THE BEST CONSTANT IN AN INEQUALITY OF
OSTROWSKI TYPE

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Abstract. We prove that the constant $\frac{1}{2}$ in Dragomir-Wang’s inequality [2] is best.

1 Introduction

The classical inequality of Ostrowski, [1, p. 469] is

**Theorem 1.1.** Let $I$ be an interval in $\mathbb{R}$, $I^o$ the interior of $I$, $f : I \to \mathbb{R}$ be differentiable on $I^o$. Let $a, b \in I^o$ with $a < b$ and $\|f'\|_\infty = \sup_{t \in [a,b]} |f'(t)| < \infty$. Then

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \frac{(x - a + b)^2}{(b - a)^2} \right] (b - a) \|f'\|_\infty
\]

for all $x \in [a,b]$. The constant $\frac{1}{4}$ in (1.1) is the best possible.

For, suppose that

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \left[ k + \frac{(x - a + b)^2}{(b - a)^2} \right] (b - a) \|f'\|_\infty
\]

for all $x \in [a,b]$. Taking $f(x) = x$, gives $\|f'\|_\infty = 1$ and (1.2) becomes

\[
\left| x - \frac{a + b}{2} \right| \leq \left[ k + \frac{(x - a + b)^2}{(b - a)^2} \right] (b - a)
\]

for all $x \in [a,b]$. With $x = a$ this becomes

\[
\frac{b - a}{2} \leq \left( k + \frac{1}{4} \right) (b - a)
\]

giving $k \geq \frac{1}{4}$.

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2 The Results

In [2], Dragomir and Wang obtained a related inequality:

**Theorem 2.1.** Let \( I, f, a, b \) be as above and \( f' \in L_1 [a, b] \). Then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|^2}{b-a} \right] \| f' \|_1
\]

for all \( x \in [a, b] \),

but did not prove that the constant \( \frac{1}{2} \) is the best possible one.

In [3], S.S. Dragomir gave an extension of Theorem 2.1 for mappings with bounded variation, i.e., he proved the result:

**Theorem 2.2.** Let \( f : [a, b] \to \mathbb{R} \) be a mapping with bounded variation on \([a, b]\). Then for all \( x \in [a, b] \), we have the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|^2}{b-a} \bigtriangledown_a^b (f)
\]

where \( \bigtriangledown_a^b (f) \) denotes the total variation of \( f \) on \([a, b]\).

The constant \( \frac{1}{2} \) is the best possible one.

For the sake of completeness and as the paper [3] is not published yet, we give here a short proof of Theorem 2.2.

Using the integration by parts formula for Riemann-Stieltjes integral, we have

\[
\int_a^b p(x,t) \, df(t) = f(x)(b-a) - \int_a^b f(t) \, dt
\]

where

\[
p(x,t) := \begin{cases} 
  t - a & \text{if } t \in [a,x) \\
  t - b & \text{if } t \in [x,b]. 
\end{cases}
\]

for all \( x, t \in [a,b] \).

It is well known that if \( p : [a,b] \to \mathbb{R} \) is continuous on \([a,b]\) and \( v : [a,b] \to \mathbb{R} \) is with bounded variation on \([a,b]\), then

\[
\left| \int_a^b p(x) \, dv(x) \right| \leq \sup_{x \in [a,b]} |p(x)| \bigtriangledown_a^b (v).
\]

Applying the inequality (2.4) for \( p(x,\cdot) \) and \( f \), we get

\[
\left| \int_a^b p(x,t) \, df(t) \right| \leq \sup_{t \in [a,b]} |p(x,t)| \bigtriangledown_a^b (f)
\]
Best Constant

\[
\max \left\{ x - a, b - x \right\} \frac{b}{a} (f) = \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right] \frac{b}{a} (f).
\]

Using the identity (2.3), we deduce the desired result (2.2).

To prove the sharpness of the constant \( \frac{1}{2} \) in the class of mappings with bounded variation, assume that the inequality (2.2) holds with a constant \( C > 0 \), i.e.,

\[
(2.5) \quad \left| \int_a^b f(t) \, dt - f(x) (b - a) \right| \leq \left[ C (b - a) + \left| x - \frac{a + b}{2} \right| \right] \frac{b}{a} (f),
\]

for all \( x \in [a,b] \).

Consider the mapping \( f : [a,b] \to \mathbb{R} \) given by

\[
f(x) = \begin{cases} 
0 & \text{if } x \in [a,b] \setminus \left\{ \frac{a+b}{2} \right\} \\
1 & \text{if } x = \frac{a+b}{2}
\end{cases}
\]

in (2.5). Then \( f \) is with bounded variation on \([a,b]\) and

\[
\frac{b}{a} (f) = 2, \quad \int_a^b f(t) \, dt = 0
\]

and for \( x = \frac{a+b}{2} \) we get in (2.5), \( 1 \leq 2C \); which implies that \( C \geq \frac{1}{2} \) and the theorem is completely proved.

Now, it is clear that if \( f \) is differentiable on \((a,b)\) and \( f' \in L_1 [a,b] \), then \( f \) is with bounded variation on \([a,b]\) and applying Theorem 2.2 we get Theorem 2.1. But we are not sure that the constant \( \frac{1}{2} \) is best in the class of differentiable mappings whose derivatives are in \( L_1 (a,b) \). We give an example showing that the constant \( \frac{1}{2} \) remains best for this class of mappings, too.

Suppose that

\[
(2.6) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ k + \left| x - \frac{a+b}{2} \right| \right] \| f' \|_1, \quad x \in [a,b].
\]

Let \( C \) be any positive real and let

\[
f(x) = \frac{C}{C^2 + x^2} - \tan^{-1} \left( \frac{1}{C} \right)
\]

with \( a = -1 \) and \( b = 1 \).

Direct calculation shows that \( \int_a^b f(t) \, dt = 0 \).

Also, since \( f'(x) \leq 0 \) for all \( x \geq 0 \),

\[
\| f' \|_1 = 2 \int_0^1 |f'(t)| \, dt = -2 \int_0^1 f'(t) \, dt = 2 [f(0) - f(1)]
\]
\[
\frac{1}{C} - \tan^{-1}\left(\frac{1}{C}\right) \leq k \frac{2}{C(C^2 + 1)}
\]

Substituting these into (2.6) and taking \(x = 0\) then gives
\[
\left| \frac{1}{C} - \tan^{-1}\left(\frac{1}{C}\right) \right| \leq k \frac{2}{C(C^2 + 1)}
\]
so that
\[
k \geq \frac{C^2 + 1}{2} \left[ 1 - C \tan^{-1}\left(\frac{1}{C}\right) \right].
\]

Since the right side tends to \(\frac{1}{2}\) as \(C \to 0^+\), we get \(k \geq \frac{1}{2}\), which shows that the constant \(\frac{1}{2}\) is the best possible in Theorem 2.1. 

REFERENCES


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