ON SIMPSON’S QUADRATURE FORMULA FOR DIFFERENTIABLE MAPPINGS WHOSE DERIVATIVES BELONG TO $L_p$-SPACES AND APPLICATIONS

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Abstract. An estimation of remainder for Simpson’s quadrature formula for differentiable mappings whose derivatives belong to $L_p$-spaces and applications in theory of special means (logarithmic mean, identric mean etc...) are given.

1 Introduction

The following inequality is well known in the literature as the Simpson’s inequality:

$$\left| \int_a^b f(x)dx - \frac{b-a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] \right| \leq \frac{1}{2880} \left\| f^{(4)} \right\|_{\infty} (b - a)^5$$

where the mapping $f : [a, b] \to \mathbb{R}$ is supposed to be four time continuous differentiable on the interval $(a, b)$ and having the fourth derivative bounded on $(a, b)$, that is

$$\left\| f^{(4)} \right\|_{\infty} := \sup_{x \in (a, b)} \left| f^{(4)}(x) \right| < \infty.$$

Now, if we assume that $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a, b]$ and $f$ is as above, then we have the Simpson’s quadrature formula:

$$\int_a^b f(x)dx = AS(f, I_h) + RS(f, I_h)$$

where $AS(f, I_h)$ is the Simpson’s rule

$$AS(f, I_h) := \frac{1}{6} \sum_{i=0}^{n-1} \left[ f(x_i) + f(x_{i+1}) \right] h_i + \frac{2}{3} \sum_{i=0}^{n-1} f \left( \frac{x_i + x_{i+1}}{2} \right) h_i$$

and the remainder term $RS(f, I_h)$ satisfies the estimation

$$|RS(f, I_h)| \leq \frac{1}{2880} \left\| f^{(4)} \right\|_{\infty} \sum_{i=0}^{n-1} h_i^5$$

where $h_i := x_{i+1} - x_i$ for $i = 0, ..., n - 1$.

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When we have an equidistant partitioning of \([a, b]\) given by

\[
I_n : x_i := a + \frac{b - a}{n} i, \quad i = 0, \ldots, n
\]

then we have the formula

\[
\int_a^b f(x)dx = A_{S,n}(f) + R_{S,n}(f)
\]

where

\[
A_{S,n}(f) := \frac{b - a}{6n} \sum_{i=0}^{n-1} \left[ f \left( a + \frac{b - a}{n} i \right) + f \left( a + \frac{b - a}{n} (i + 1) \right) \right]
\]

\[
+ \frac{2(b - a)}{3n} \sum_{i=0}^{n-1} f \left( a + \frac{b - a}{n} \cdot \frac{2i + 1}{2} \right)
\]

and the remainder satisfies the estimation

\[
|R_{S,n}(f)| \leq \frac{1}{2880} \cdot \frac{(b - a)^5}{n^4} \left\| f^{(4)} \right\|_{\infty}.
\]

In the recent paper [1] the author proved the following result for Lipschitz mappings

**Theorem 1.1.** Let \( f : [a, b] \to \mathbb{R} \) be an \( L \)-Lipschitzian mapping on \([a, b]\). Then we have the inequality

\[
\int_a^b f(x)dx - \frac{b - a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right]
\]

\[
\leq \frac{5}{36} L(b - a)^2.
\]

The following corollary is useful in practice:

**Corollary 1.2.** Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\), whose derivative is bounded on \((a, b)\), i.e.,

\[
\left\| f' \right\|_{\infty} := \sup_{x \in (a, b)} |f'(x)| < \infty.
\]

Then we have the inequality

\[
\int_a^b f(x)dx - \frac{b - a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right]
\]

\[
\leq \frac{5}{36} \left\| f' \right\|_{\infty} (b - a)^2.
\]

In [2], S.S. Dragomir proved a version of Simpson's inequality for mappings with bounded variation as follows:

**Theorem 1.3.** Let \( f : [a, b] \to \mathbb{R} \) be a mapping with bounded variation on \([a, b]\). Then we have the inequality

\[
\int_a^b f(x)dx - \frac{b - a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right]
\]

\[
\leq \frac{1}{3} (b - a)^2 \sqrt{(f)}
\]
where $\sqrt{b}_{a}(f)$ denotes the total variation of $f$ on the interval $[a, b]$. The constant $\frac{1}{b-a}$ is the best possible.

The following corollary is useful in practice

**Corollary 1.4.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$, whose derivative is integrable on $(a, b)$, i.e.,

$$
\|f'\|_1 \geq \int_{a}^{b} |f'(x)| \, dx < \infty.
$$

Then we have the inequality

$$
(1.12) \quad \left| \int_{a}^{b} f(x) \, dx - \frac{b - a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] \right| \\
\leq \frac{1}{3} \|f'\|_1 (b - a)^2.
$$

For some other integral inequalities see the recent book [3].

The main aim of this paper is to point out some bounds of the remainder in terms of $p$-norm of the derivative $f'$ and apply them for composite quadrature formulae and for special means.

2  **Simpson’s Inequality in Terms of $p$-Norms**

The following result holds:

**Theorem 2.1.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$, whose derivative belongs to $L_p(a, b)$. Then we have the inequality

$$
(2.1) \quad \left| \int_{a}^{b} f(x) \, dx - \frac{b - a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] \right| \\
\leq \frac{1}{6} \left\{ \frac{2^{q+1} + 1}{3(q+1)} \right\}^{\frac{1}{q}} (b - a)^{1 + \frac{1}{q}} \|f'\|_p,
$$

where $\frac{1}{p} + \frac{1}{q} = 1, p > 1$.

**Proof.** Using the integration by parts formula we have:

$$
(2.2) \quad \int_{a}^{b} s(x)f'(x) \, dx \\
= \frac{b - a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] - \int_{a}^{b} f(x) \, dx
$$

where

$$
s(x) := \begin{cases} 
    x - \frac{5a + b}{6}, & x \in [a, \frac{a+b}{2}) \\
    x - \frac{a + 5b}{6}, & x \in [\frac{a+b}{2}, b].
\end{cases}
$$
Indeed,
\[
\int_{a}^{b} s(x)f'(x)dx = \int_{a}^{\frac{a+1}{2}} \left(x - \frac{5a + b}{6}\right) f'(x)dx + \int_{\frac{a+1}{2}}^{b} \left(x - \frac{a + 5b}{6}\right) f'(x)dx
\]

\[
= \left(x - \frac{5a + b}{6}\right) f(x) \bigg|_{a}^{\frac{a+1}{2}} + \left(x - \frac{a + 5b}{6}\right) f(x) \bigg|_{\frac{a+1}{2}}^{b} - \int_{a}^{b} f(x)dx
\]

\[
= \frac{b-a}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \int_{a}^{b} f(x)dx
\]

and the identity is proved.

Applying Hölder’s integral inequality we get

\[
\left| \int_{a}^{b} s(x)f'(x)dx \right| \leq \left( \int_{a}^{b} \left| s(x) \right|^q dx \right)^{\frac{1}{q}} \| f' \|_p.
\]

(2.3)

Let us compute

\[
\int_{a}^{b} \left| s(x) \right|^q dx = \int_{a}^{\frac{a+1}{2}} \left|x - \frac{5a + b}{6}\right|^q dx + \int_{\frac{a+1}{2}}^{b} \left|x - \frac{a + 5b}{6}\right|^q dx
\]

\[
= \int_{a}^{\frac{a+1}{2}} \left(\frac{5a + b}{6} - x\right)^q dx + \int_{\frac{a+1}{2}}^{b} \left(x - \frac{5a + b}{6}\right)^q dx
\]

\[
+ \int_{\frac{a+1}{2}}^{b} \left(\frac{a + 5b}{6} - x\right)^q dx + \int_{\frac{a+1}{2}}^{b} \left(x - \frac{a + 5b}{6}\right)^q dx
\]

\[
= \frac{1}{q+1} \left[ \left(\frac{5a + b}{6} - a\right)^{q+1} \bigg|_{a}^{\frac{a+1}{2}} + \left(x - \frac{5a + b}{6}\right)^{q+1} \bigg|_{\frac{a+1}{2}}^{b} - \left(\frac{a + 5b}{6} - x\right)^{q+1} \bigg|_{a}^{\frac{a+1}{2}} + \left(x - \frac{a + 5b}{6}\right)^{q+1} \bigg|_{\frac{a+1}{2}}^{b} \right]
\]

\[
= \frac{1}{q+1} \left[ \left(\frac{5a + b}{6} - a\right)^{q+1} + \left(\frac{a + b}{2} - \frac{5a + b}{6}\right)^{q+1} \right]
\]

\[
+ \left( \frac{a + 5b}{6} - \frac{a + b}{2} \right)^{q+1} \right) + \left( b - \frac{a + 5b}{6} \right)^{q+1} \right]
= \frac{(2^{q+1} + 1)(b - a)^{q+1}}{3(q + 1)^{6q}}.
\]

Now, using the inequality (2.3) and the identity (2.2) we deduce the desired result (2.1).

The following corollary for Simpson’s composite formula holds:

**Corollary 2.2.** Let \( f \) and \( I_h \) be as above. Then we have Simpson’s rule (1.2) and the remainder \( R_S(f, I_h) \) satisfies the estimation

\[
|R_S(f, I_h)| \leq \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q + 1)} \right]^{\frac{1}{q}} \| f' \|_p \left( \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \right)^{\frac{1}{q}}.
\]

**Proof.** Apply Theorem 2.1 on the interval \([x_i, x_{i+1}]\) \((i = 0, \ldots, n - 1)\) to get

\[
\left| \int_{x_i}^{x_{i+1}} f(x)dx - \frac{h_i}{3} \left[ f(x_i) + f(x_{i+1}) + 2f \left( \frac{x_i + x_{i+1}}{2} \right) \right] \right|
\]

\[
\leq \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q + 1)} \right]^{\frac{1}{q}} \| f' \|_p \left( \int_{x_i}^{x_{i+1}} |f'(t)|^p dt \right)^{\frac{1}{p}}.
\]

Summing the above inequalities over \(i\) from 0 to \(n - 1\), using the generalized triangle inequality and Hölder’s discrete inequality, we get

\[
|R_S(f, I_h)| \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx - \frac{h_i}{3} \left[ f(x_i) + f(x_{i+1}) + 2f \left( \frac{x_i + x_{i+1}}{2} \right) \right] \right|
\]

\[
\leq \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q + 1)} \right]^{\frac{1}{q}} \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \left( \int_{x_i}^{x_{i+1}} |f'(t)|^p dt \right)^{\frac{1}{p}}.
\]

\[
\leq \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q + 1)} \right]^{\frac{1}{q}} \left( \sum_{i=0}^{n-1} \left( h_i^{1+\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \times \left( \sum_{i=0}^{n-1} \left( \int_{x_i}^{x_{i+1}} |f'(t)|^p dt \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}
\]

\[
= \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q + 1)} \right]^{\frac{1}{q}} \| f' \|_p \left( \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \right)^{\frac{1}{q}}
\]

and the corollary is proved.

The case of equidistant partitioning is embodied in the following corollary:

**Corollary 2.3.** Let \( f \) be as above and if \( I_n \) is an equidistant partitioning of \([a, b]\), then we have the estimation

\[
|R_{S,n}(f)| \leq \frac{1}{6n} \left[ \frac{2^{q+1} + 1}{3(q + 1)} \right]^{\frac{1}{q}} (b - a)^{1+\frac{1}{q}} \| f' \|_p.
\]
Remark 2.1. If we want to approximate the integral \( \int_a^b f(x)dx \) by Simpson’s formula \( A_{s,n}(f) \) with an accuracy less that \( \varepsilon > 0 \), we need at least \( n_\varepsilon \in \mathbb{N} \) points for the division \( I_n \), where

\[
n_\varepsilon := \left[ \frac{1}{6\varepsilon} \left( \frac{2q+1}{3(q+1)} \right)^{\frac{q}{2}} (b-a)^{1+\frac{q}{2}} \|f''\|_p \right] + 1
\]

and \([r]\) denotes the integer part of \( r \in \mathbb{R} \).

Remark 2.2. If the mapping \( f : [a, b] \rightarrow \mathbb{R} \) is neither four time differentiable nor the fourth derivative is bounded on \((a, b)\), then we can not apply the classical estimation in Simpson’s formula using the fourth derivative. But if we assume that \( f' \in L_p(a, b) \), then we can use instead of (1.4), the formula (2.4).

We give here a class of mappings whose first derivatives belong to \( L_p(a, b) \) but having the fourth derivatives unbounded on the given interval.

Let \( f_s : [a, b] \rightarrow \mathbb{R} \), \( f_s(x) := (x - a)^s \) where \( s \in (3, 4) \). Then obviously

\[
f'_s(x) := s(x - a)^{s-1}, \quad x \in (a, b)
\]

and

\[
f^{(4)}_s(x) = \frac{s(s-1)(s-2)(s-3)}{(x-a)^{4-s}}, \quad x \in (a, b).
\]

It is clear that \( \lim_{x \to a^+} f^{(4)}_s(x) = +\infty \) but \( \|f''\|_p = s \frac{(b-a)^{s-1+\frac{1}{p}}}{((s-1)p+1)^\frac{1}{p}} < \infty \).

3 Applications for Special Means

Let us recall the following means:

1. Arithmetic mean

\[
A = A(a, b) := \frac{a + b}{2}, \quad a, b \geq 0;
\]

2. Geometric mean

\[
G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;
\]

3. Harmonic mean

\[
H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0;
\]

4. Logarithmic mean

\[
L = L(a, b) := \frac{b-a}{\ln b - \ln a}, \quad a, b > 0, a \neq b;
\]

5. Identric mean

\[
I = I(a, b) := \frac{1}{e} \left( \frac{b}{a} \right)^{\frac{b}{a}}, \quad a, b > 0, a \neq b;
\]

6. \( p \)-Logarithmic mean

\[
S_p = S_p(a, b) := \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}, a, b > 0, a \neq b.
\]
It is well known that \( S_p \) is monotonous nondecreasing over \( p \in \mathbb{R} \) with \( S_{-1} := L \) and \( S_0 := I \). In particular, we have the following inequalities:

\[
(3.1) \quad H \leq G \leq L \leq I \leq A.
\]

In what follows, by the use of Theorem 2.1, we point out some new inequalities for the above means:

1. Let \( f : [a, b] \rightarrow \mathbb{R} \) \((0 < a < b), f(x) = x^s, s \in \mathbb{R} \setminus \{-1, 0\} \).

Then

\[
\frac{1}{b - a} \int_a^b f(x)dx = S^s_p(a, b), \quad f\left(\frac{a + b}{2}\right) = A^s(a, b),
\]

\[
\frac{f(a) + f(b)}{2} = A(a^s, b^s) \quad \text{and} \quad ||f'||_p = |s| S^s_{p-1}(a, b)(b - a).
\]

Using the inequality (2.1) we get

\[
(3.2) \quad \left| S^s_p(a, b) - \frac{1}{3} A(a^s, b^s) - \frac{2}{3} A^s(a, b) \right| \leq \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q+1)} \right] \frac{1}{2^{q+1}} |s| S^{s-1}_{p-1}(a, b)(b - a).
\]

where \( \frac{1}{p} + \frac{1}{q} = 1, p > 1 \).

2. Let \( f : [a, b] \rightarrow \mathbb{R} \) \((0 < a < b), f(x) = x^s.\)

Then

\[
\frac{1}{b - a} \int_a^b f(x)dx = L^{-1}(a, b), \quad f\left(\frac{a + b}{2}\right) = A^{-1}(a, b),
\]

\[
\frac{f(a) + f(b)}{2} = H^{-1}(a, b) \quad \text{and} \quad ||f'||_p = S^{-1}_{-2p}(a, b)(b - a)^{\frac{1}{p}}.
\]

Using the inequality (2.1) we get

\[
(3.3) \quad |3HA - LA - 2LH| \leq \frac{1}{2} AHL \left[ \frac{2^{q+1} + 1}{3(q+1)} \right] S^{-1}_{-2p}(b - a).
\]

3. Let \( f : [a, b] \rightarrow \mathbb{R} \) \((0 < a < b), f(x) = \ln x.\)

Then

\[
\frac{1}{b - a} \int_a^b f(x)dx = \ln I(a, b), \quad f\left(\frac{a + b}{2}\right) = \ln A(a, b),
\]

\[
\frac{f(a) + f(b)}{2} = \ln A(a, b) \quad \text{and} \quad ||f'||_p = S^{-1}_{-p}(a, b)(b - a)^{\frac{1}{p}}.
\]

Using the inequality (2.1) we get

\[
(3.4) \quad \ln \left[ \frac{I}{G^{1/3} A^{3/2}} \right] \leq \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q+1)} \right] S^{-1}_{-p}(a, b)(b - a).
\]
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