. The material properties of concrete such as the Young's modulus ( $E_c$ ) and Poisson's ratio ( $\nu$ ) must be input for a linear elastic analysis. The Young's modulus of concrete can be determined in accordance with AS 3600 (1994) by

$$E_c = 0.043\rho^{1.5}\sqrt{f_{cm}} \quad (6.8)$$

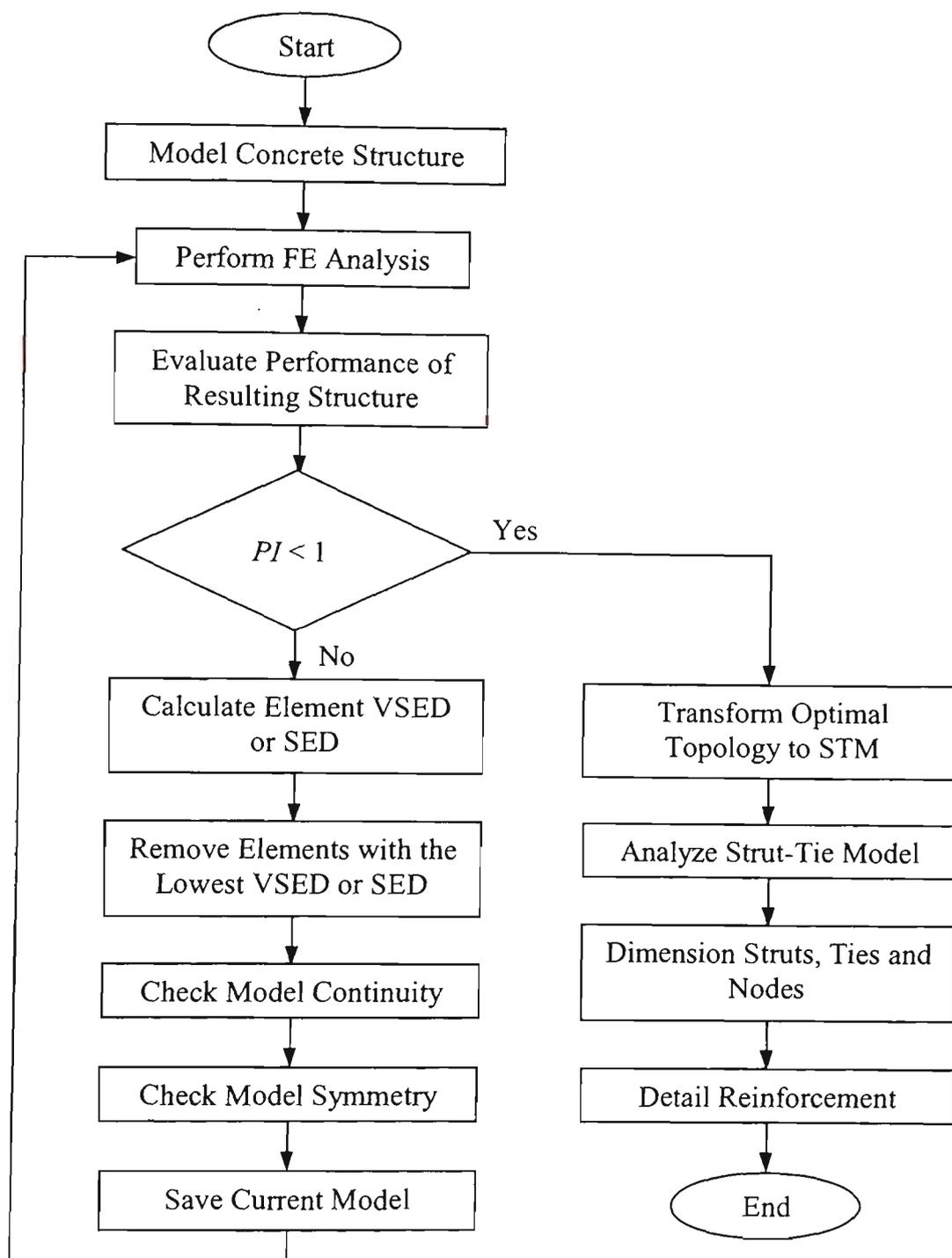
where  $\rho$  = the density of concrete ( $\text{kg/m}^3$ );

$f_{cm}$  = the mean value of the compressive strength of concrete at the relevant age.

The values of Poisson's ratio ( $\nu$ ) for concrete are between 0.11 and 0.21. It is suggested that the value of 0.15 should be used in the optimization of strut-and-tie models.

- (2) Perform a linear elastic finite element analysis on the concrete structure. If displacement constraints are considered, the concrete structure must also be

analyzed for virtual unit loads. The finite element analysis provides element stresses and nodal displacements.



**Fig. 6.2** Flowchart of design optimization procedure for strut-and-tie models in structural concrete

- (3) Evaluate the performance of resulting structure by using performance indices for plane stress structures. Eq. (4.14) should be used for structures subject to displacement constraints, whilst Eq. (5.14) should be used for structures with overall stiffness constraint. The strain energy of the structure under each load cases must be calculated in order to evaluate its performance when using Eq. (5.14).
- (4) Calculate the virtual strain energy density or strain energy density of elements for each loading case.  $R$  (%) Elements with the lowest virtual strain energy density or strain energy density are accounted and grouped together.
- (5) Remove  $R$  (%) elements with the lowest virtual strain energy density or strain energy density from the concrete structure. It has been found that the element removal ratio of 1% or 2% provides reasonable results for use in engineering practice.
- (6) Check model continuity. This is to ensure that the strut-and-tie model obtained by the optimization method must be a continuous model and satisfies the equilibrium condition.
- (7) Check model symmetry for a concrete structure with an initial symmetrical loading, geometry and support condition.
- (8) Save current model. The resulting models generated at each iteration are saved to files for use in latter stage.
- (9) Repeat step (2) to (8) until the performance index is less than unity.
- (10) Transform the optimal topology to the discrete strut-and-tie model. After the optimal topology has been obtained, it is a straightforward matter to transform it to a discrete strut-and-tie model for designing the structure. The optimal topology is a continuum structure that represents the actual stress fields in cracked concrete.

It should be noted that the flow of stress fields in cracked structural concrete might not be straight in many situations. The arch-tie action that is the most efficient load-carrying mechanism is frequently found in structural concrete members (Marti 1985; Liang et al. 2000c). Therefore, the strut-and-tie modeling is extended to include straight struts, arches and ties. The discrete strut-and-tie model is an idealization of the optimal topology. It may be necessary to make some modifications to the discrete strut-and-tie model for practical purposes.

- (11) Analyze the discrete strut-and-tie model. Internal force in each member of the strut-and-tie model is determined.
- (12) Dimension struts, ties and nodes. More details are given in Section 6.6.
- (13) Detail steel reinforcement based on the strut-and-tie model obtained. Additional reinforcement should be provided to ensure that the concrete structure has adequate serviceability and ductility performance and to prevent concrete in highly stressed regions from splitting.

## **6.6 DIMENSIONING STRUTS, TIES AND NODES**

### **6.6.1 General**

Dimensioning struts, ties and nodes is of significant importance to the overall performance of a concrete structure. Dimensioning a strut-and-tie model includes not only sizing the struts and reinforcing the ties based on the forces they carry, but also checking nodal zones for safe transmission of the loads between them. The detailing of nodes directly affects the strength performance of concrete struts connected to them and of the ties anchored in them. Moreover, the details of nodes influence the flow of forces in a concrete structure. As a result of this, it is necessary to ensure that the optimal strut-

and-tie model predicted by the PBO technique can still be realized at ultimate after detailing.

### 6.6.2 Concrete Struts

The compressive strength of concrete in struts is influenced by its state of stresses, cracks and the arrangement of steel reinforcement. The transverse compression considerably improves the compressive strength performance of concrete. It may be provided by the transverse reinforcement that confines the concrete. Transverse tensile stresses and cracks induced by them detrimentally reduce the compressive strength of concrete. If steel reinforcement is not provided to carry these tensile stresses, the concrete may fail at below its cylinder compressive strength. The compressive strength of concrete in struts is also reduced by cracks that are not parallel to compressive stresses. For safety, the effective compressive strength of concrete should be used in design of concrete struts.

Marti (1985) suggested that the effective compressive strength of concrete ( $f_{c,cal}$ ) in struts should be taken as  $0.6f'_c$ . This value may be increased or decreased depending on the state of transverse stress and the arrangement of steel reinforcement. The effective compressive strength of concrete in struts has also been proposed by Schlaich et al. (1987) as follows:

For an uncracked and uniaxial state of compressive stresses,

$$f_{c,cal} = 0.85f'_c \quad (6.9)$$

where  $f'_c$  = the characteristic compressive cylinder strength of concrete at 28 days.

When transverse compressive stresses are present,

$$f_{c.cal} = 0.935f_c' \quad (6.10)$$

With transverse tensile reinforcement that may cause cracking parallel to the strut,

$$f_{c.cal} = 0.68f_c' \quad (6.11)$$

When cracks or steel bars are skewed to the direction of the strut,

$$f_{c.cal} = 0.51f_c' \quad (6.12)$$

In AS 3600 (1994), the effects of transverse stress, cracking and steel reinforcement are not taken into account in the determination of the effective compressive strength of concrete in struts. In AS 3600, the effective compressive strength of concrete in struts is calculated by

$$f_{c.cal} = \left(0.8 - \frac{f_c'}{200}\right)f_c' \quad (6.13)$$

The effective width of a concrete strut should be determined by three-dimensional conditions in the regions. It should not be greater than the width of any adjacent bearing plates or supports. The effective depth of a concrete strut measured perpendicular to the longitudinal axis of the strut depends on the geometry of end nodes.

*It should be noted that the PBO method produces the optimal strut-and-tie model, which indicates the locations of struts, ties and nodes but not necessarily their exact dimensions.* The optimal strut-and-tie model in a concrete structure is developed on the basis of stiffness performance criteria without consideration of strength performance criteria. Therefore, dimensioning the strut-and-tie model should be based on the bearing conditions and strength performance criteria.

### 6.6.3 Ties

Reinforcing steel should be provided to carry tensile forces in ties. The strength of a tensile tie reinforced by steel bars and tendon is determined by

$$T = A_{st}f_{yr} + A_p f_{yp} \quad (6.14)$$

where

$A_{st}$  = the cross-sectional area of reinforcing steel;

$f_{yr}$  = the yield strength of reinforcing steel;

$A_p$  = the cross-sectional area of prestressing steel; and

$f_{yp}$  = the effective yield strength of prestressing steel for tensile ties.

In the proposed method, *prestressed forces are treated as external loads in the analysis, optimization and dimensioning of a structural system*, as suggested by Schlaich et al. (1987). Since part of the strength of the prestressed steel has been utilized by prestressing, only the rest is effective in resisting tensile forces. The effective yield strength of prestressing steel is used in the calculation of the strength of the tensile tie.

Sufficient anchorage of reinforcing bars should be provided to ensure that the stress in bars could be developed to their yield strength. The bar anchorage length measured from the intermost boundary of the strut or node should not be less than the stress development length of the bar. If the space is not available for anchorage, cogs, bends and U-bars should be used.

### 6.6.4 Nodes

Nodes are the interaction points of three or more struts and ties in a strut-and-tie system.

The flow of forces changes its direction at a node in the model. Nodes can be classified into *singular nodes*, where one of the struts or ties represents a concentrated stress field, and *smearred nodes*, where wide compressive stress fields join together or with tensile ties reinforced with closely distributed steel bars.

Singular nodes are usually formed due to point loads, support reactions, concentrated forces induced by steel reinforcement through anchor plates, and geometrical discontinuities. It is clear that singular nodes are the most highly stressed regions in a concrete structure, and thus the most critical regions that need to be designed carefully. The sizes of struts are likely to be governed by the shapes and sizes of adjacent nodes. Therefore, AS 3600 (1994) requires that the sizes of struts and ties shall not be larger than the capacity of nodes to transfer forces between struts and ties in a strut-and-tie model.

Dimensioning a node in a strut-and-tie system usually includes the following steps:

- (a) Choose the geometry of the node with applied forces;
- (b) Check whether the compressive stresses of concrete within the node exceed the limits;
- (c) Ensure that each reinforcing bar has an adequate anchorage.

For a node joining with three concrete struts, the borders of the node can be designed in a way such that they are perpendicular to the longitudinal axis of the struts. Moreover, the dimensions of the three borders ( $a_1, a_2, a_3$ ) can be proportioned to the forces in the struts (Schlaich et al. 1987), such as:

$$a_1 : a_2 : a_3 = F_1 : F_2 : F_3 \quad (6.15)$$

For nodes joining with tensile ties, the dimensions of nodes are governed by the sizes of any anchorage plates for reinforcing bars as well as the concrete cover to the bars.

## 6.7 OPTIMAL STRUT-AND-TIE MODELS IN REINFORCED CONCRETE MEMBERS

### 6.7.1 General

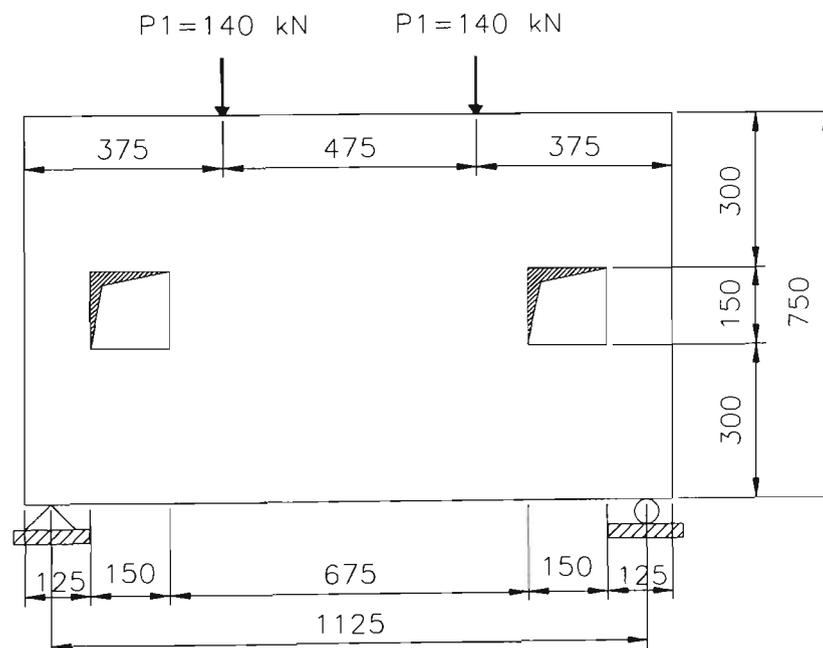
The proposed method for developing optimal strut-and-tie models in structural concrete is based on a very simple engineering design concept that by gradually removing lowly strained elements from a discretized concrete structure, the actual load transfer mechanism of the structure can be found. In this section, numerical examples are provided to demonstrate the effectiveness and validity of the proposed PBO method for automatically generating optimal strut-and-tie models in reinforced concrete structures, such as deep beams with openings, continuous beams, beams with various span-to-depth ratios, and corbels. Optimal strut-and-tie models obtained by the PBO method are compared with experimental observation as well as existing analytical solutions. Dimensioning struts, ties and nodes according to codes of practice is so straightforward that it is not considered in illustrative examples.

### 6.7.2 Verification by Experimental Evidence

To verify the proposed PBO method for developing optimal strut-and-tie models in structural concrete, two deep beams with web openings are investigated by using the PBO technique, and the results are compared with experimental observations.

A simply supported deep beam with two web openings based on the test specimen (O-O.3/3) conducted by Kong and Sharp (1977) is shown in Fig. 6.3. In the tested specimen, one 20-mm diameter deformed bar was used as the bottom longitudinal steel reinforcement, and no web reinforcement was provided. Two point loads of  $P_1 = 140$  kN are applied to the top of the deep beam. By neglecting the effect of the bottom longitudinal reinforcement, the plain concrete beam is modeled using 25-mm square

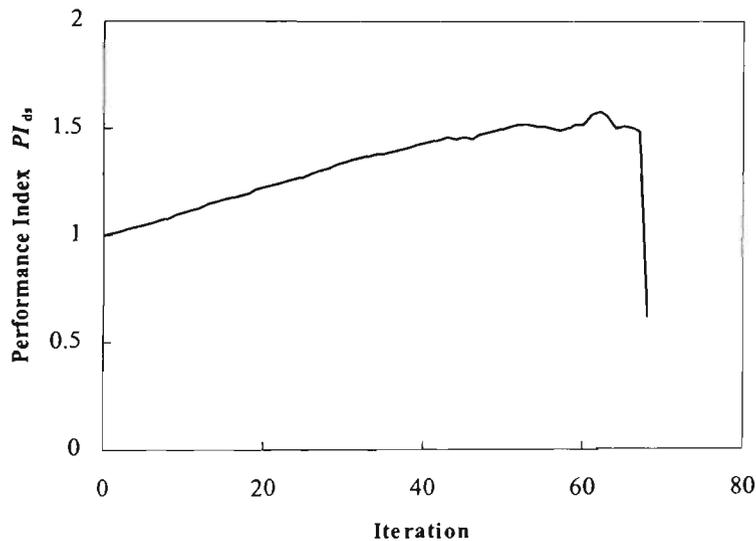
four-node plane stress elements. The displacement constraints of the same limit are imposed at the two loaded points in the vertical direction. The compressive cylinder strength of concrete  $f'_c = 35.5$  MPa, Young's modulus of concrete  $E_c = 30088$  MPa, Poisson's ratio  $\nu = 0.15$  and the width of the beam  $b = 100$  mm are used in the analysis. Plane stress conditions are assumed. The element removal ratio  $R = 1\%$  is adopted in the optimization process.



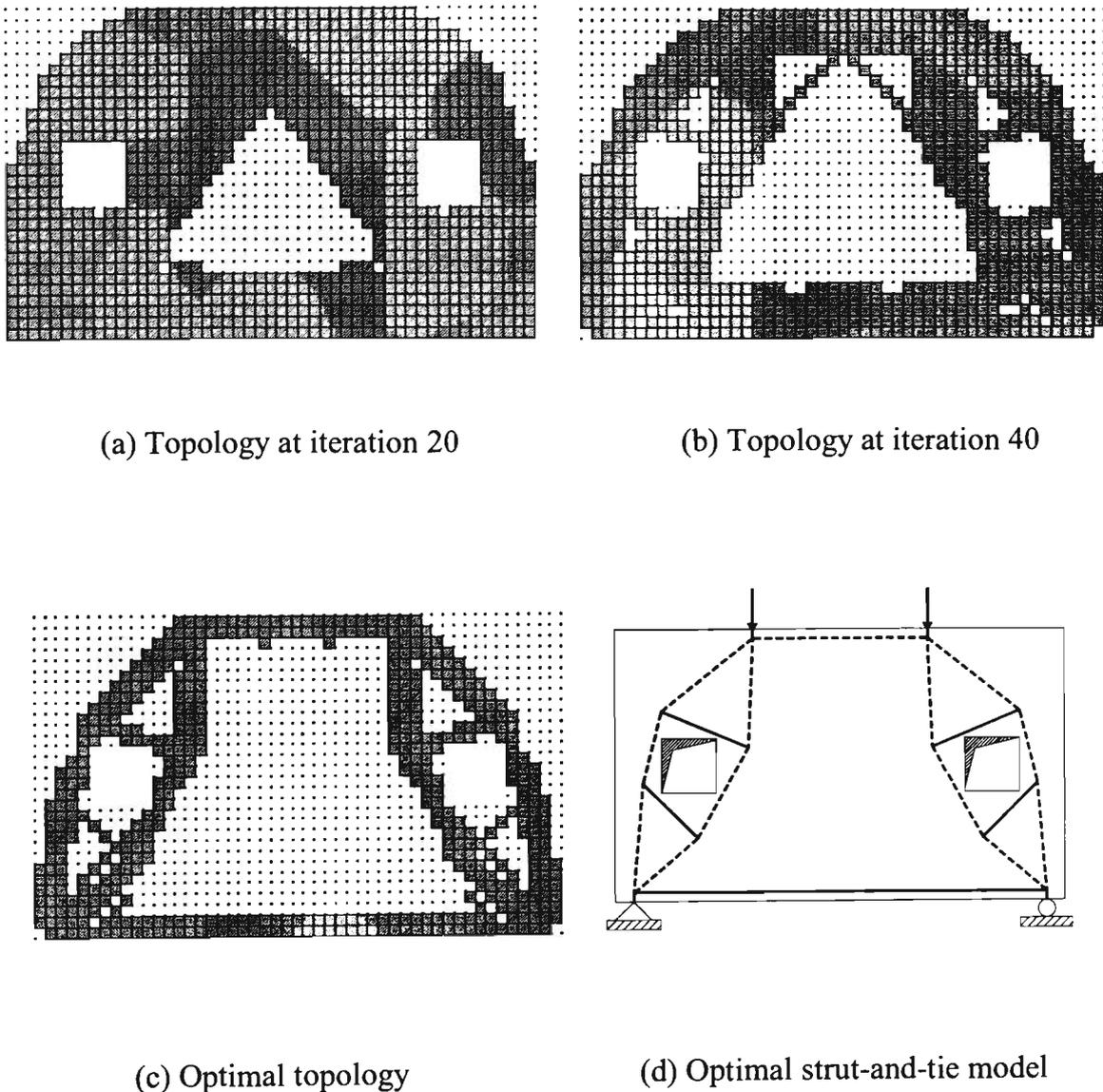
**Fig. 6.3** Deep beam with web openings

The performance index history of the deep beam with web openings generated by the PBO method is presented in Fig. 6.4. The maximum performance index is 1.58, which corresponds to the optimal topology of the deep beam. The topology optimization history is shown in Fig. 6.5, from which it can be observed that the load transfer mechanism of the deep beam is gradually manifested by the remaining elements in the

deep beam. Since the underutilized portions of concrete are removed from the deep beam, the concrete deep beam as shown in Fig. 6.5(c) is fully cracked. This means that loads are mainly carried by the resulting structural system in the deep beam at a fully cracked state under the ultimate condition. The optimal topology shown in Fig. 6.5(c) is interpreted as the optimal strut-and-tie model in the deep beam shown in Fig. 6.5(d), where solid bold lines represent concrete struts and dotted lines represent tensile ties.



**Fig. 6.4** Performance index history of deep beam with openings

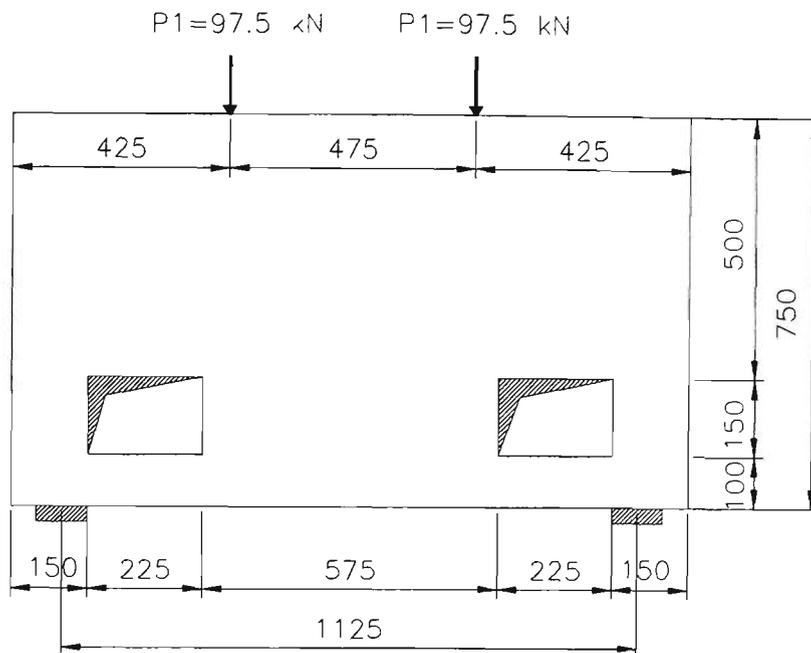


**Fig. 6.4** Optimization history of strut-and-tie model in deep beam with web openings

In nature, the loads are usually transmitted along the shortest natural load paths between the loading and support points to minimize the associated strain energy of the load-carrying system. If the opening intercepts the natural load path, the load is to be re-routed around the opening (Kumar 1978). This is confirmed by the optimal strut-and-tie model shown in Fig. 6.5(d), which indicates that loads are transmitted to supports by the upper and lower struts around the opening. The test conducted by Kong and Sharp

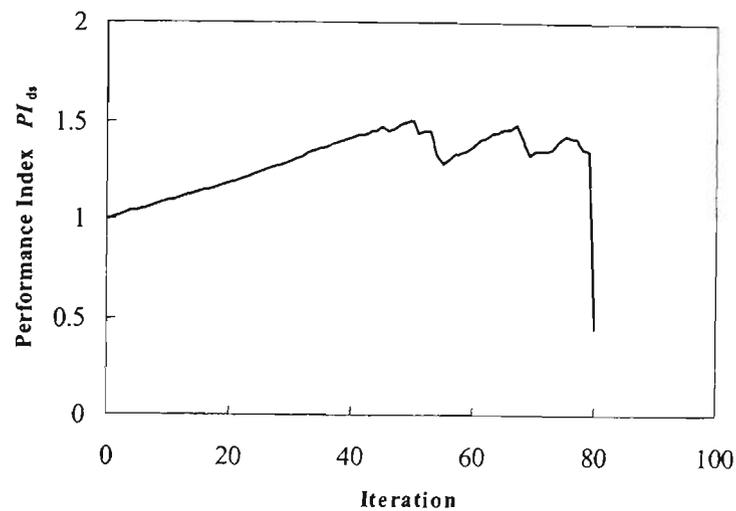
(1977) shows that diagonal cracking occurred at above and below the opening. It is clear that tensile stresses were developed across the corners of the openings. The presence of two inclined tensile ties that connect the upper and lower struts around the opening in Fig. 6.5(d) is confirmed by experimental observations. Optimal strut-and-tie model clearly indicates the location of tensile ties. In dimensioning the optimal strut-and-tie model, inclined web reinforcement should be provided to carry inclined tensile forces. Inclined web reinforcement has been proved to be the most effective for increasing the ultimate strength and for crack control of such deep beams with web openings (Kong and Sharp 1973, 1977).

A further verification of the proposed method is conducted on a simply supported lightweight-concrete deep beam with two openings located below the axis of the depth of the beam as shown in Fig. 6.6. This deep beam is based on the test specimen (O-O.3/16) presented by Kong and Sharp (1977). In the tested specimen, one 20-mm diameter deformed bar was used as the longitudinal tensile steel reinforcement placed at the bottom of the beam. No web reinforcement was used in the tested specimen. In this study, two concentrated loads of  $P_1 = 97.5$  kN are applied to the top of the beam. The Young's modulus of concrete  $E_c = 30966$  MPa, Poisson's ratio  $\nu = 0.15$  and the width of the beam  $b = 100$  mm are adopted in the analysis. The plain concrete beam is divided into 25-mm square, four-node plane stress elements. Two displacement constraints of the same limit are imposed at the two loaded points in the vertical direction. The element removal ratio  $R = 1\%$  is used in the optimization process.



**Fig. 6.6** Simply supported deep beam with web openings

The performance index history of this deep beam is shown in Fig. 6.7. It can be seen from Fig. 6.7 that when inefficiently used concrete is gradually removed from the beam, the performance index will increase from unity to the maximum value of 1.52. This maximum performance index corresponds to the optimal topology of the deep beam for the given geometry and loading conditions. Topologies obtained at iterations 20, 40, and 50 are shown in Figs. 6.8 to 6.10. It is observed from these figures that the actual load transfer mechanism in the cracked concrete deep beam is gradually represented by the resulting topology when elements with the least contribution to the stiffness of the beam are deleted from the finite element model.



**Fig. 6.7** Performance index history of deep beam with web openings

The optimal topology shown in Fig. 6.10 can be idealized by the optimal strut-and-tie model presented in Fig. 6.11. By comparison of the optimal strut-and-tie model with the tested specimen shown in Fig. 6.12, it can be observed that tensile ties in the optimal model exactly indicate the patterns and locations of cracks in the tested specimen. This strongly proves that the proposed PBO method can predict extremely well the actual load transfer mechanism of structural concrete members at the ultimate limit states. Therefore, it is appropriate to develop strut-and-tie models in structural concrete based on the linear elastic theory of cracked concrete for system performance criteria, and to design concrete structures based on the theory of plasticity for strength performance criteria, as pointed out previously.

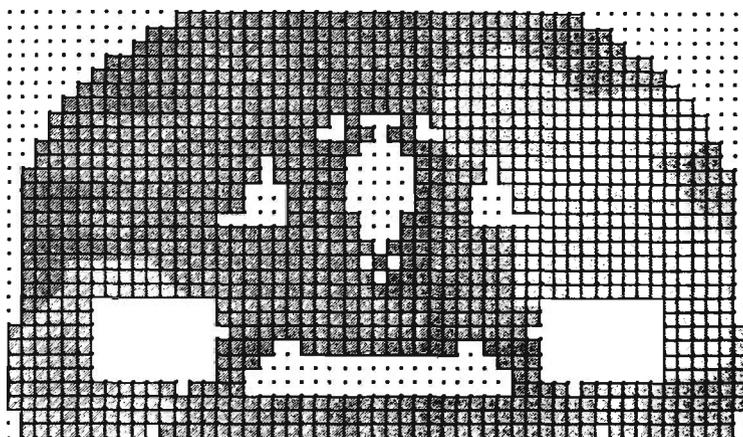


Fig. 6.8 Topology at iteration 20

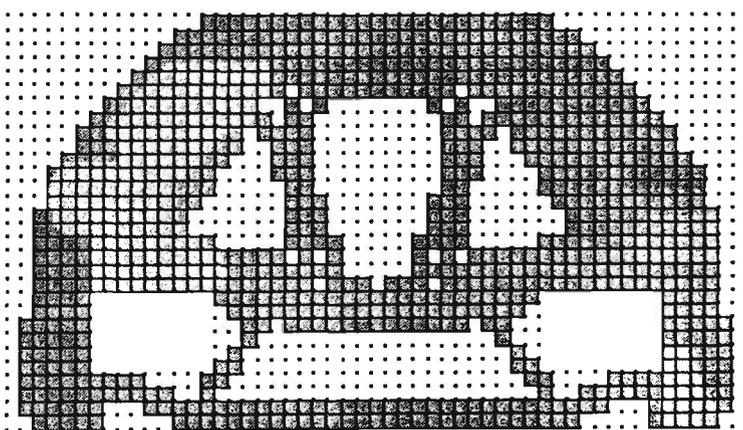


Fig. 6.9 Topology at iteration 40

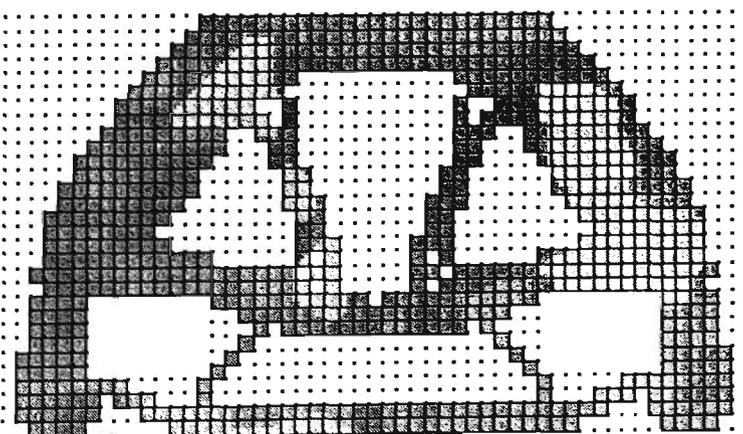
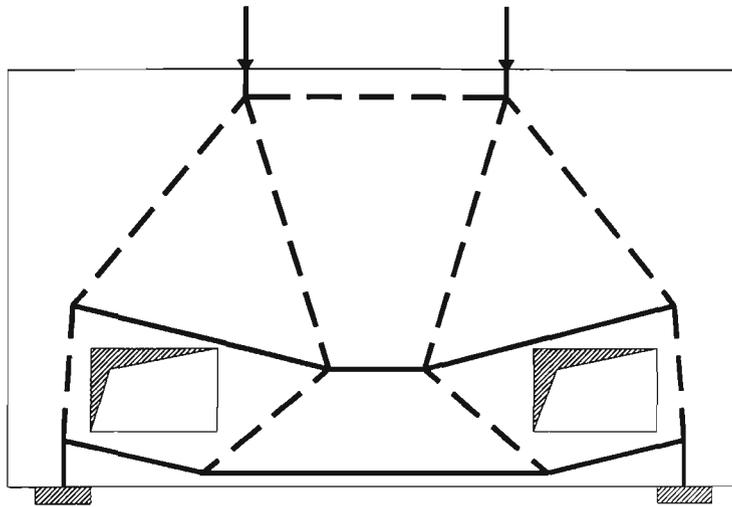
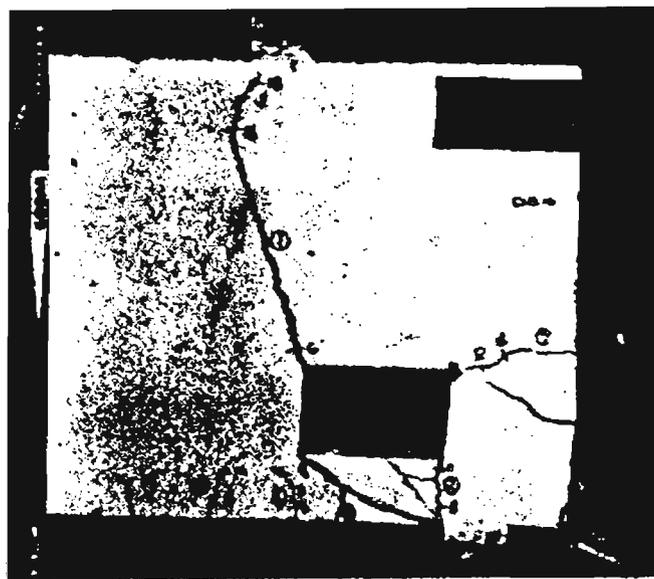


Fig. 6.10 Optimal Topology at iteration 50



**Fig. 6.11** Optimal strut-and-tie model



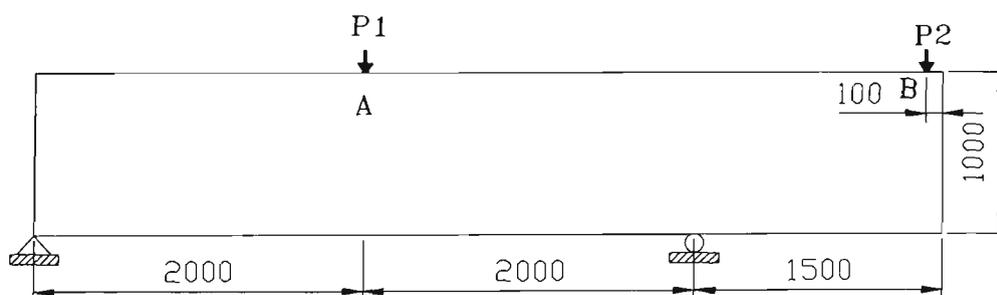
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**Fig. 6.12** Test result by Kong and Sharp (1977)

### 6.7.3 Continuous Beam

Continuous reinforced concrete beams are commonly used in building construction. More efforts are usually needed to develop strut-and-tie models in continuous concrete beams than in single span beams if conventional methods are used. This example is to show the efficiency of the PBO method in dealing with the strut-and-tie modeling of continuous concrete beams.

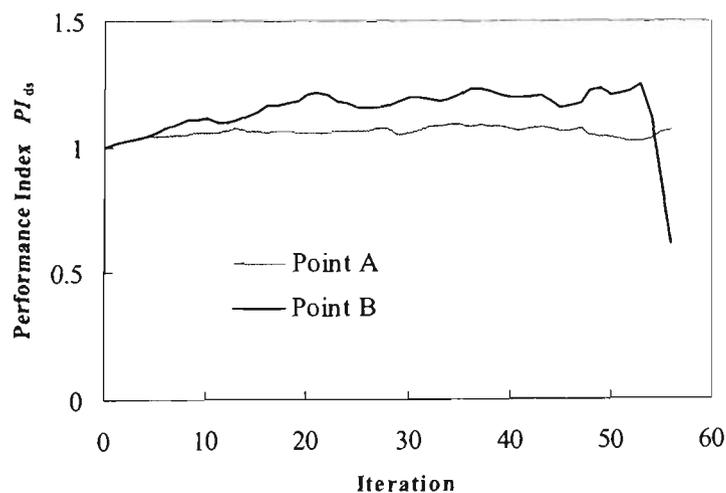
The PBO technique is used to find the optimal strut-and-tie model in a continuous concrete beam under two point loads of  $P_1 = 1000$  kN and  $P_2 = 550$  kN, as shown in Fig. 6.13. The continuous plain concrete beam is modeled using 50-mm square four-node plane stress elements in the finite element analysis. The compressive cylinder strength of concrete  $f'_c = 32$  MPa, the Young's modulus of concrete  $E_c = 28600$  MPa, Poisson's ratio  $\nu = 0.15$  and the initial width of the beam  $b_0 = 200$  mm are assumed in the analysis. The same deflection limit is imposed at the two loaded points A and B. The  $R = 1\%$  is adopted in the optimization process. It should be noted that the beam width is uniformly sizing during the optimization process to keep the displacement constraints active. The width of the beam can be adjusted to meet the actual strength and deflection performance requirements after the optimal strut-and-tie model has been obtained.



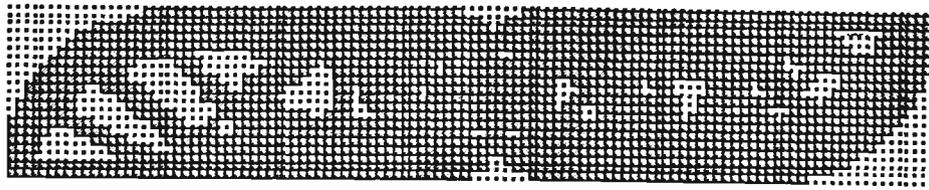
**Fig. 6.13** Continuous beam

The performance index history of the continuous beam is presented in Fig. 6.14, where the performance index curves are obtained by using Eq. (4.14) with the deflections at points A and B. It can be seen from Fig. 6.14 that the performance index at the point A is less than that at the point B from iterations 1 to 55. This is because the deflection is more critical at point A than at point B. After iteration 55, however, the performance index at point B drops sharply since further element removal leads to the large deflection at point B, which becomes the most critical displacement. By comparing these two curves, it is obvious that the topology of the continuous beam reaches the optimum at iteration 55.

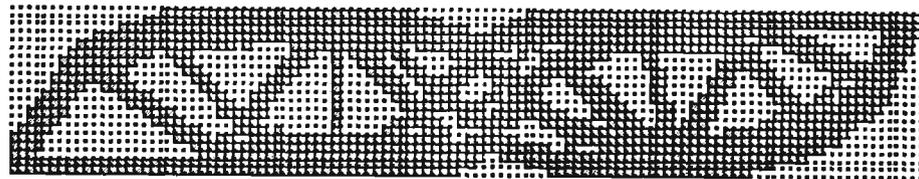
Fig. 6.15 presents the optimization history of the strut-and-tie model in this beam. The strut-and-tie idealization illustrated in Fig. 6.15(d) indicates a complex model, which is difficult to be found if using conventional methods. It is seen that loads are transferred through struts and ties to supports. Inclined tensile ties are developed in shear spans, as shown in Fig. 6.15(d). This optimal strut-and-tie model suggests that the inclined



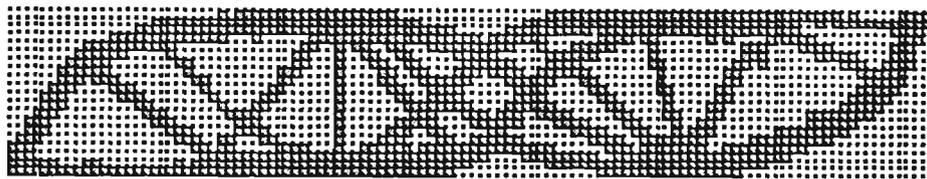
**Fig. 6.14** Performance index history of the continuous beam



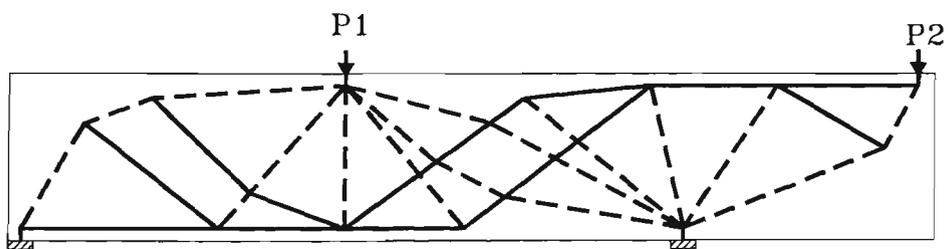
(a) Topology at iteration 20



(b) Topology at iteration 40



(c) Optimal topology at iteration 55



(d) Optimal strut-and-tie model

Fig. 6.15 Optimization history of strut-and-tie model in continuous beam

reinforcement bent up from the bottom steel reinforcement should be used to resist tensile forces developed in shear spans and extended to the end of the cantilever to carry tensile forces induced by the load  $P_2$ . A layout arrangement of steel reinforcement in the continuous beam is illustrated in Fig. 6.16. It should be noted that steel reinforcement in the bottom or the top of the beam could be in one layer if they can fit in.

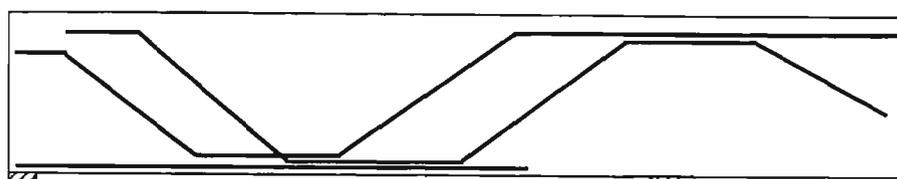
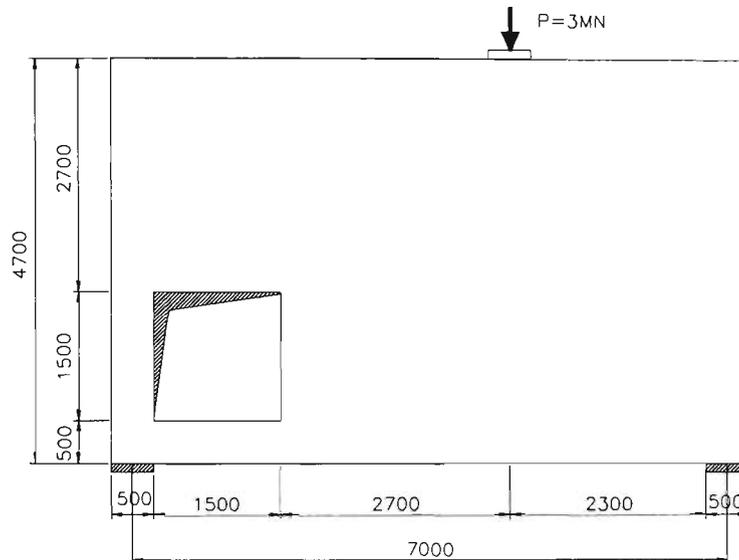


Fig. 6.16 Reinforcement layout of the continuous beam

#### 6.7.4 Deep Beam with a Large Hole

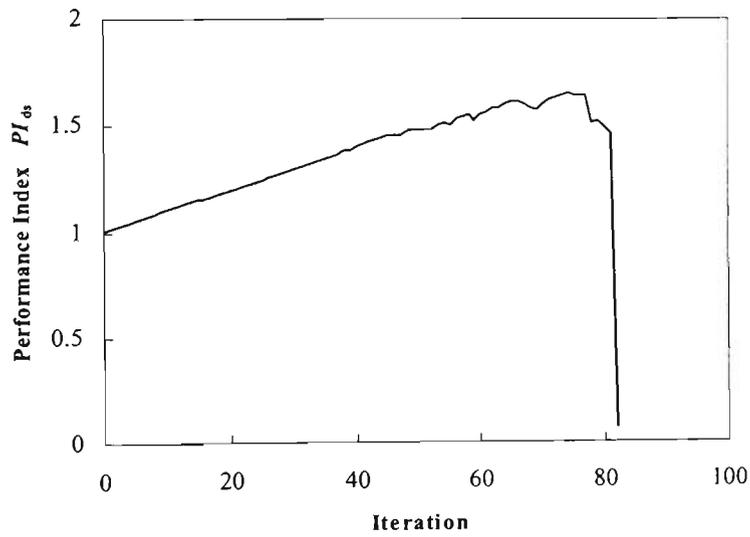
In this example, the strut-and-tie model in a simply supported deep beam with a large hole as shown in Fig. 6.17 is investigated by the PBO method and the result obtained is compared with the solution given by Schlaich et al. (1987). The concrete beam is discretized by using 100-mm square four-node plane stress elements. A displacement constraint is imposed at the loaded point in the vertical direction. The compressive design strength of concrete  $f_c = 17$  MPa, Young's modulus of concrete  $E_c = 20820$  MPa, Poisson's ratio  $\nu = 0.15$  and the initial width of the beam  $b_0 = 400$  mm are adopted in this study. The  $R = 1\%$  is used in the optimization process.



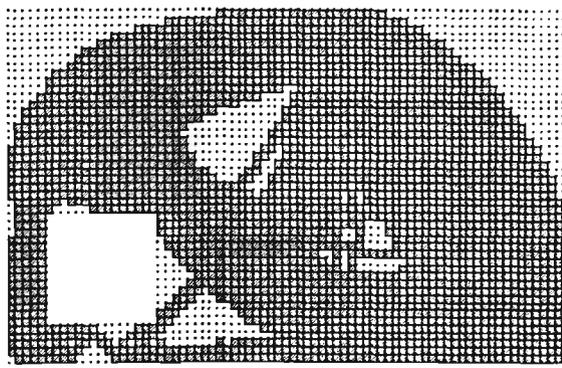
**Fig. 6.17** Deep beam with a large hole

Fig. 6.18 shows the performance index history of the deep beam with a large hole. After reaching the peak value, the performance index drops sharply. This is caused by the breakdown of the load-carrying mechanism. The maximum performance index is 1.65 that corresponds to the optimal topology shown in Fig. 6.19(c). The topologies obtained at different iterations in the optimization process are presented in Fig. 6.19. It is seen that the load is to be re-routed around the opening even if the opening is very close to the support. The inclined tensile tie is developed across the upper right corner of the opening, which will tend to crack under the applied load. The basic layout of the load transfer system is clearly shown by the topology obtained at iteration 40, as presented in Fig. 6.19(b). Further element removal only sizes the model. The two internal concrete struts join together at iteration 40, but depart from each other at the optimum shown in Fig. 6.19(c). This may be caused by the checkerboard pattern, which results in the stiffest structure. The optimal strut-and-tie model interpreted from the optimal topology is illustrated in Fig. 6.19(d), where the two internal struts have been joined together. The optimal strut-and-tie model obtained by the present study is similar to the model given by Schlaich et al. (1987). The layout of main steel reinforcement for this deep beam is

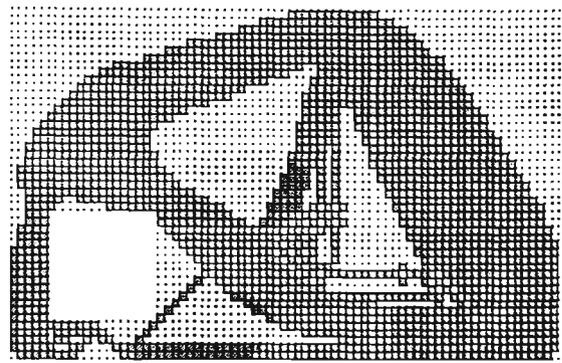
illustrated in Fig. 6.20. It is worth noting that additional reinforcing meshes should be provided for crack control in accordance with codes of practice.



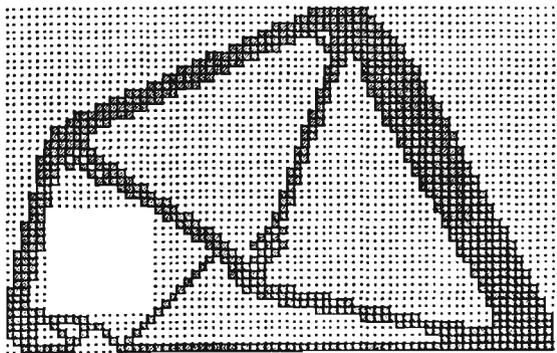
**Fig. 6.18** Performance index history of the deep beam with a large hole



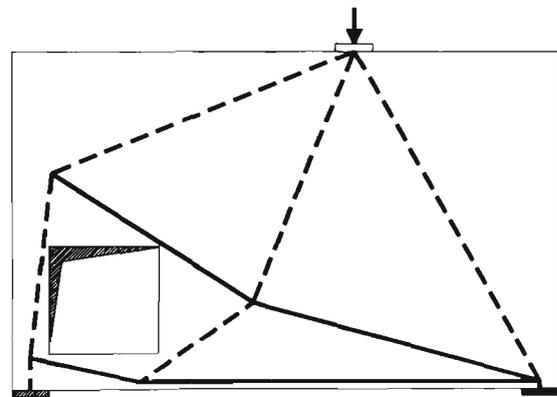
(a) Topology at iteration 20



(b) Topology at iteration 40

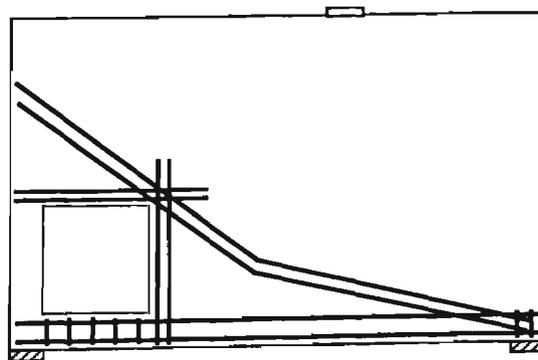


(c) Optimal topology



(d) Optimal strut-and-tie model

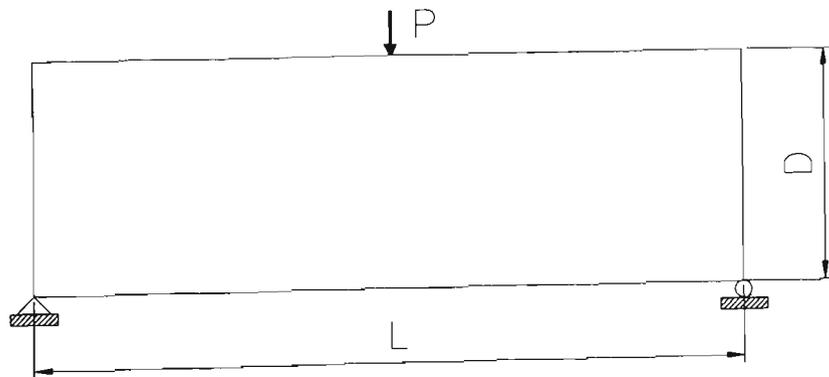
**Fig. 6.19** Optimization history of strut-and-tie model in the deep beam with a large hole



**Fig. 6.20** Reinforcement layout of the deep beam with a large hole

### 6.7.5 Effects of Span-to-Depth Ratios

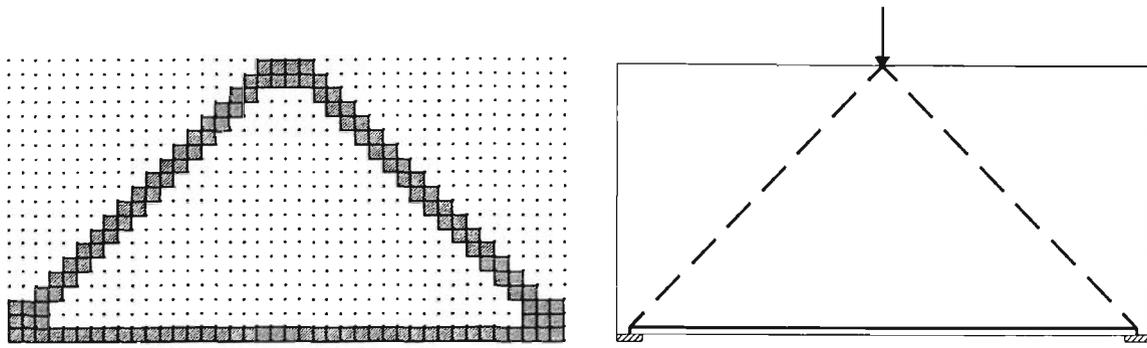
The load transfer mechanism in a concrete structure depends on the geometry, loading and support conditions of the structure. This example is to investigate the effects of span-to-depth ratios on optimal strut-and-tie models in reinforced concrete beams when other conditions are fixed. Simply supported concrete beams under a concentrated load acting at the mid-span of the beams as shown in Fig. 6.21 are considered here. The depth of the beams  $D$  is 1000 mm for all cases whilst the span-to-depth ratio for cases (a) to (d) is 2, 3, 4, and 5. The applied point load  $P = 1200$  kN and the initial width of the beam  $b_0 = 250$  mm are assumed for all cases. The concrete beams are modeled using 50-mm square four-node plane stress elements. A displacement constraint imposed at the loaded point in the vertical direction is considered. The compressive cylinder strength of concrete  $f'_c = 32$  MPa, Young's modulus of concrete  $E_c = 28567$  MPa, Poisson's ratio  $\nu = 0.15$  are used for all cases. The element removal ratio  $R = 1\%$  is employed in the optimization.



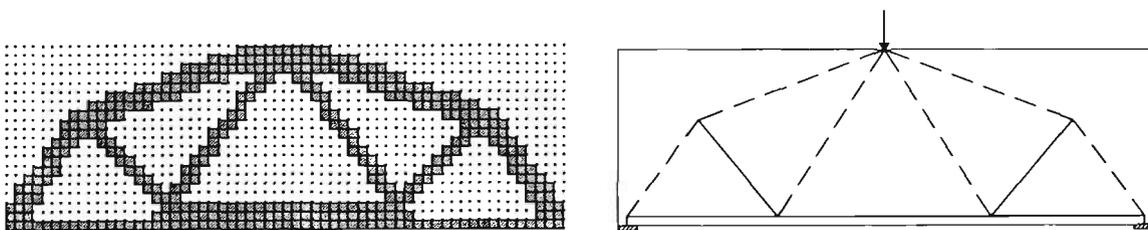
**Fig. 6.21** Simply supported beams with various span-to-depth ratios

The maximum performance indices obtained by using Eq. (4.14) for case (a) to (d) are 1.88, 1.3, 1.23 and 1.21. The optimal topology and corresponding strut-and-tie model for each case are presented in Fig. 6.22. It is demonstrated that the load transfer mechanism in concrete beams changes with the changes in span-to-depth ratios of beams. When the span-to-depth ratio of the beam is equal to 2, the load is transferred from the loaded point to the supports through straight struts. For beams with a span-to-depth ratio greater than 3, inclined tensile ties connecting compressive concrete struts are necessary to form the strut-and-tie model as shown in Fig. 6.22(b) to (d). For very slender beams, optimal topologies obtained by the PBO method are continuum-like structures, in which strut-and-tie actions are difficult to be identified, such as that shown in Fig. 6.22(d). For such cases, the flexural beam theory or standard truss models may be used to design these concrete beams. These optimal strut-and-tie models indicate that the angles between compressive concrete struts and longitudinal ties are equal to or larger than  $45^{\circ}$ . In detail design, some of the bottom steel bars may be bent up to carry the forces in inclined tensile ties.

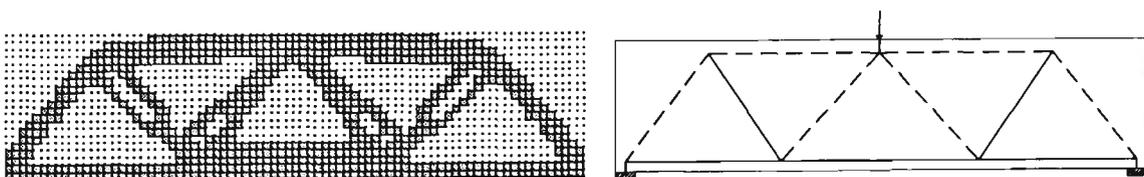
It is clearly shown that the load transfer mechanism in a structural concrete member adapts to its geometry. Without modification, a strut-and-tie system developed for a specific structural concrete member cannot be used in the design of members with different geometry, loading and support conditions.



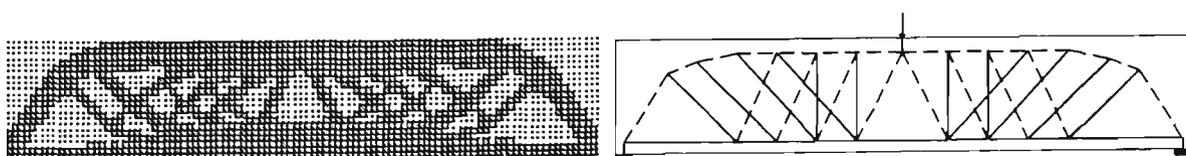
(a)  $L/D = 2$



(b)  $L/D = 3$



(c)  $L/D = 4$



(d)  $L/D = 5$

**Fig. 6.22** Optimal topologies and strut-and-tie models showing the transition from deep beams to slender beams

### 6.7.6 Corbel

The PBO technique is used to find the strut-and-tie model for the design of a corbel joining with a column as shown in Fig. 6.23. To look at how the load is transmitted from the corbel to the column, the corbel and column are treated as a whole structure in the development of the strut-and-tie system. The corbel is designed to support a point load of 500 kN as illustrated in Fig. 6.23. The column is fixed at both ends. This structure is discretized using 25-mm square, four-node plane stress elements. A displacement constraint is imposed at the loaded point in the vertical direction. The compressive cylinder strength of concrete  $f'_c = 32$  MPa, Young's modulus of concrete  $E_c = 28567$  MPa, Poisson's ratio  $\nu = 0.15$ , and the initial width of the corbel and column  $b_0 = 300$  mm are assumed. The element removal ratio  $R = 1\%$  is used in the optimization process.

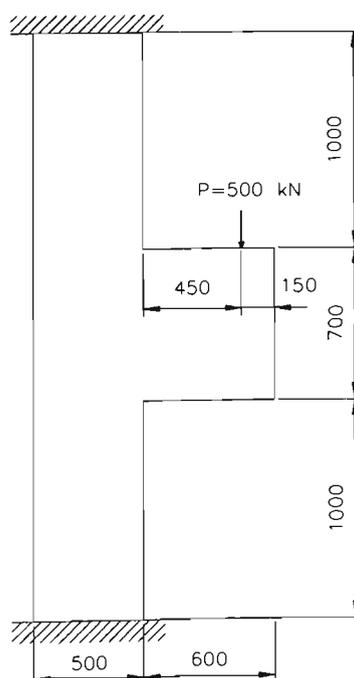
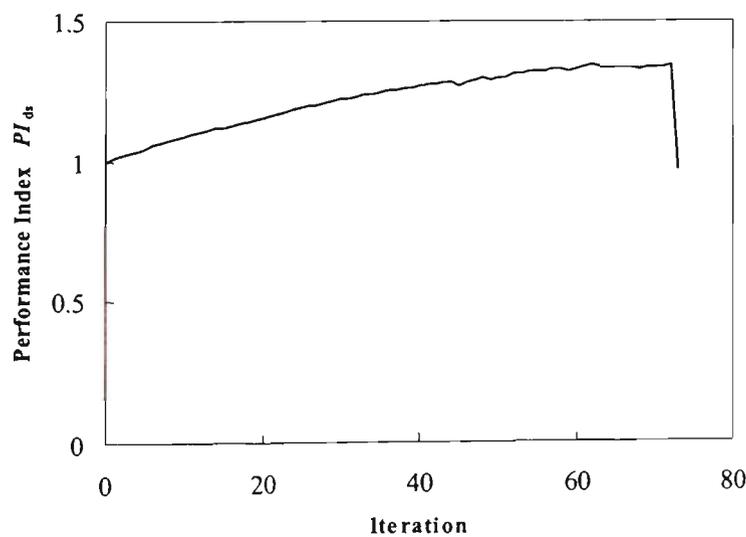
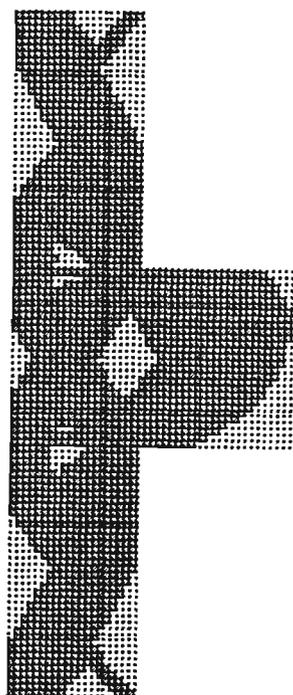


Fig. 6.23 Corbel connecting with a column

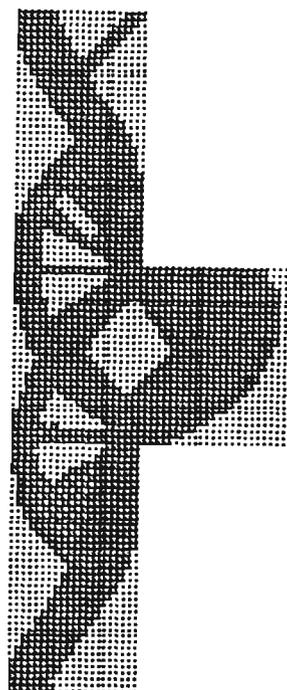
Fig. 6.24 shows the performance index history of the structure in the optimization process. The maximum performance index is 1.34 and the corresponding optimal strut-and-tie topology is shown in Fig. 6.25(c). From the optimization history of the structure presented in Fig. 6.25, the checkerboard pattern is observed in the topology obtained at iteration 40. The load is transferred from the corbel to the whole range of the column along the paths of concrete struts and tensile ties. The strut-and-tie model obtained is rather complicated. It is suggested that the corbel and column should be considered as a whole structure in developing the strut-and-tie model. The optimal strut-and-tie model illustrated in Fig. 6.25(d) is supported by the solution obtained by Schlaich et al. (1987) using the load path method.



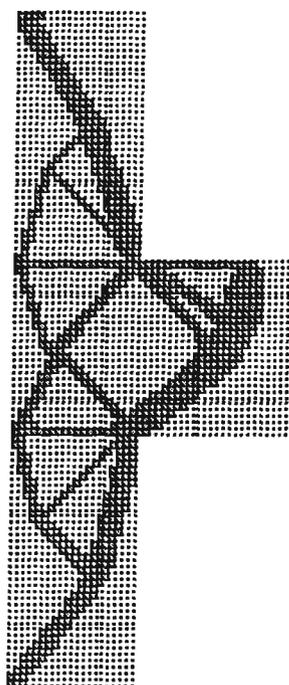
**Fig. 6.24** Performance index history of the corbel



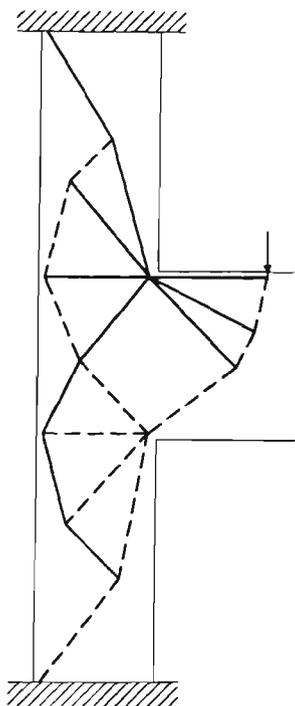
(a) Topology at iteration 20



(b) Topology at iteration 40



(c) Optimal topology



(d) Optimal strut-and-tie model

Fig. 6.25 Optimization history of strut-and-tie model in corbel

## 6.8 OPTIMAL STRUT-AND-TIE MODELS IN PRESTRESSED CONCRETE BEAMS

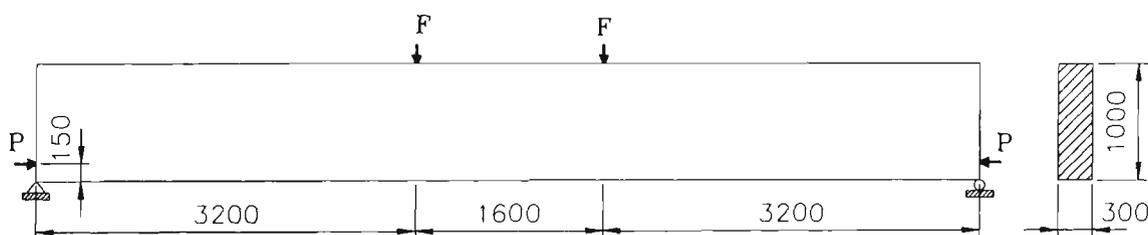
### 6.8.1 General

In prestressed concrete beams, compressive prestressing forces are artificially applied to the concrete beam with the help of hydraulic jacks in order to reduce or eliminate cracking due to high tensile stresses and deflection induced by applied loads. Choosing the appropriate tendon profile, the type and the magnitude of prestressing forces can favorably and efficiently alter the load transfer mechanism in concrete beams. A fully prestressed concrete beam has no tensile chord at its bottom whilst a partially prestressed concrete beam has a tensile chord at its bottom, which is shorter than that of a nonprestressed concrete beam. Schlaich et al. (1987) suggested that by treating prestressing forces as external loads, prestressed concrete beams could be analyzed and dimensioned like reinforced concrete ones with strut-and-tie systems. Ramirez (1994) applied the strut-and-tie model approach to the design of pretensioned concrete members.

Kirsch (1993b) proposed a procedure for optimizing the member size, initial prestressing force and tendon profile of a prestressed concrete system. However, no work has been undertaken so far on optimization of strut-and-tie models in prestressed concrete beams by continuum topology optimization methods. In this section, therefore, optimal strut-and-tie models in nonprestressed, partially prestressed and fully prestressed concrete beams are investigated by using the PBO method for structures with displacement constraints. *It is proposed here that strut-and-tie models in prestressed concrete structures can also be optimized by treating prestressing forces as external loads.*

## 6.8.2 Nonprestressed Concrete Beam

Fig. 6.26 shows a simply supported prestressed concrete beam with a rectangular cross-section under two concentrated loads of  $F = 500$  kN and the prestressing force  $P$ . When the prestressing force  $P = 0$ , the beam is a nonprestressed concrete beam, which is considered herein for comparison purposes. This concrete beam is modeled using a  $160 \times 20$  mesh with four-node plane stress elements. The depth of the beam is 1000 mm. The initial width of the concrete beam  $b_0 = 300$  mm is assumed. The values of the Young's modulus of concrete  $E_c = 31940$  MPa, Poisson's ratio  $\nu = 0.15$  are used in the analysis. Two displacement constraints of the same limit are imposed at the points of load  $F$  in the vertical direction. The element removal ratio  $R = 1\%$  is used in the optimization process.



**Fig. 6.26** Prestressed concrete beam

The performance index history of the nonprestressed concrete beam obtained by the PBO method is shown in Fig. 6.27. When a small number of elements with the lowest virtual strain energy density is removed from the beam, the performance index increases from unity to the maximum value of 1.38. After reaching the peak, it drops sharply and this means that further element removal will result in large deflections. The performance

index may jump in the optimization history as shown in Fig. 6.27 because the element removal ratio used is still high. As presented in Chapter 4, it is possible to achieve smoother solutions to nonprestressed members by using a smaller element removal ratio, but the computational cost will be increased considerably.

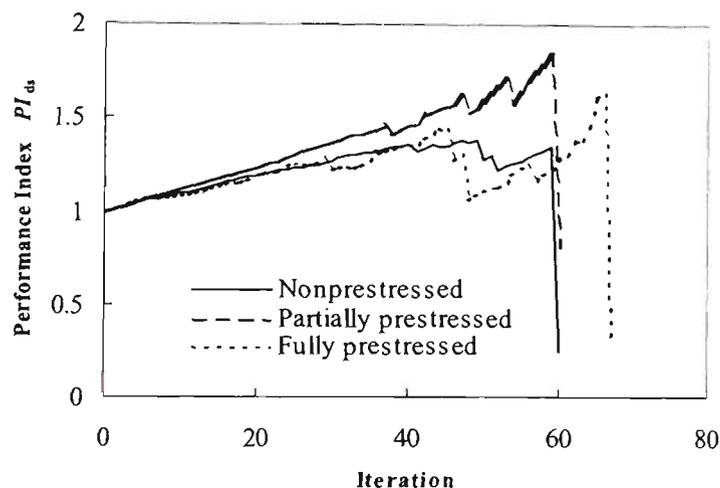
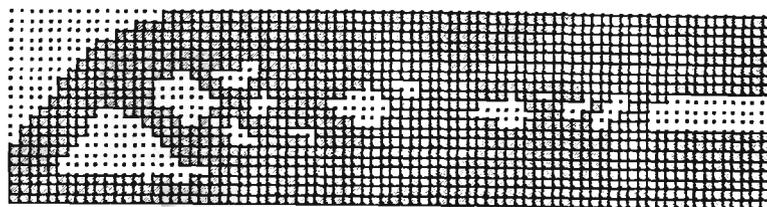
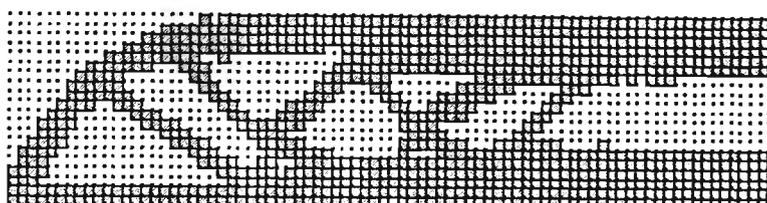


Fig. 6.27 Performance index history of prestressed concrete beams

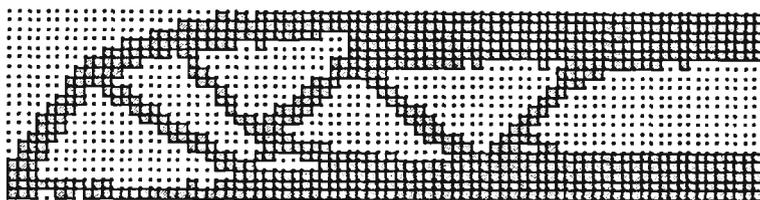
The optimization history of the strut-and-tie model in the nonprestressed concrete beam is presented in Fig. 6.28, in which only half of the model is shown by taking the advantage of symmetry. It can be seen from Fig. 6.28 that when inefficiently used concrete is removed from the beam, the strut-and-tie model is gradually characterized by the remaining elements. This optimal strut-and-tie model indicates that inclined reinforcing steel bent up from the bottom reinforcement should be used to resist inclined tensile forces developed in shear spans. The strut-and-tie model of this reinforced concrete beam shown in Fig. 6.28(e) was obtained by Schlaich et al. (1987) using the load path method. In their strut-and-tie model, vertical ties were assumed to form the model.



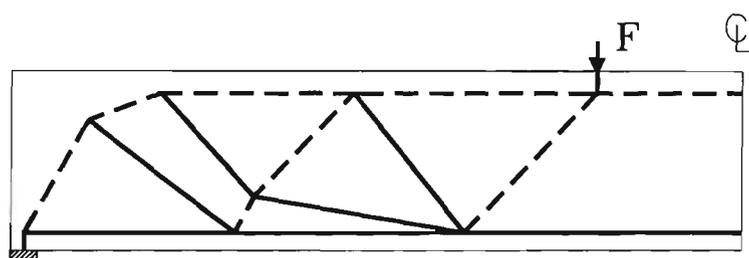
(a) Topology at iteration 20



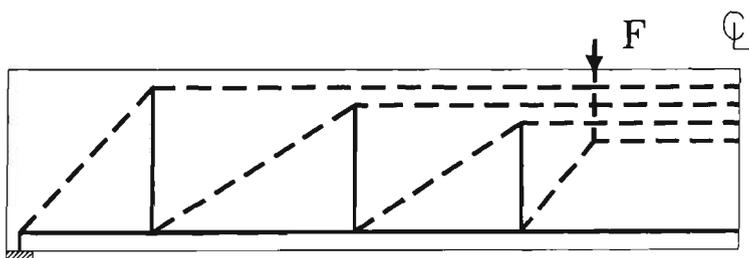
(b) Topology at Iteration 40



(c) optimal topology



(d) Optimal strut-and-tie model



(e) Strut-and-tie model by Schlaich et al. (1987)

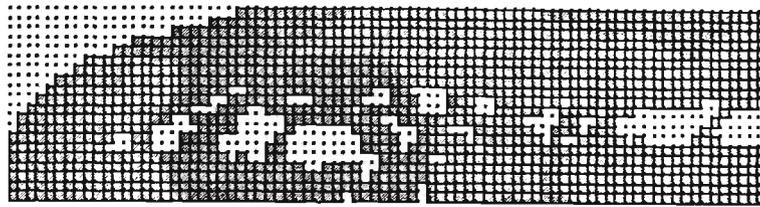
Fig. 6.28 Optimization history of strut-and-tie model in nonprestressed concrete beam

### 6.8.3 Partially Prestressed Concrete Beam

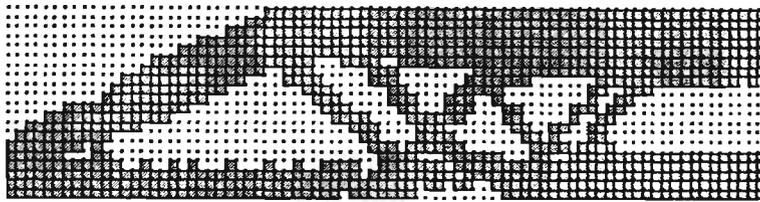
Cracking and deflections of a concrete beam can be reduced or eliminated by applying prestressing forces to the beam. Prestressing reduces the length of the tension chord along the bottom of a concrete beam. By treating prestressing forces as external loads, prestressed concrete beams can be analyzed, designed and dimensioned with strut-and-tie models in the same manners as reinforced concrete beams as suggested by Schlaich et al. (1987).

A prestressing force of  $P = 1650$  kN is applied to the concrete beam shown in Fig. 6.26. Two equal vertical displacement constraints are imposed at the points of loading  $F$  since the deflections of the beam are to be reduced. The  $R = 1\%$  is adopted in the performance-based optimization process. The performance index history of this prestressed concrete beam is also shown in Fig. 6.27. The maximum performance index of the prestressed concrete beam is 1.85, which is higher than that of the non-prestressed concrete beam.

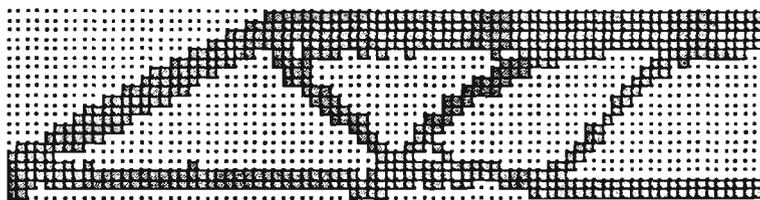
Fig. 6.29 shows the optimization history of this prestressed concrete beam. The load transfer mechanism in the beam becomes more and more clear when elements are systematically eliminated from the model. It can be seen from Fig. 6.29(d) that there is a tension chord at the bottom of the beam that is shorter than that of the nonprestressed concrete beam shown in Fig. 6.28(d). Thus, this concrete beam is partially prestressed. It can be observed from a comparison of Figs. 6.28(d) and 6.29(d) that prestressing loads significantly affect the strut-and-tie model in the concrete beam and loads transmit along a more direct load path. Furthermore, Fig. 6.27 shows that the partially prestressed concrete member has the highest performance index. This means that the most economic design can be achieved by using partial prestressing. The strut-and-tie model of a partially prestressed concrete beam given by Schlaich et al. (1987) is illustrated in Fig. 6.29(e). In Schlaich et al.'s model, however, the strut at the bottom of the beam is absent.



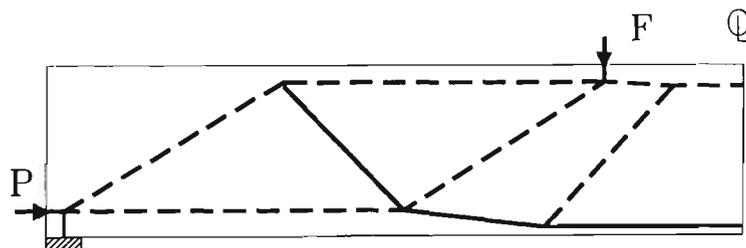
(a) Topology at iteration 20



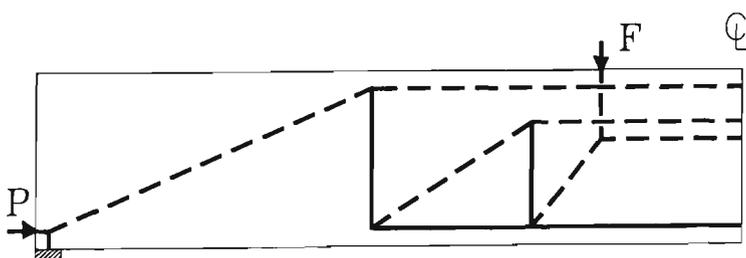
(b) Topology at iteration 40



(c) Optimal topology



(d) optimal strut-and-tie model



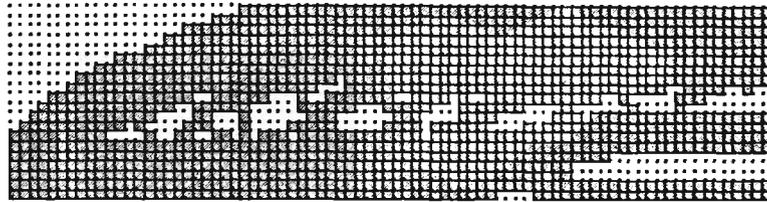
(e) Strut-and-tie model by Schlaich et al. (1987)

**Fig. 6.29** Optimization history of strut-and-tie model in partially-prestressed concrete beam

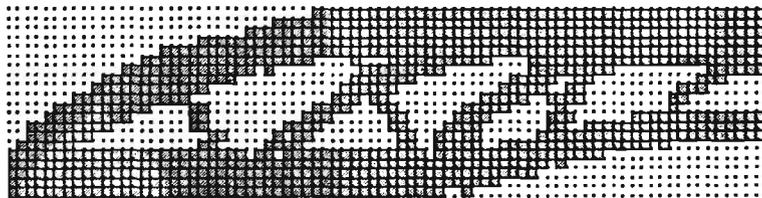
### 6.8.4 Fully Prestressed Concrete Beam

The strut-and-tie model of a fully prestressed concrete beam has no tension chord at the bottom of the beam as demonstrated by Schlaich et al. (1987). By choosing the prestressing force  $P = 2500$  kN, the PBO technique is used to generate the strut-and-tie model in the prestressed concrete beam illustrated in Fig. 6.26. The maximum performance index obtained is 1.62, as seen from Fig. 6.27. It can be observed from Fig. 6.27 that the performance index increases after decreasing at a few iterations. This is because further element removal results in a more direct load transfer mechanism in the beam.

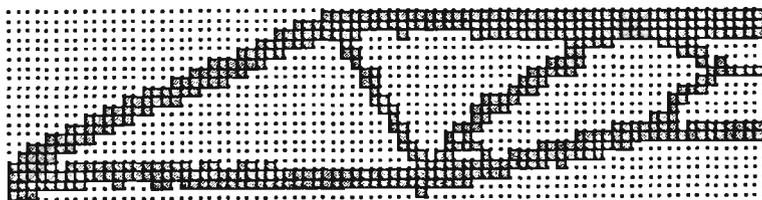
The optimization history is shown in Fig. 6.30, from which it can be seen that the strut-and-tie model has no tensile chord at the bottom of the beam because the full prestressing transforms the beam under applied loads into a beam-column. However, the inclined tensile tie still exists in the shear span since a tensile force is developed in shear spans. Schlaich et al. (1987) presented the strut-and-tie model of a fully prestressed concrete beam as illustrated in Fig. 6.30(e). They suggested that in a fully prestressed condition, the resultant of the prestressed force and the support force meet the line of action of the load  $F$  within the kern of the section. However, the optimal strut-and-tie model shown in Fig. 6.30(d) indicates that the full prestressing condition may be achieved without the resultant meeting the action line of the load  $F$ . Moreover, the fully prestressing is characterized by the absence of a tension chord along the bottom of a prestressed beam.



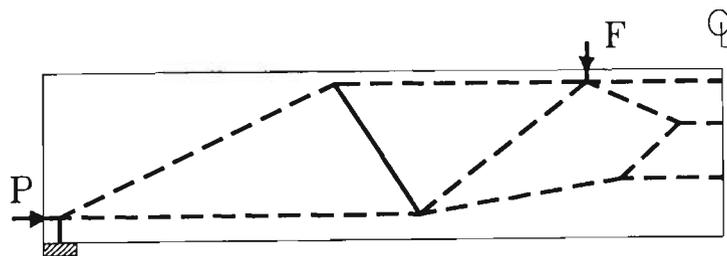
(a) Topology at iteration 20



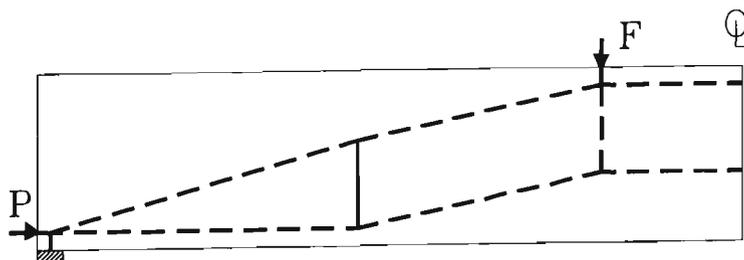
(b) Topology at iteration 40



(c) Optimal topology



(d) Optimal strut-and-tie model



(e) Strut-and-tie model by Schlaich et al. (1987)

**Fig. 6.30** Optimization history of strut-and-tie model in fully-prestressed concrete beam

## 6.9 OPTIMAL STRUT-AND-TIE MODELS IN LOW-RISE SHEARWALLS

### 6.9.1 General

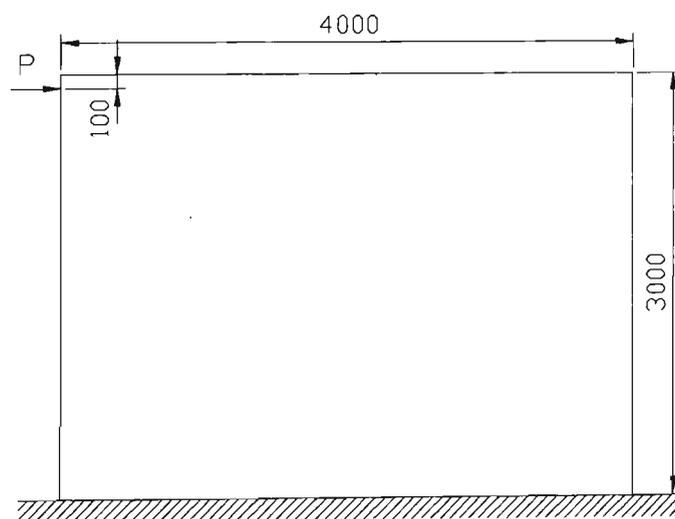
Reinforced concrete shearwalls are commonly used in buildings to resist lateral loads arising from wind or earthquakes. High-rise shearwalls in tall buildings behave essentially in the same manner as flexural reinforced concrete members, which can be designed by using the flexural beam theory. Low-rise shearwalls in low-rise buildings more often have the height-to-length ratio less than 1.5 and thus their behavior cannot be predicted by conventional methods applied to tall shearwalls. The design of low-rise shearwalls in the past practice was largely based on the findings of experimental work on low-rise shearwalls (Benjamin and Williams 1957) and on deep beams. The truss model theory considering the softening of concrete has been used to predict the load-deformation responses of low-rise reinforced concrete shearwalls with boundary elements.

Low-rise reinforced concrete shearwalls are actually deep beams, which can be designed by using strut-and-tie models. Marti (1985) has used the load path method to develop strut-and-tie models in reinforced concrete shearwalls with openings. However, the utility of strut-and-tie models in engineering practice is often limited by the designer's ability to develop appropriate models for structures with complicated loading and geometry conditions using conventional methods, which usually involve a trial-and-adjustment iterative process.

In this section, the Performance-Based Optimization (PBO) method formulated on the basis of overall stiffness performance criteria is used to develop strut-and-tie models in low-rise reinforced concrete shearwalls with and without openings. The PBO method is efficient in dealing with complex shearwalls under a large number of point loads as well as multiple loading cases.

### 6.9.2 Shearwall under One Loading Case

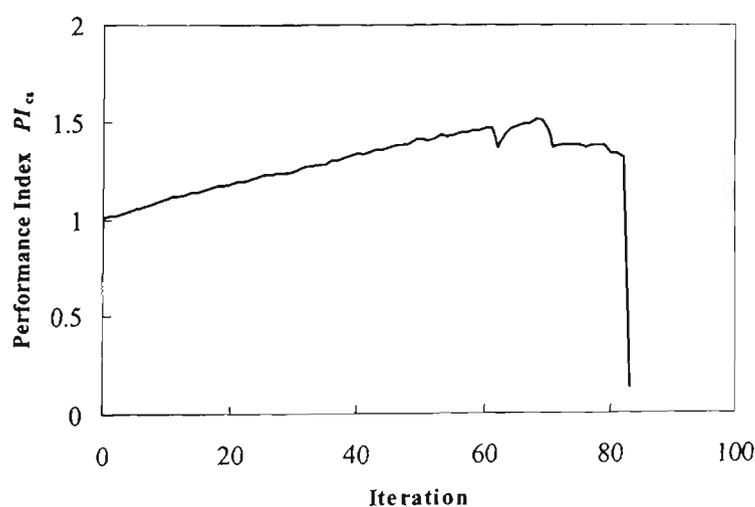
The PBO method is used to develop the optimal strut-and-tie model in a low-rise concrete shearwall with a rectangular cross section, as shown in Fig. 6.31. The applied load at the top is 1200 kN. The shearwall is fixed on the foundation. The compressive cylinder strength of concrete  $f'_c = 32$  MPa, Young's modulus of concrete  $E_c = 28600$  MPa, Poisson's ratio  $\nu = 0.15$  and the initial thickness of the shearwall  $t_0 = 150$  mm are assumed. The concrete shearwall is modeled using  $40 \times 30$  four-node plane stress elements. The mean compliance limit is set to a large value in order to obtain the global optimum. The  $R = 1\%$  is used in the optimization process.



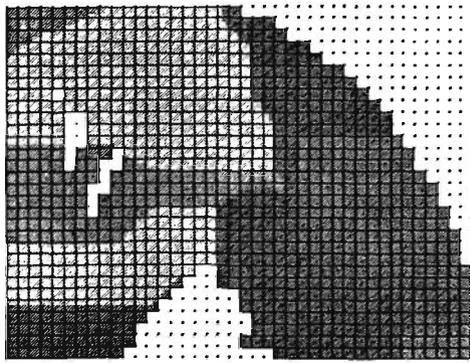
**Fig. 6.31** Low-rise shearwall under one loading case

Fig. 6.32 shows the performance index history of the low-rise shearwall obtained by the optimization procedure. It is seen that the performance of the resulting topology is gradually improved by eliminating a small number of elements with the lowest strain energy density from the model at each iteration. The maximum performance index is 1.51. At the final stage, the performance index drops sharply. This means that further element removal leads to a large increase in strain energy, which may be caused by the breakdown of the load transfer mechanism.

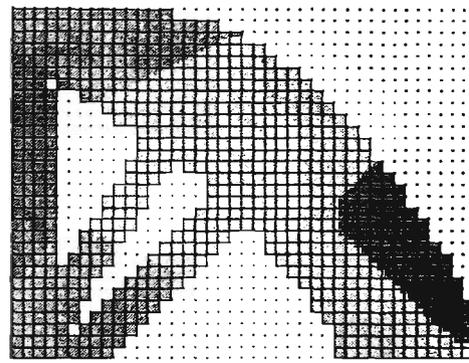
The optimization history of strut-and-tie model in the low-rise shearwall is demonstrated in Fig. 6.33. It can be observed that the load transfer mechanism in the shearwall gradually become clear by removing underutilized material from the model. The optimal topology shown in Fig. 6.33(c) is obtained while the performance index is maximized. This optimal topology can be interpreted as the optimal strut-and-tie model of the shearwall shown in Fig. 6.33(d). Although the shearwall considered here has no columns and beams as boundary elements, the optimal strut-and-tie model provides insight into the crack patterns of tested reinforced concrete shearwalls conducted by Benjamin and Williams (1957).



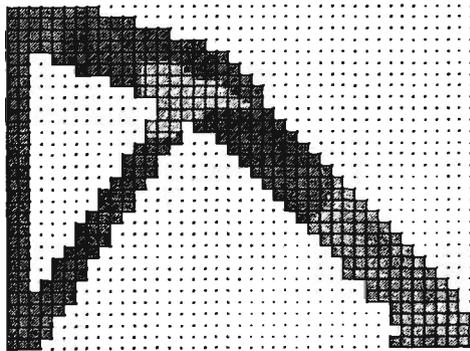
**Fig. 6.32** Performance index history of shearwall under one loading case



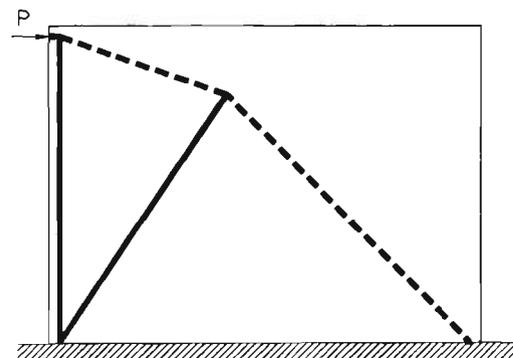
(a) Topology at iteration 20



(b) Topology at iteration 40



(a) Topology at iteration 20



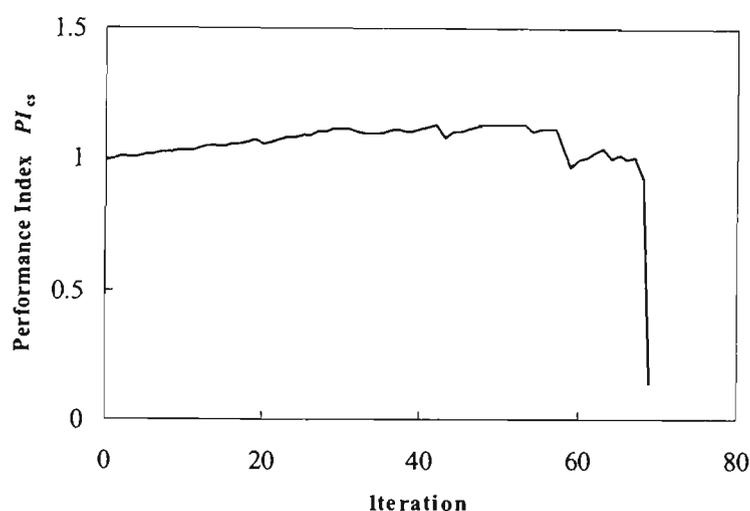
(b) Topology at iteration 40

Fig. 6.33 Optimization history of shearwall under one loading case

### 6.9.3 Shearwall under Multiple Loading Cases

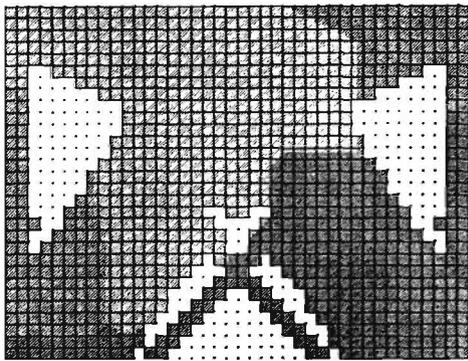
This example is to show the capacities of the computer-based topology optimization method in producing optimal strut-and-tie models in concrete shearwalls under multiple loading cases. The concrete shearwall shown in Fig. 6.32 is now considered under a reversible loading. The material properties and modeling are the same as used in previous example. The  $R = 1\%$  is also adopted in the optimization process. The

performance index of the concrete shearwall under two load cases is presented in Fig. 6.34, where the maximum performance index is 1.14.

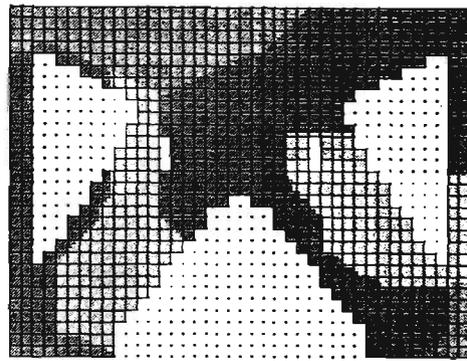


**Fig. 6.34** Performance index history of shearwall under multiple loading cases

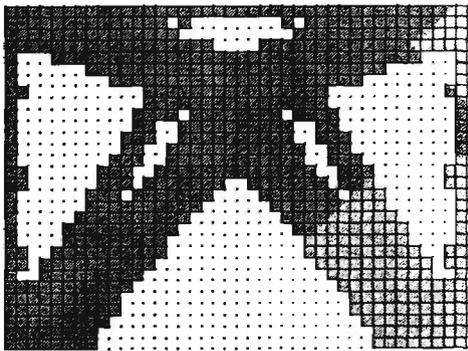
Fig. 6.35 shows the topology optimization history of strut-and-tie model in the shearwall. By removing elements whose strain energy density is the lowest under two load cases from the structure, actual load paths in the shearwall under the fully cracked condition can be represented by remaining elements in the model. The optimal topology is obtained at iteration 52 as shown in Fig. 6.35(d). By inspection of this optimal topology, it is seen that the load carried by the horizontal compressive strut at the top is very small. At iteration 55, this strut is removed as shown in Fig. 6.35(e), but the performance index only has a minor change as seen from Fig. 6.34. Hence, the topology obtained at iteration 55 can be used as the optimal strut-and-tie model for the design of the shearwall under two load cases and is interpreted in Fig. 6.35(f). By comparison of the strut-and-tie model illustrated in Fig. 6.35(f) with the one shown in



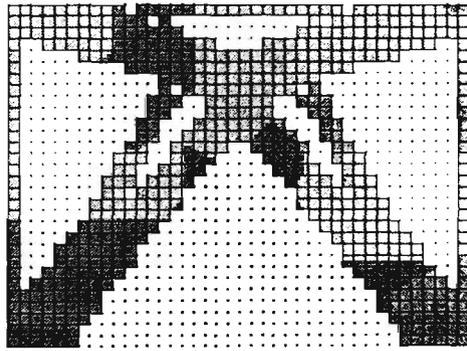
(a) Topology at iteration 20



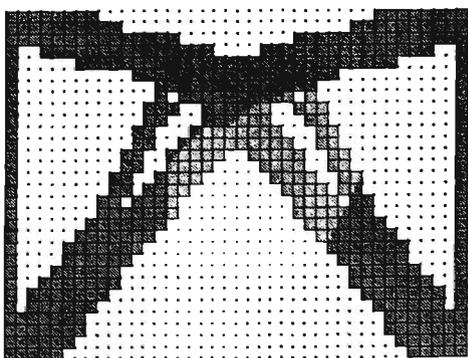
(b) Topology at iteration 30



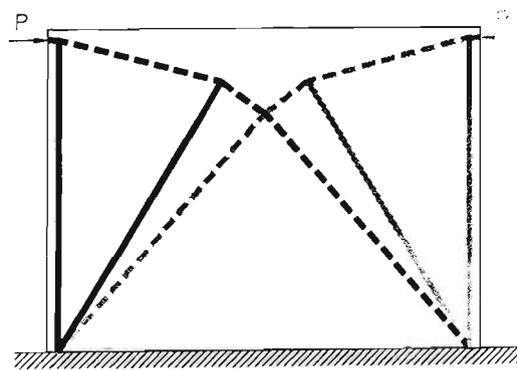
(c) Topology at iteration 40



(d) Optimum at iteration 52



(e) Topology at iteration 55



(f) Strut-tie model showing superposition

**Fig. 6.35** Optimization history of shearwall under multiple loading case

Fig. 6.35(d), the superposition of the model under the reversible loading can be observed. It should be noted that a member in a strut-and-tie model could be in compression under one load case and in tension under another load case. Therefore, it is important to reinforce all tension members with steel bars under each load case.

### 6.9.4 Shearwall with Openings

This example is to demonstrate the efficiency of the PBO method in dealing with concrete shearwalls with complex geometry and loading conditions. Fig. 6.36 shows a low-rise concrete shearwall with openings under many concentrated loads in one load case. This shearwall is based on the example presented by Marti (1985). In the present study, the loads  $P_1 = 1000$  kN and  $P_2 = 500$  kN are assumed. The compressive cylinder strength of concrete  $f'_c = 32$  MPa, Young's modulus of concrete  $E_c = 28600$  MPa, Poisson's ratio  $\nu = 0.15$  and the initial thickness of the shearwall  $t_0 = 200$  mm are used in the analysis. The shearwall is modeled using 100-mm square, four-node plane stress elements. The mean compliance constraint is considered. The  $R = 1\%$  is used in the optimization process.

Fig. 6.37 shows the performance index history of the shearwall with openings. It can be seen from Fig. 6.37 that even if there are a large portion of openings in the shearwall, the performance of the shearwall is still improved by eliminating unnecessary concrete from the model. The maximum performance index of 1.2 occurs at iteration 35. The optimization history of strut-and-tie model in the shearwall with openings is presented in Fig. 6.38. When elements are removed from the shearwall, the resulting topology evolves towards a frame-like structure. Optimal topology is obtained at iteration 35 as shown in Fig. 6.38(d), which can be idealized as the optimal model shown in Fig. 6.38(e). This model consists of only struts. The optimal strut model in the shearwall with openings generated by the PBO method agrees extremely well with the solution shown in Fig. 6.38(f), which was presented by Marti (1985). Since the strut-and-tie

model obtained has no tensile ties, it is not necessary to provide main steel reinforcement to resist tensile forces in the shearwall. However, reinforcing meshes must be provided in the shearwall to control cracking that may be induced by shrinkage and temperature effects.

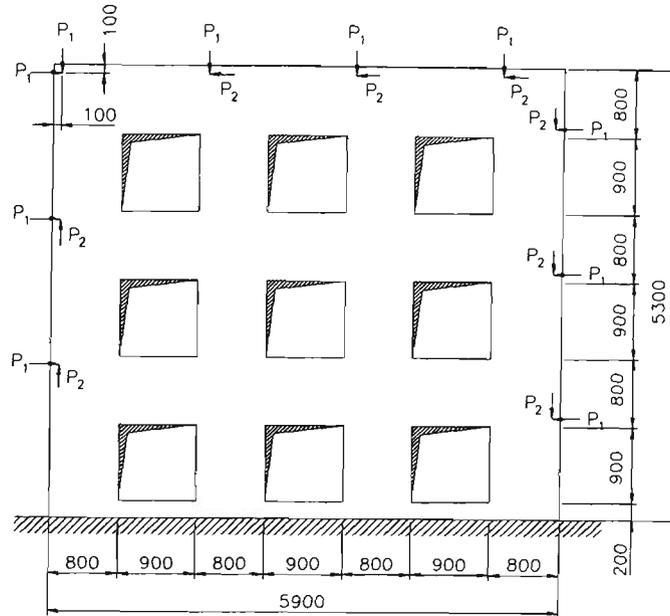


Fig. 6.36 Shearwall with openings

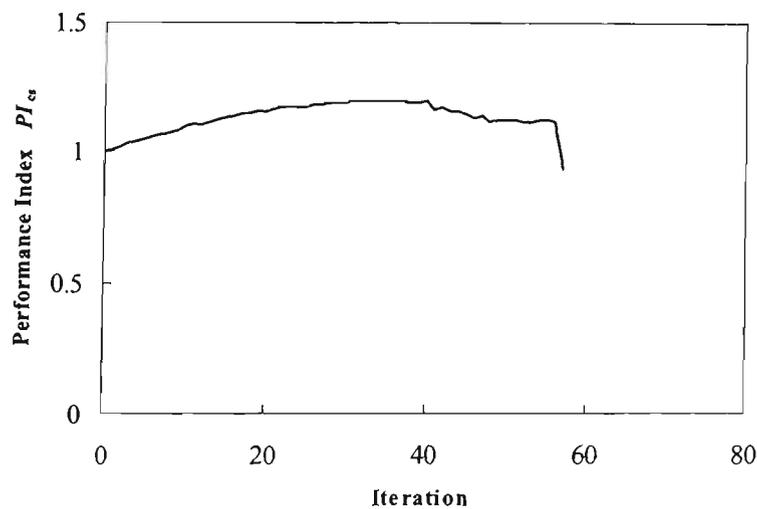
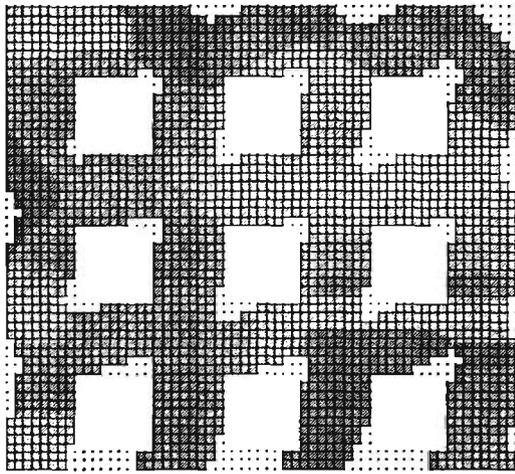
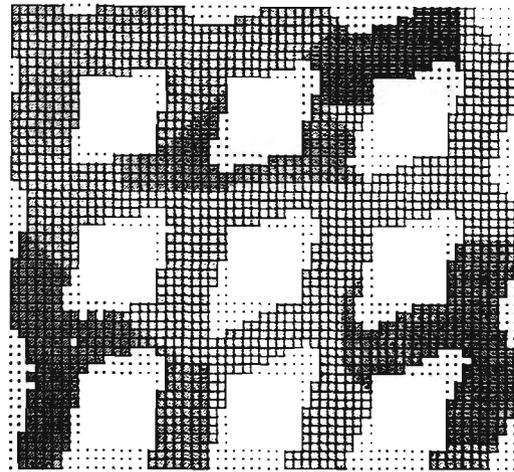


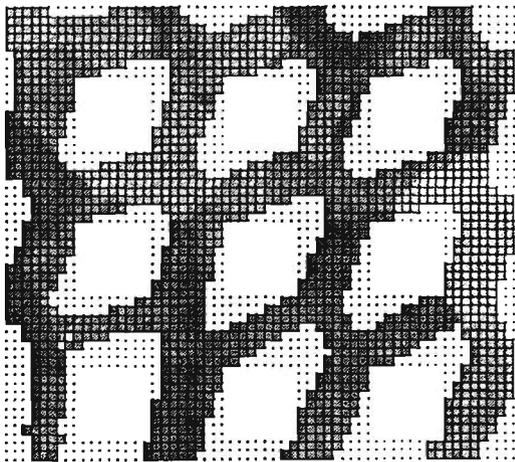
Fig. 6.37 Performance index history of shearwall with openings



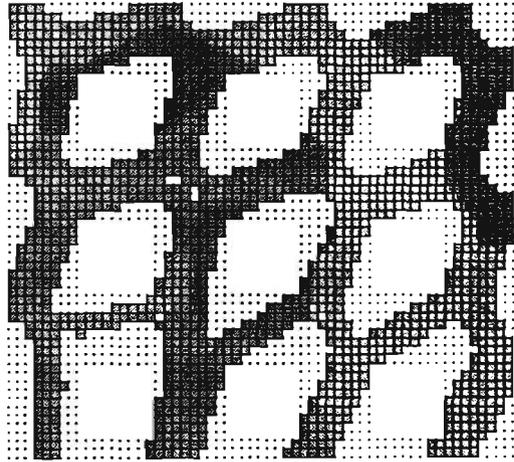
(a) Topology at iteration 10



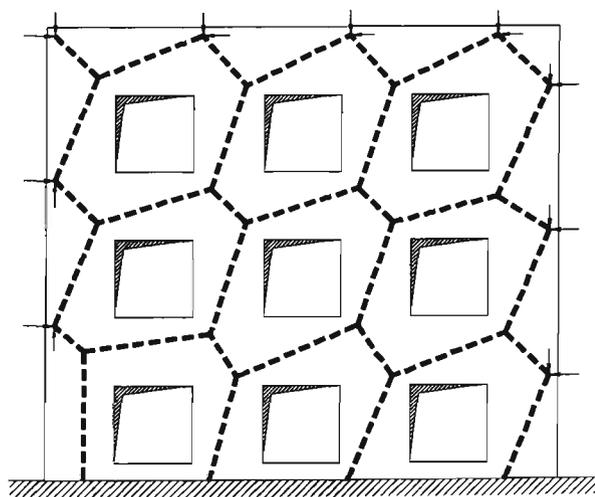
(b) Topology at iteration 20



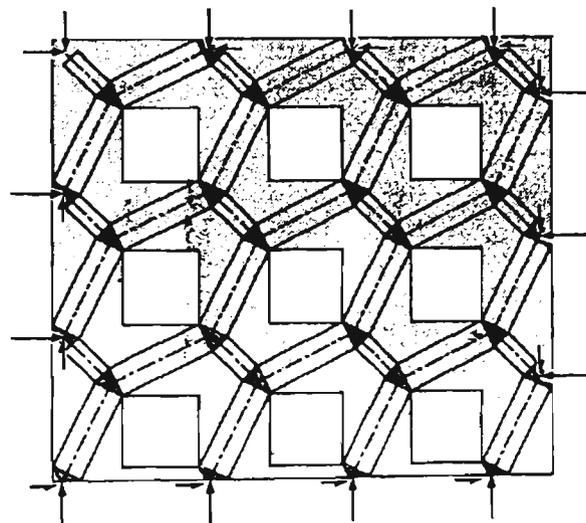
(c) Topology at iteration 30



(b) Optimum at iteration 35



(e) Optimal strut model



(f) Strut model by Marti (1985)

Fig. 6.38 Optimization history of strut-and-tie model in shearwall with openings

## 6.10 CONCLUDING REMARKS

In this chapter, the Performance-Based Optimization (PBO) methods formulated on the basis of stiffness performance criteria have been extended and developed for automatically generating optimal strut-and-tie models in reinforced and prestressed concrete structures. In the proposed methods, the development of strut-and-tie models in structural concrete is transformed to the topology optimization problem of continuum structures. The optimal topology of a concrete structure obtained by the PBO method is treated as the optimal strut-and-tie model for the design of the concrete structure. An integrated design optimization procedure has been proposed for optimizing and dimensioning structural concrete with strut-and-tie systems. The proposed methods have been used to develop strut-and-tie models in reinforced concrete members, prestressed concrete beams and low-rise shearwalls. Optimal strut-and-tie models generated by the proposed design optimization procedure have been verified by existing analytical solutions and experimental observations.

Based on the work presented in this chapter, the following conclusions are drawn:

- (1) The PBO method is a rational, efficient and reliable tool for concrete designers for automatically generating optimal strut-and-tie models for the design and detailing of structural concrete, which include reinforced and prestressed concrete structures, especially for complex concrete structures where no previous experience is available.
- (2) The PBO method is also a useful tool for concrete researchers for quantifying the shear transfer mechanism in structural concrete.
- (3) It has been proved to be appropriate to develop strut-and-tie systems in structural concrete based on the linear elastic theory of cracked concrete for system performance criteria (stiffness) and to design the concrete structures based on the theory of plasticity for component performance criteria (strength).

- (4) Performance-based optimization concepts are consistent with performance-based design concepts being adopted in current building codes of practice in many countries. The PBO method for strut-and-tie modeling would significantly improve the performance of structural concrete, and is appropriate to be adopted in concrete model codes, such as the Asian Concrete Model Code (ICCMC 1999) and AS 3600 (1994).
- (5) When openings intercept natural load paths, the load is to be re-routed around the openings where inclined tensile ties join the upper and lower struts. It is important to provide inclined reinforcement at the top and bottom of the opening based on the strut-and-tie model. This inclined reinforcement is efficient for increasing the ultimate load capacity of the deep beam and for crack control.
- (6) The load transfer mechanism in concrete structure relates to its geometry, loading and support condition. Without modification, a strut-and-tie model developed for a specific structure cannot be used for the design of other structures.
- (7) For reinforced concrete beams with  $L/D \geq 3$ , inclined reinforcement bent up from bottom steel bars is most efficient in resisting inclined tensile forces developed in shear spans. It should be noted that a structural concrete member reinforced with inclined steel bars must be designed by using the correct strut-and-tie model developed for the member. Some of the reinforced concrete structures built in the past using inclined reinforcement have shown signs of distress and cracking due to insufficient flexural reinforcement. The reasons for this are that the inclined and flexural reinforcements in these structures were designed by using empirical equations, rules of thumb, guess work and past experience rather than by strut-and-tie modeling. This problem can be overcome by using optimal strut-and-tie models generated by the PBO technique.
- (8) For very slender concrete beams, optimal topologies obtained by the PBO method are continuum-like structures in which strut-and-tie actions are difficult to be

- identified. For such cases, standard truss models with vertical ties or sectional methods can be used to design these concrete beams. The PBO technique is most appropriately used for developing optimal strut-and-tie models in complex non-flexural concrete structures, where standard truss models with vertical ties are not applicable.
- (9) In the structural idealization of corbels, the column connecting with the corbel should be considered together with the corbel in developing the strut-and-tie model.
  - (10) By treating prestressed forces as external loads, prestressed concrete structures can be analyzed, optimized and dimensioned with strut-and-tie systems like reinforced concrete structures.
  - (11) The PBO method can be used to develop strut-and-tie models for the design of low-rise reinforced concrete shearwalls under multiple loading conditions.
  - (12) For concrete structures under multiple loading cases, a member of a strut-and-tie model may be in compression under one loading case and in tensile under other loading case. Therefore, reinforcing steel must be provided to carry tensile forces in tension members under each load case.
  - (13) The strut-and-tie modeling corresponds to a lower-bound limit analysis. It is of significant importance to ensure the detailing of the reinforcement so that the load transfer mechanism predicted by the PBO method can be realized at ultimate. Adequate anchorage of steel reinforcement must be provided.
  - (14) Optimal strut-and-tie models generated by the PBO method indicate only the location of struts, ties and nodes, but not necessarily the exact dimensions. Dimensioning the strut-and-tie model should be based on the limit analysis and strength performance criteria.

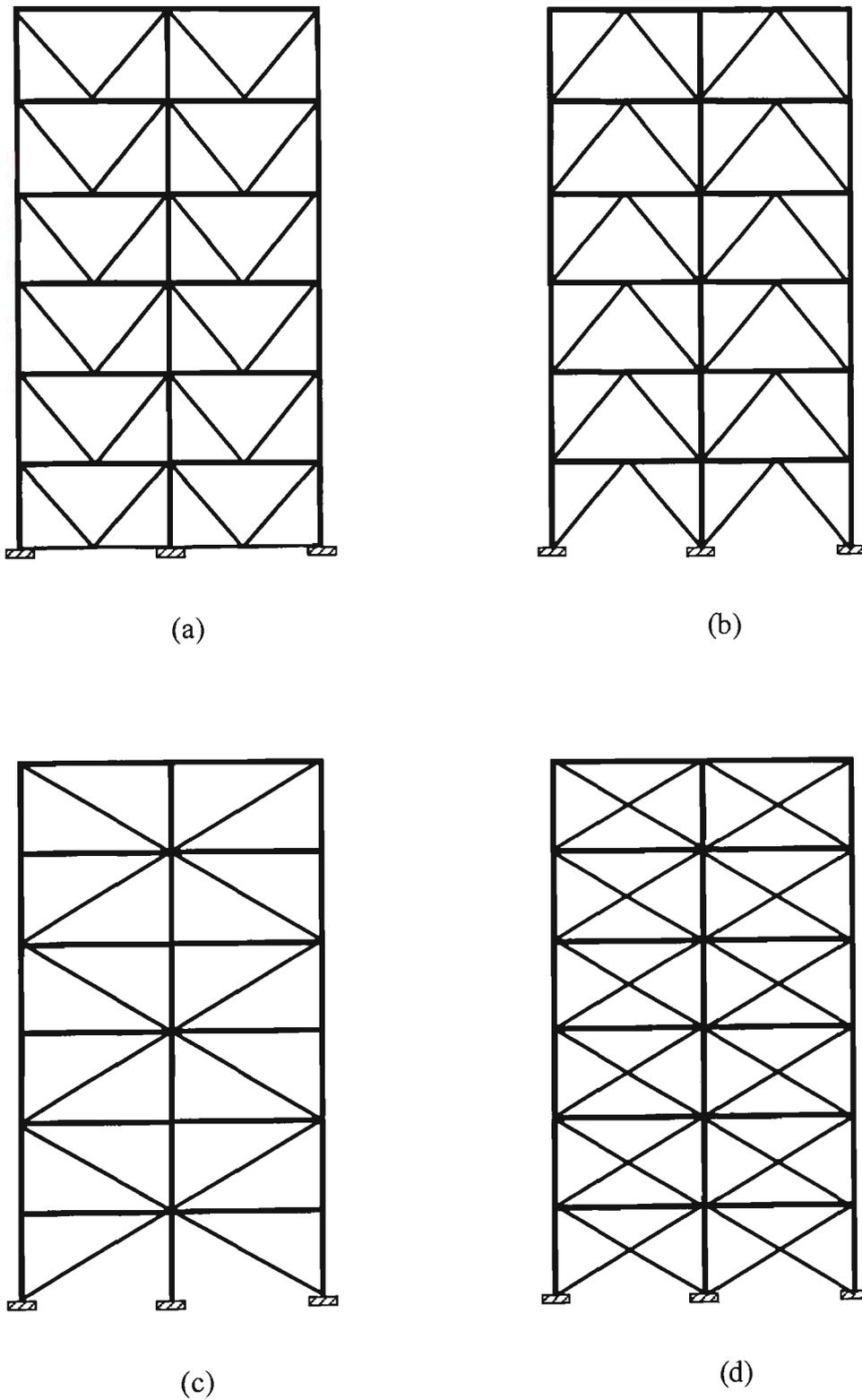
## Chapter 7

# LAYOUT DESIGN OF BRACING SYSTEMS FOR MULTISTORY STEEL FRAMES

### 7.1 INTRODUCTION

The design of a multistory steel building under lateral loads is usually governed by system performance criteria (overall stiffness) rather than by component performance criteria (strength). An important task in the design of a tall steel building for structural designers is to select cost-efficient lateral load resistance systems. Pure rigid frame systems alone are not efficient in resisting lateral loads for tall steel buildings since the shear racking component of deflections induced by the bending of columns and girders will cause the building drift too large (Taranath 1988). Braced frameworks can significantly improve the performance of pure rigid frame actions by eliminating the bending effects of columns and girders. This is achieved by using truss members such as diagonals to brace steel frameworks so those diagonals absorb the shear. The braced framework is an efficient lateral-load resistance system as all members are subjected to axial forces only. In the absence of an efficient optimization technique, the selection of lateral bracing systems for multistory steel frameworks is usually undertaken by the designer based on trial-and-error and previous experience. Traditional design methods for lateral-bracing systems in a multistory steel building are highly iterative and time-consuming. The optimal layout design of bracing systems is a challenging task for structural designers as it involves a large number of possibilities for the arrangement of bracing members. Fig. 7.1 shows the possible layouts of bracing systems for a six-story

steel building framework.



**Fig. 7.1** Possible layouts of bracing systems for multistory steel frameworks

Various researchers have developed stiffness-based sizing techniques for the minimum-weight design of lateral load resistance systems in multistory steel buildings. Baker (1990) presented a sizing technique based on energy methods for lateral load resistance systems in multistory steel buildings. The discretized optimality criteria method proposed by Zhou and Rozvany (1992) is shown to be efficient in sizing large structural systems subject to stress and displacement constraints under multiple loading conditions. Chan et al. (1995) developed an automatic resizing technique for the optimal design of tall steel building frameworks under lateral loads. The optimality criteria method is employed to solve the minimum-weight design problem of a tall steel building framework subject to multiple inter-story drifts and member strength and sizing constraints in accordance with building codes and construction requirements. In these approaches mentioned above, all members of a lateral load resistance system are resized on the basis of uniform strain energy density criteria. Kim et al. (1998) presented a method for the design of tall steel buildings where steel frameworks are designed for strength criteria, and only bracing members are resized for stiffness performance criteria. They suggested that it is most efficient to increase the lateral stiffness of lower-stories in a tall building to improve the performance of lateral load resistance systems. However, all of these sizing techniques only work for lateral bracing systems with fixed topologies. The efficiency of a resized structural system in resisting lateral loads is obviously limited by the chosen topology of bracing systems.

Continuum shape and topology optimization has received extensive developments in the last few decades. Review papers on these topics have been presented by Haftka and Grandhi (1986) and Rozvany et al. (1995). Several methods have been developed for the topology design of continuum structures, such as the homogenization-based optimization method (Bendsøe and Kikuchi 1988; Suzuki and Kikuchi 1991; Tenek and Hagiwara 1993; Bendsøe et al. 1995), density function approaches (Meljnek and Schirmacher 1993; Yang and Chuang 1994), hard kill optimization methods (Rodriguez and seireg 1985; Atrek 1989; Xie and Steven 1993,1997), and the soft kill option method (Mattheck 1998). However, these continuum topology optimization methods focus mainly on theoretical aspects rather than practical applications. In

addition, further work on developing performance-based optimization criteria for obtaining globally optimal designs in continuum topology optimization is needed.

The homogenization-based optimization method has been used by Diaz and Kikuchi (1992) to find optimal reinforcement layouts, which improve the natural frequency of a plane stress continuum structure. Walther and Mattheck (1993) used the soft kill option method to generate efficient frameworks for supporting floor systems in construction engineering. The layout design of bracing systems for multistory steel building frames under one lateral load case has been attempted by Mijar et al. (1998) using a topology optimization method based on classical Voigt-Resuss mixing rules. In this method, the objective is to minimize the compliance of a steel frame braced by a continuum design domain under the given loading and boundary condition. The constraint is imposed on an amount of solid material used for the bracing system. Obviously, the bracing system produced by this method largely depends on the material volume constraint, which is arbitrarily specified by the designer.

In this chapter, the Performance-Based Optimization (PBO) method formulated on the basis of system performance criteria is extended and proposed for the topology design of bracing systems for multistory steel building frameworks under multiple lateral loading conditions. In the PBO methods described in previous chapters, either plane stress elements or plate elements are used to model the structure for optimization. However, in dealing with practical design problems, different element types have to be used to model a structure. The capacity of the PBO method is extended to include both beam and plane stress elements in one model. In the proposed design optimization procedure, unbraced frameworks are firstly designed for component performance criteria by selecting standard steel sections from databases. The optimal topology of a bracing system is generated by gradually removing underutilized plane stress elements from a continuum design domain that braces the framework. Two design examples are provided to demonstrate the effectiveness and validity of the PBO method for layout design of bracing systems. Results obtained by the present study are compared with existing solutions. Some of the results have been reported by Liang et al. (2000b).

## 7.2 TOPOLOGY DESIGN PROBLEM FORMULATION

Bracing systems are used to reinforce steel building frameworks so that lateral drifts are maintained within acceptable performance levels. In the proposed PBO method, a continuum design domain under plane stress conditions is used to stiffen a multistory steel building framework. The continuum design domain is modeled using plane stress finite elements. The framework itself with a fixed topology is treated as a non-design domain, which is modeled by beam elements. Beam elements are not removed during the optimization process. A steel framework fully braced by a continuum design domain is used as a starting point for deriving the optimal bracing system for the framework. This is achieved by removing underutilized elements from the continuum design domain. Therefore, the performance objective of the layout design for bracing systems is to minimize the weight of the continuum design domain while maintaining the overall stiffness constraint of the braced framework within an acceptable limit. The performance objective can be expressed as follows:

$$\text{minimize } W = \sum_{e=1}^n w_e \quad (7.1)$$

$$\text{subject to } C \leq C^* \quad (7.2)$$

where  $W$  is the total weight of the continuum design domain,  $w_e$  is the weight of the  $e$ th element in the continuum design domain,  $C$  is the absolute value of the mean compliance of a braced framework,  $C^*$  is the prescribed limit of  $C$  and  $n$  is the total number of elements in the discretized continuum design domain. It is noted that the mean compliance of a structure is usually used as an inverse measure of its overall stiffness. As a result of this, the maximization of the overall stiffness of a structure is equivalent to minimizing its mean compliance.

### 7.3 ELEMENT REMOVAL CRITERIA

Some portions of a continuum reinforcing system are not effective in resisting lateral loads, and are thus removed from the continuum design domain to improve the performance of the lateral load resistance system. Element removal criteria are used to identify these underutilized portions in the optimization algorithm, and can be derived by undertaking a sensitivity analysis. The sensitivity analysis in the proposed method is to investigate the effects of element removal on the change of the mean compliance of a braced framework.

In finite element analysis, the equilibrium equation of a braced framework can be written by

$$[K]\{u\} = \{P\} \quad (7.3)$$

where  $[K]$  is the stiffness matrix of a braced framework,  $\{u\}$  is nodal displacement vector of the braced framework, and  $\{P\}$  is nodal load vector that is not changed. When the  $e$ th element is removed from a discretized continuum design domain, the stiffness and displacements will be changed accordingly, and Eq. (7.3) can be rewritten as

$$[K + \Delta K]\{u + \Delta u\} = \{P\} \quad (7.4)$$

in which  $\Delta K$  is the changes of the stiffness matrix and  $\{\Delta u\}$  is the change of nodal displacement vector of the braced framework. When only the  $e$ th element is removed from the continuum design domain, the change of the stiffness matrix can be derived as

$$[\Delta K] = [K_r] - [K] = -[k_e] \quad (7.5)$$

in which  $[K_r]$  is the stiffness matrix of the resulting structure and  $[k_e]$  is the stiffness matrix of the  $e$ th element in the continuum design domain.

The change of displacement vector can be obtained approximately by subtracting Eq. (7.3) from Eq. (7.4) and neglecting higher order terms as

$$\{\Delta u\} = -[K]^{-1}[\Delta K]\{u\} \quad (7.6)$$

The mean compliance or strain energy of a braced framework is calculated by

$$C = \frac{1}{2}\{P\}^T\{u\} \quad (7.7)$$

The change of the strain energy of the braced framework due to the elimination of the  $e$ th element can approximately be expressed by

$$\Delta C = \frac{1}{2}\{P\}^T\{\Delta u\} = -\frac{1}{2}\{P\}^T[K^{-1}][\Delta K]\{u\} = -\frac{1}{2}\{u\}^T[\Delta K]\{u\} = \frac{1}{2}\{u_e\}^T[k_e]\{u_e\} \quad (7.8)$$

where  $\{u_e\}$  is the displacement vector of the  $e$ th element in the continuum design domain. It is seen from Eq. (7.8) that the change of the strain energy of a braced framework due to the removal of the  $e$ th element can be approximately evaluated by the strain energy of the  $e$ th element. Therefore, the element strain energy can be used as a measure of the efficiency of an element in contribution to the overall stiffness of a lateral bracing system and is denoted as

$$c_e = \frac{1}{2}\{u_e\}^T[k_e]\{u_e\} \quad (7.9)$$

To achieve the performance objective, elements with the lowest strain energy should be gradually removed from the continuum design domain. If a continuum design domain is divided into different size elements, the lowest strain energy density of elements should be used as element removal criteria. The strain energy density of the  $e$ th element is calculated by  $\zeta_e = |c_e|/v_e$ .

Multistory buildings are often subjected to reverse wind loads, which can be treated as multiple loading cases. For a braced framework subject to multiple loading cases, a logical AND scheme is used in the proposed method to take account of the effect of different loading cases, as described in Chapter 5. In the logical AND scheme, an element is removed from the design domain only if its strain energy density is the lowest for all load cases.

Elements with the lowest strain energy density are counted by a loop until they make up the specified amount, which is the element removal ratio times the number of elements in the initial design domain. The element removal ratio ( $R$ ) for each iteration is defined by the ratio of the number of elements to be removed to the total number of elements in the initial continuum design domain.

#### **7.4 PERFORMANCE-BASED OPTIMALITY CRITERIA**

In the proposed method, braced systems are gradually modified by removing elements with the lowest strain energy density from a continuum design domain that braces a multistory steel framework. To obtain the optimal bracing system, the performance of resulting bracing system at each iteration must be evaluated using some sort of performance indicators. Performance-based optimization criteria in terms of performance indices have been proposed in previous chapters using the scaling design concept for selecting the optimum from the optimization history. These performance indices are also useful tools for ranking the performance of structural topologies and shapes generated by different optimization methods.

For the layout optimization of bracing systems, a continuum design domain is structurally connected to a steel framework. The overall stiffness of a braced framework is not a linear function of the thickness of the continuum design domain. As a result of this, the thickness of elements cannot be linearly scaled to keep the mean compliance constraint active at each iteration. However, it is known that the best structure is the one

that has the maximum stiffness at minimum weight, as pointed out by Hemp (1973). Therefore, the performance index in the form of Eq. (5.14) can still be used to evaluate the performance of a bracing system for a steel building framework under the mean compliance constraint, and is given by

$$PI_{cs} = \frac{C_0 W_0}{C_i W_i} \quad (7.10)$$

where  $C_0$  is the absolute values of the mean compliance of the initial braced framework,  $C_i$  is the absolute values of the mean compliance of the current braced framework at the  $i$ th iteration,  $W_0$  is the weight of the initial continuum design domain and  $W_i$  is the weight of the current continuum design domain. It is not necessary to take the weight of the steel framework into account in the calculation of the performance index since beam elements are not removed in the optimization process.

The performance index can indicate the efficiency of a lateral bracing system in terms of the material usage and the overall stiffness. The performance of a bracing system is improved when elements with the lowest strain energy density are gradually removed from the continuum design domain. By systematically eliminating underutilized elements, the most uniform distribution of strain energy density within a continuum design domain can be achieved. However, in some cases the uniformity of element strain energy density may not be achieved even if the mean compliance constraint is active. As a result of this, the uniform strain energy density condition is not used in optimization algorithms as a termination criterion. In the proposed method, the performance-based optimization criterion is maximizing the performance index of a braced framework, which is written as

$$\text{maximize } PI_{cs} = \frac{C_0 W_0}{C_i W_i} \quad (7.11)$$

The performance index is used in the optimization algorithm to monitor the

optimization process, from which the optimal topology can be identified.

## 7.5 DESIGN OPTIMIZATION PROCEDURE

The design optimization process of bracing systems is divided into two main stages. In the first stage, after analyzing the unbraced steel framework with finite elements, the members of a steel framework are designed for strength performance criteria by selecting commercial standard steel sections from databases. In the second stage, a repeated finite element analysis and topology optimization cycle is undertaken for the framework braced by a continuum design domain until the termination criterion is satisfied. The main steps of the design optimization procedure are illustrated in Fig. 7.2, and summarized as follows:

- (1) Model the unbraced steel framework with beam elements. The loads, support conditions and material properties of the assumed steel sections are specified.
- (2) Perform a linear elastic finite element analysis on the unbraced steel framework.
- (3) Size the members of the unbraced framework for strength performance criteria by selecting commercial standard steel sections from databases.
- (4) Model the continuum design domain with plane stress elements. The discretization of the continuum design domain must be consistent with that of the steel framework.
- (5) Perform a linear elastic finite element analysis on the braced framework.
- (6) Evaluate the performance of resulting bracing system using Eq. (7.10). For a braced framework under multiple loading cases, the strain energy density of the braced framework under the most critical loading case must be used in Eq. (7.10).

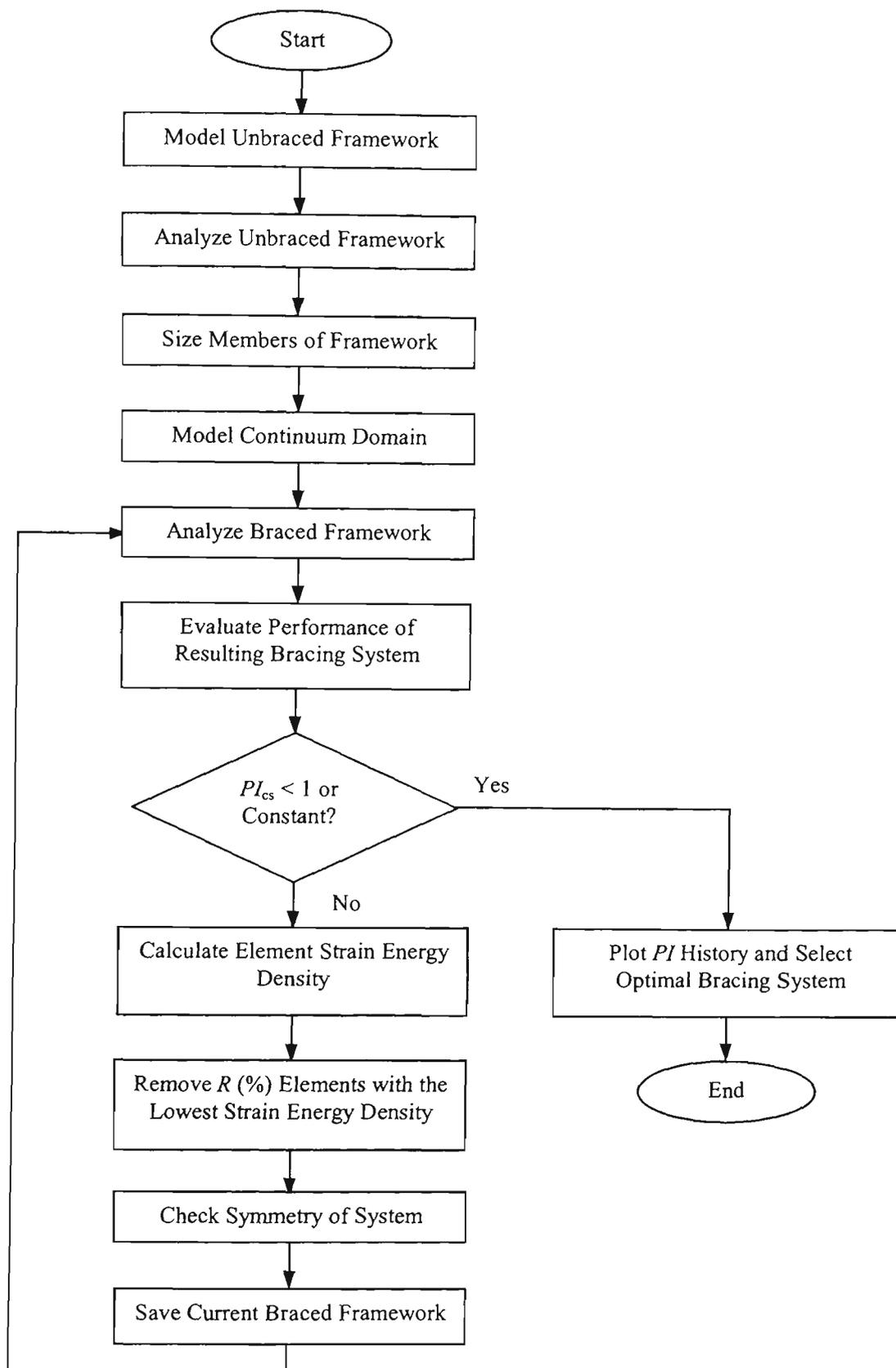


Fig. 7.2 Flowchart of design optimization procedure for bracing systems

- (7) Calculate the strain energy density of elements for each loading case.
- (8) Remove  $R$  (%) elements with the lowest strain energy density from the continuum design domain.
- (9) Check the symmetry of the bracing system under an initially symmetrical condition.
- (10) Save information for the current braced framework.
- (11) Repeat step (5) to (11) until the performance index is less than unity.
- (12) Plot the performance index history and select the optimal bracing system.

If the performance index of a bracing system is less than unity, its performance of resisting lateral loads is lower than that of the initial braced steel framework. Therefore, the iterative optimization process can be terminated when the performance index is less than unity. This termination criterion ensures that the optimal bracing system is included in the optimization history. It is desirable that the mean compliance constraint in terms of the drifts of the building is active at the optimum. However, it may not always be the case because the thickness of the continuum design domain significantly influences the efficiency of resulting systems. To deal with this problem, shape and sizing optimization techniques can be used to further optimize the bracing system until lateral drifts reach prescribed limits. Another way to handle this is to uniformly change the element thickness that leads to the satisfaction of the required system performance level. This can also be done by uniformly changing the thickness of the continuum design domain to keep the mean compliance constraint active in the optimization process. However, the derivatives of the mean compliance constraint with respect to the thickness have to be calculated.

It is noted that continuum topology optimization is a generalized layout design method

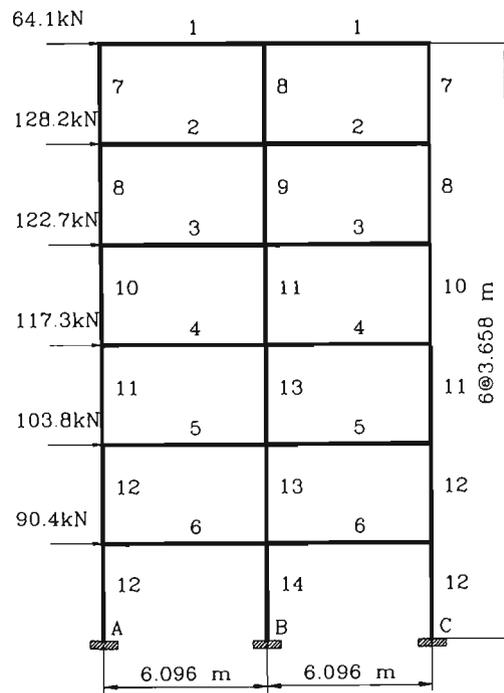
for structures. It may or may not result in truss-like optimal structures. For practical purposes, some modifications to the optimal bracing system obtained may be necessary. It is also possible to select a bracing system that meets the construction and architectural requirements from the optimization history as the final design proposal. However, the performance of the selected bracing system must be within an acceptable level. The performance index is a useful tool in the selection of bracing systems for the design of multistory steel building frameworks.

## 7.6 ILLUSTRATIVE EXAMPLES

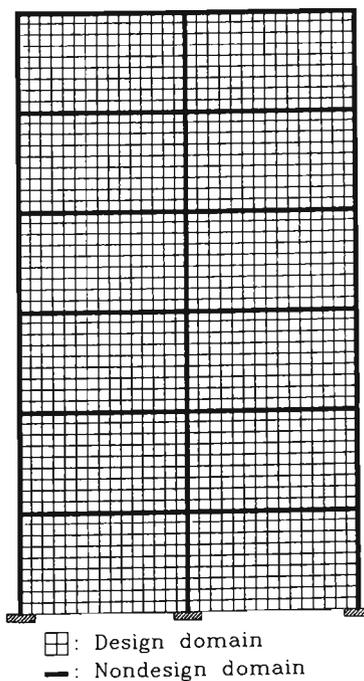
### 7.6.1 Bracing System for Six-Story Steel Framework

This example is to demonstrate the effectiveness of the proposed design optimization procedure for producing the best layout of a bracing system for a six-story steel building framework under multiple lateral loading cases. The result obtained is compared with the solution given by Mijar et al. (1998). A two-bay, six-story plane steel building framework shown in Fig. 7.3(a) is to be designed to control the lateral drifts. This unbraced framework was initially designed by Huang (1995) using standard steel sections under stress constraints according to the American Institute of Steel Construction design code. In Huang's design, the uniformly distributed load applied to floor beams was 14.59 kN/m and the wind loads of 40.05 kN were applied as horizontal point loads at each floor level. The wide flange sections used for 14 member groups shown in Fig. 7.3(a) are listed as W 8 × 21, W 8 × 28, W 10 × 26, W 12 × 26, W 14 × 26, W 14 × 19, W 10 × 17, W 8 × 10, W 12 × 19, W 12 × 14, W 14 × 22, W 16 × 26, W 16 × 31 and W 24 × 62. The lateral wind loads shown in Fig. 7.3(a) were used by Mijar et al. (1998) to find the bracing system for this framework. Under this lateral loading, stresses in the members of the unbraced framework may exceed the allowable stress.

Since wind loads are often reversible, two lateral loading cases are considered in the



(a) Geometry and loading



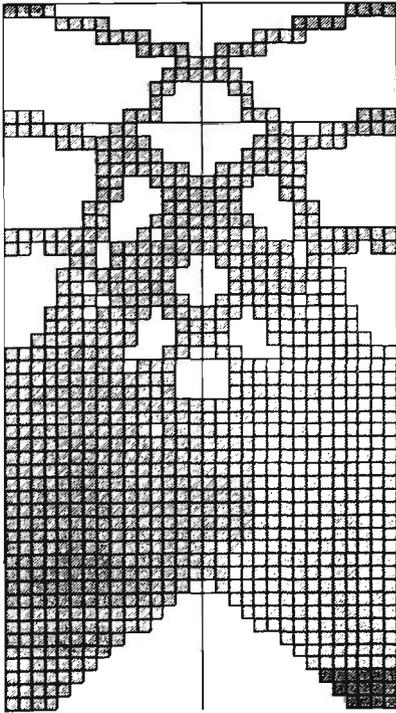
(b) Design and non-design domains

**Fig. 7.3** Two-bay, six-story steel building framework

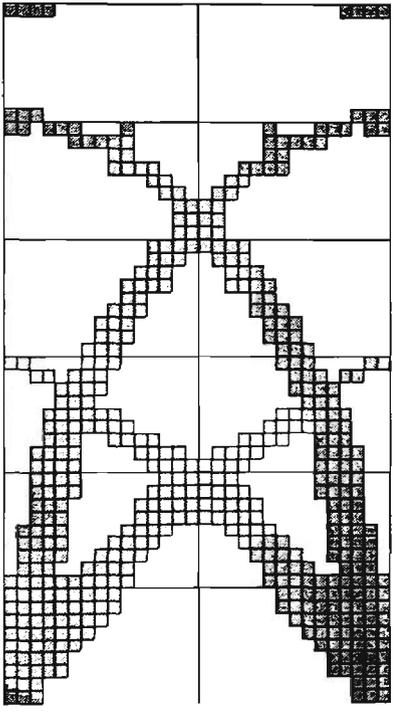
present study. Lateral bracing systems in multistory steel buildings are mainly designed to resist lateral loads. Floor loads that are carried by beams and columns have a negligible effect on the layout of bracing systems so that they are not considered in the analysis. The steel framework itself is modeled using 342 linear beam elements with all moment connections. The unbraced framework is treated as a non-design domain in which beam elements are not removed during the optimization process. The continuum design domain is discretized by 1620 four-node plane stress elements as shown in Fig. 7.3(b). The supports of the framework at points A, B and C are fixed. The Young's modulus of material  $E = 200$  GPa, Poisson's ratio  $\nu = 0.3$  and a uniform thickness  $t = 0.0254$  m are used for the continuum design domain. The maximum lateral displacement of the unbraced steel framework is 0.56 m. The element removal ratio  $R = 1\%$  is used in the optimization process.

The optimization history of the bracing system for the six-story steel framework is shown in Fig. 7.4. It can be seen that the bracing system gradually evolves towards a truss-like structure when elements with the lowest strain energy density are removed from the continuum design domain. All topologies obtained are symmetrical about the vertical axis of the frame as expected under the reversible wind loading conditions. The topology shown in Fig. 7.4(b) is similar to that shown in Fig. 7.4(c).

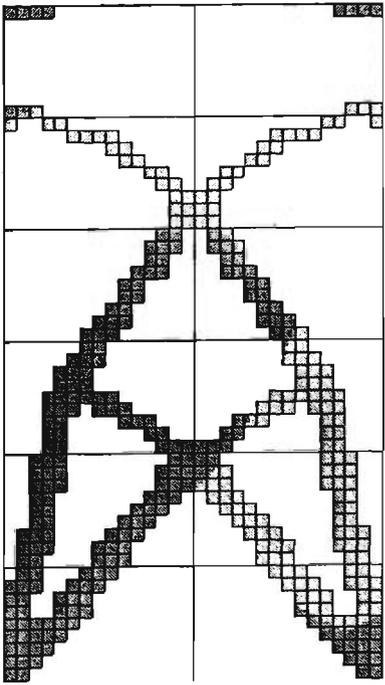
The optimized topology of the bracing system for this steel building frame given by Mijar et al. (1998) is regenerated here, as shown in Fig. 7.4(d). This topology was obtained by minimizing the mean compliance of the braced framework under a volume constraint that allowed the solid material to occupy up to 30% of the initial continuum design domain. The maximum lateral displacement of the optimized bracing system shown in Fig. 7.4(d) is 0.07 m. The material volume of the bracing system shown in Fig. 7.4(c) is 22% of the initial continuum design domain, but its maximum lateral displacement is only 0.024 m. By using Eq. (7.10), the performance index of the bracing system shown in Fig. 7.4(c) is 1.15 whilst it is only 0.32 for the topology presented in Fig. 7.4(d). This indicates that the layout of bracing systems for multistory steel frames significantly affects the structural performance of lateral resistance systems.



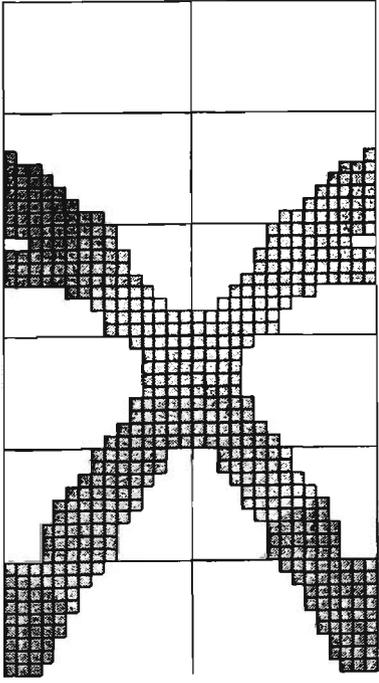
(a) Topology with  $V = 63\%V_0$



(b) Topology with  $V = 30\%V_0$



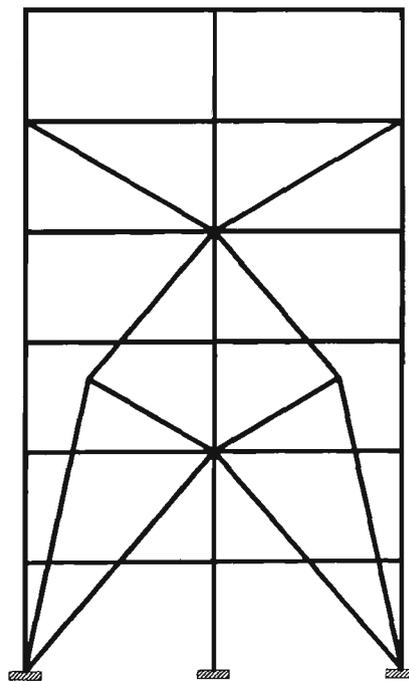
(c) Topology with  $V = 22\%V_0$



(d) Topology with  $V = 30\%V_0$  by Mijar et al. (1998)

Fig. 7.4 Optimization history of bracing system for the six-story steel framework

It is possible to achieve minimum-weight designs for bracing systems by using topology optimization techniques while lateral drifts are maintained within acceptable limits. The topology of the bracing system shown in Fig. 7.4(c) is interpreted as the layout arrangement of bracing members illustrated in Fig. 7.5. This bracing system can be constructed by using available standard steel sections from databases.



**Fig. 7.5** Layout of bracing system for the six-story steel framework

### 7.6.2 Bracing System for the 12-Story Steel Framework

The PBO method is used to generate an optimal bracing system for a 3-bay, 12-story tall steel building framework shown in Fig. 7.6. The tall steel framework is subjected to two lateral wind-loading cases, i.e. one from the left and the other from the right. Gravity loads are not considered in the analysis. The framework is fixed at points A, B, C and D. All beams and columns are rigidly connected. The framework itself is modeled using

684 linear beam elements. The Young's modulus  $E = 200$  GPa, shear modulus  $G = 7690$  MPa, and the material density  $\rho = 7850$  kg / m<sup>3</sup> are used for steel sections. A linear elastic finite element analysis on the unbraced framework is performed. The BHP hot rolled standard steel sections are selected from databases to size the members of the framework based on strength performance criteria. For practical purposes, beams are grouped together as having a common section for each floor whilst columns are grouped for every two stories. Sized members are summarized in Table 7.1.

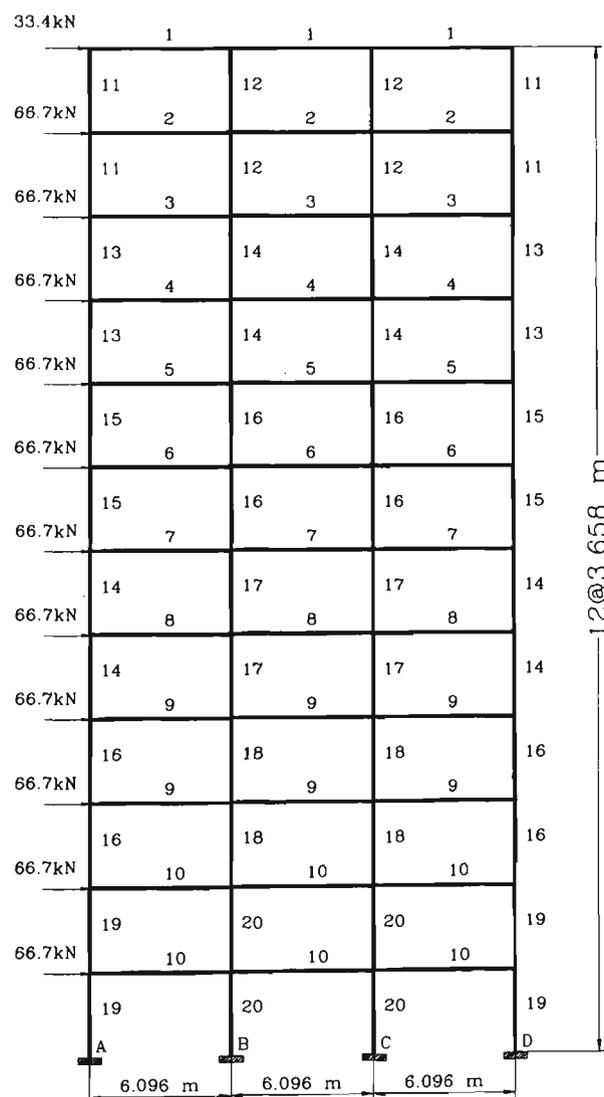


Fig. 7.6 3-bay, 12-story steel building framework

**Table 7.1 Member Sizes of the 12-Story Steel Building Framework**

| Member No.<br>(1) | Section<br>(2) | Member No.<br>(3) | Section<br>(4) |
|-------------------|----------------|-------------------|----------------|
| 1                 | 150 UB 18.0    | 11                | 150 UC 23.7    |
| 2                 | 180 UB 18.1    | 12                | 150 UC 37.2    |
| 3                 | 200 UB 29.8    | 13                | 200 UC 46.2    |
| 4                 | 250 UB 27.3    | 14                | 200 UC 59.5    |
| 5                 | 310 UB 40.4    | 15                | 200 UC 52.2    |
| 6                 | 360 UB 50.7    | 16                | 250 UC 72.9    |
| 7                 | 360 UB 56.7    | 17                | 250 UC 89.5    |
| 8                 | 410 UB 53.7    | 18                | 310 UC 96.8    |
| 9                 | 460 UB 67.1    | 19                | 310 UC 118     |
| 10                | 460 UB 74.6    | 20                | 310 UC 137     |

A continuum design domain with a uniform thickness of  $t = 0.025$  m is used to brace the framework. The continuum design domain is divided into a  $45 \times 108$  mesh using four-node plane stress elements. The discretization of the continuum design domain is consistent with that of the steel framework. The Young's modulus  $E = 200$  GPa and Poisson's ratio  $\nu = 0.3$  are used for the continuum design domain. The  $R = 2\%$  is adopted in the optimization process.

The maximum lateral displacement of the unbraced framework is 0.618 m, which exceeds the drift limit of  $H / 400$  ( $H$  is the total height of the framework). The overall behavior of a braced steel building framework is like that of a cantilever structure where beams and columns can be treated as stiffeners. The performance index history of the bracing system obtained by the proposed method is shown in Fig. 7.7. By eliminating elements with the lowest strain energy density from the continuum design domain, the performance index of the bracing system is gradually increased from unity to the maximum value in the optimization process. After reaching the peak, the performance index decreases if further elements are removed from the continuum design domain. The maximum performance index is 1.51.

Fig. 7.8 shows the topology obtained at iteration 20 and the optimal topology obtained

at iteration 31. The optimal layout of the bracing system exhibits a large-scale discrete structure. The optimal topology provides very useful information for the structural designer on which member of the framework should be stiffened by resizing. Exterior columns from the ground level up to the fifth level are needed to be resized. The optimal topology of the bracing system for this 12-story steel building framework can be interpreted as the bracing layout illustrated in Fig. 7.9, where columns that need to be resized are not shown. Since the mean compliance constraint in terms of the lateral drift does not reach the actual limit at the optimum, sizing techniques can be employed to further optimize the design using available standard steel sections.

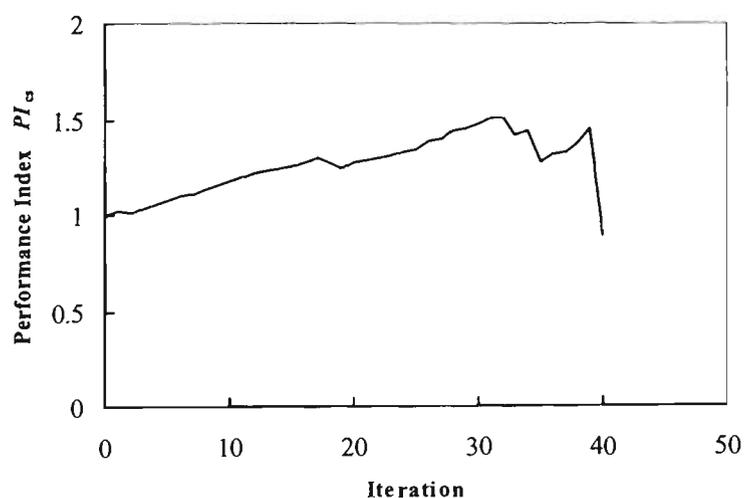
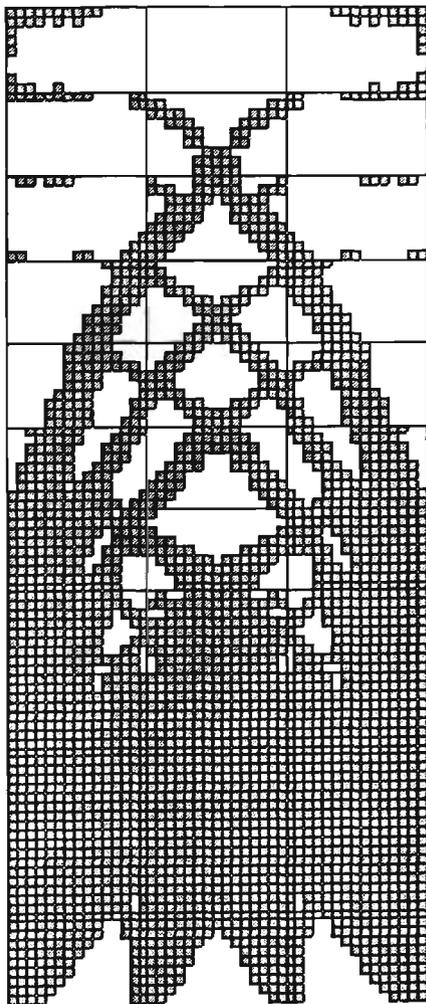
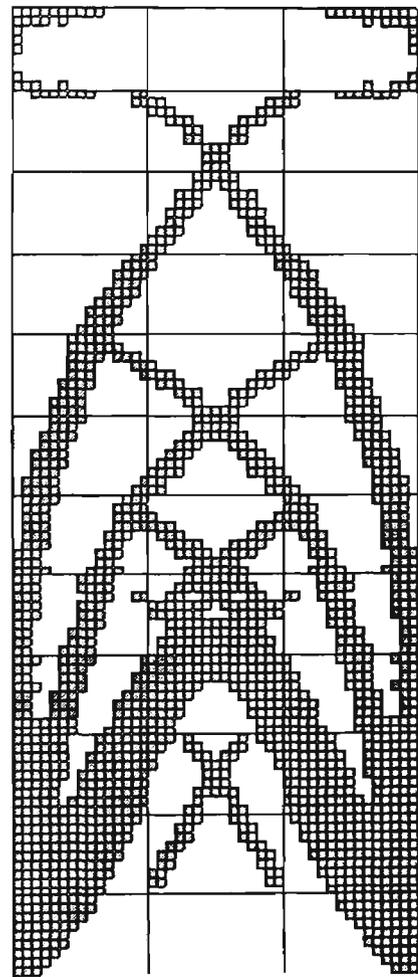


Fig. 7.7 Performance index history of bracing system

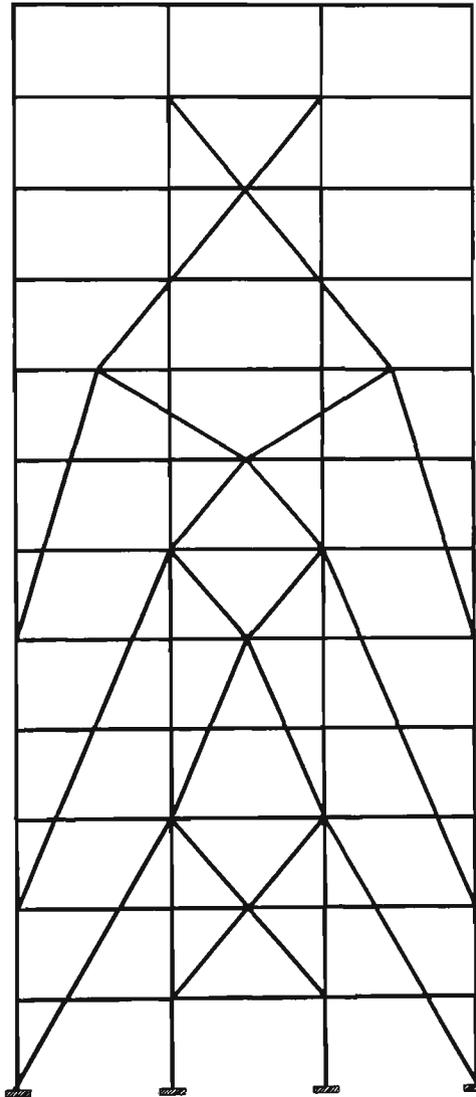


(a) Topology at iteration 20



(b) Optimal topology at iteration 31

**Fig. 7.8** Optimization history of bracing system for the 12-story steel framework



**Fig. 7.9** Layout of bracing system for the 12-story steel framework

## 7.7 CONCLUDING REMARKS

In this chapter, the Performance-Based Optimization method tailored for the minimum-weight topology design of bracing systems for multistory steel building frameworks under multiple lateral loading conditions has been presented. The proposed method allows for an unbraced steel building framework to be initially sized for strength performance criteria by selecting commercially available standard steel sections from databases. Bracing systems for multistory steel frameworks are developed on the basis of system performance criteria. The optimal topology of the bracing system is obtained by systematically removing elements with the lowest strain energy density from a continuum design domain that is used to stiffen the framework while the performance-based optimization criterion is satisfied.

The proposed performance index is a useful tool for the structural designer in assisting the selection of the best topology for lateral bracing systems when considering the structural performance, aesthetic and construction requirements. Examples presented have demonstrated that the design method can produce efficient bracing systems, which provide the structural designer with useful information on bracing and stiffening multistory steel building frameworks.

The performance-based topology optimization method proposed has attractive features such as clear in concept and simple in mathematical formulation compared to other continuum topology optimization methods. The PBO method is suitable for use in engineering practice in the conceptual layout design of bracing systems for multistory steel building frameworks under lateral loads.

# Chapter 8

## CONCLUSIONS

### 8.1 SUMMARY

The systematic development of the Performance-Based Optimization (PBO) method for topology and shape design of continuum structures subject to stress, displacement and overall stiffness constraints have been presented in this thesis. The performance-based design concept has been incorporated into continuum topology and shape optimization, which is treated as the problem of improving the performance of continuum design domains. Based on the design sensitivity analysis, the element effective stress level, virtual strain energy density and strain energy density have been proposed as element removal criteria. A set of performance indices has been developed for ranking the performance of structural topologies and shapes produced by different structural optimization methods. These performance indices are used to monitor the optimization process and as termination criteria in optimization algorithms. Performance-based optimality criteria have been developed and proposed as the maximization of performance indices in the optimization process.

The PBO method has been further developed for automatically generating optimal strut-and-tie models for the design and detailing of structural concrete, which includes

reinforced and prestressed concrete structures. Developing strut-and-tie models in structural concrete is transformed into the topology optimization problem of continuum structures. It has been proposed to develop strut-and-tie models in structural concrete based on the linear elastic theory of cracked concrete for system performance criteria (stiffness) and to design concrete structures based on component performance criteria (strength). Optimal strut-and-tie models in reinforced concrete members, prestressed concrete beams and low-rise shearwalls have been investigated by the PBO technique. The proposed PBO method for strut-and-tie modeling has been verified by analytical solutions and experimental evidence.

The PBO method has also been further developed for optimal topology design of bracing systems for multistory steel building frameworks. The proposed method allows for an unbraced steel building framework to be initially designed for strength performance by selecting standard steel sections from databases. The optimal bracing system is generated by gradually removing elements with the lowest strain energy density from a continuum design domain, which is used to brace the framework. The PBO method can be used in the conceptual layout design of bracing systems for multistory steel frameworks under multiple lateral loading conditions.

It has been demonstrated that the PBO method is a rational, efficient and reliable design tool for practicing engineers in the topology and shape design of continuum structures. It offers not only concrete designers an automated tool for generating optimal strut-and-tie models in structural concrete, but also steel designers an efficient tool for conceptual layout design of bracing systems. Performance indices and performance-based optimality criteria developed in this thesis can overcome problems in many continuum topology optimization methods, and can be incorporated in any continuum topology optimization methods to guarantee success in obtaining globally optimal designs.

## 8.2 ACHIEVEMENTS

This thesis has made many significant contributions to the fields of structural optimization, structural concrete and steel structures. These significant achievements are summarized as follows:

- (1) Developed the Performance-Based Optimization (PBO) method for fully stressed topology design of continuum structures. The PBO method can produce fully stressed topologies at minimum weight.
- (2) Derived a strength performance index for evaluating the performance of structural topologies with stress constraints. The performance index is used in the PBO method to monitor the optimization history.
- (3) Proposed performance-based optimality criteria based on element effective stress levels for identifying the global optimum in the optimization process. Performance-based optimality criteria can be incorporated into any stress-based continuum topology optimization methods, such as the soft kill option (Mattheck 1998), the evolutionary structural optimization (Xie and Steven 1993), and the rule-based optimization method (Seireg and Rodriguez 1997).
- (4) Developed the PBO method for topology and shape design of continuum structures for displacement performance.
- (5) Derived two performance indices for evaluating the performance of topologies and shapes of plane stress structures and of bending plates subject to displacement constraints.
- (6) Proposed performance-based optimality criteria for identifying the global optimum of structures with displacement constraints in the optimization process. Performance-based optimality criteria can be incorporated into any continuum topology optimization methods for structures with displacement constraints, such

as the hard kill optimization method (Atrek 1989), the evolutionary structural optimization method (Chu et al. 1996), and the density function approach (Yang 1997).

- (7) Developed the PBO method for topology design of continuum structures for overall stiffness performance.
- (8) Derived two performance indices for evaluating the performance of structural topologies of plane stress structures and of bending plates with overall stiffness constraint.
- (9) Proposed performance-based optimality criteria for identifying the optimum of continuum structures with overall stiffness constraint in the optimization process. Performance-based optimality criteria can be incorporated into any continuum topology optimization methods for structures with overall stiffness consideration, such as the HBO method (Bendsøe 1995; Tenek and Hagiwara 1993; Youn and Park 1997; Krog and Olhoff 1999), the density function approach (Yang 1997), and the topology optimization approach (Swan and Kosaka 1997).
- (10) Developed the PBO method for structures with displacement and overall stiffness constraints into a rational, efficient and reliable tool for automatically generating optimal strut-and-tie models for the design and detailing of structural concrete. The PBO method is also a useful tool for concrete researchers for quantifying the shear behavior of structural concrete. The PBO method for strut-and-tie modeling of structural concrete would significantly improve the performance of concrete structures, and thus is suitable for inclusion in concrete model codes, such as the Asian Concrete Model Code (ICCMC 1999) and AS 3600 (1994).
- (11) Proposed the PBO method for optimal topology design of bracing systems for multistory steel building frameworks with overall stiffness constraint under multiple lateral loading conditions.

### 8.3 FURTHER RESEARCH

The proposed PBO method is an efficient tool for topology and shape design of continuum structures. Further research is still needed to make it an integrated, general and user-friendly design tool for practicing engineers. The recommendations for further research are summarized as follows:

- (1) Further research should focus on the development of the PBO method for topology, shape and sizing design of continuum structures subject to the combination of stress, displacement and overall stiffness constraints. Optimal topology of a structural system is firstly generated on the basis of overall stiffness performance criteria. The optimal topology obtained is further modified by shape and sizing optimization techniques to meet stress and displacement requirements.
- (2) Extend the PBO method to dynamic and 3D problems.
- (3) Develop a technique that can eliminate the effect of finite element meshes on the optimal topologies and shapes.
- (4) Incorporate smoothing techniques into the PBO method for post-processing.
- (5) Develop approximate techniques that can significantly reduce the computation time.
- (6) Incorporate construction constraints into the PBO method.

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