SOME INTEGRAL INEQUALITIES OF GRÜSS TYPE

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ABSTRACT. Some classical and new integral inequalities of Grüss type are presented.

1 GRÜSS INTEGRAL INEQUALITY

In 1935, G. Grüss, proved the following integral inequality which gives an estimation for the integral of a product in terms of the product of integrals (see for example [1, p. 296])

\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \\
\leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma) ;
\]

provided that \( f \) and \( g \) are two integrable functions on \([a,b]\) and satisfying the condition

\[ \varphi \leq f(x) \leq \Phi, \quad \gamma \leq g(x) \leq \Gamma \quad (1.1) \]

for all \( x \in [a,b] \).

The constant \( \frac{1}{4} \) is the best possible and is achieved for \( f(x) = g(x) = \text{sgn}(x - \frac{a+b}{2}) \).

We give here a weighted version of Grüss’ inequality

**Theorem 1.1.** Let \( f \) and \( g \) be two functions defined and integrable on \([a,b]\). If (1.1) holds for each \( x \in [a,b] \), where \( \varphi, \Phi, \gamma, \Gamma \) are given real constants, and \( h : [a,b] \rightarrow [0,\infty) \) is integrable and \( \int_a^b h(x)dx > 0 \), then

\[
\left| \int_a^b h(x)dx \cdot \int_a^b f(x)g(x)h(x)dx - \int_a^b f(x)h(x)dx \cdot \int_a^b g(x)h(x)dx \right| \\
\leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma) \left( \int_a^b h(x)dx \right)^2
\]

and the constant \( \frac{1}{4} \) is the best possible.

For the sake of completeness we give here a simple proof of this fact which is similar with the classical one for unweighted case (compare with [1, p. 296]).
Let us note that the following equality is valid:

\[ (1.3) \]
\[
\frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} f(x) g(x) h(x) dx
\]

\[- \frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} f(x) h(x) dx \cdot \frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} g(x) h(x) dx \]

\[=\]
\[
\frac{1}{2 \left( \int_{a}^{b} h(x)dx \right)^2} \int_{a}^{b} \left( f(x) - f(y) \right) \left( g(x) - g(y) \right) h(x)h(y) dx dy.
\]

Applying Cauchy-Buniakowski-Schwarz's integral inequality for double integrals we have

\[ (1.4) \]
\[
\left[ \frac{1}{2 \left( \int_{a}^{b} h(x)dx \right)^2} \int_{a}^{b} \left( f(x) - f(y) \right) \left( g(x) - g(y) \right) h(x)h(y) dx dy \right]^2
\]

\[\leq\]
\[
\frac{1}{2 \left( \int_{a}^{b} h(x)dx \right)^2} \int_{a}^{b} \left( f(x) - f(y) \right)^2 h(x)h(y) dx dy
\]

\[\times\]
\[
\frac{1}{2 \left( \int_{a}^{b} h(x)dx \right)^2} \int_{a}^{b} \left( g(x) - g(y) \right)^2 h(x)h(y) dx dy
\]

\[=\]
\[
\left[ \frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} f^2(x) h(x)dx - \left( \frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} f(x) h(x)dx \right)^2 \right] \times \left[ \frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} g^2(x) h(x)dx - \left( \frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} g(x) h(x)dx \right)^2 \right].
\]

The following equality also holds

\[\frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} f^2(x) h(x)dx - \left( \frac{1}{b \int_{a}^{b} h(x)dx} \int_{a}^{b} f(x) h(x)dx \right)^2\]
\[= \left( \Phi - \frac{1}{\int_a^b f(x) h(x) \, dx} \right) \cdot \left( \frac{1}{\int_a^b f(x) h(x) \, dx} \right) \]

\[-\frac{1}{\int_a^b (\Phi - f(x))(f(x) - \varphi) h(x) \, dx}.\]

As, \((\Phi - f(x))(f(x) - \varphi) \geq 0\) for each \(x \in [a, b]\), then

\[(1.5) \quad \frac{1}{\int_a^b f^2(x) h(x) \, dx} - \left( \frac{1}{\int_a^b (f(x) h(x) \, dx} \right)^2 \]

\[\leq \left( \Phi - \frac{1}{\int_a^b f(x) h(x) \, dx} \right) \cdot \left( \frac{1}{\int_a^b f(x) h(x) \, dx} \right).\]

Similarly, we have

\[(1.6) \quad \frac{1}{\int_a^b g^2(x) h(x) \, dx} - \left( \frac{1}{\int_a^b g(x) h(x) \, dx} \right)^2 \]

\[\leq \left( \Gamma - \frac{1}{\int_a^b g(x) h(x) \, dx} \right) \cdot \left( \frac{1}{\int_a^b g(x) h(x) \, dx} \right).\]

Now, by (1.3), (1.4), (1.5) and (1.6) we get

\[(1.7) \quad \left| \frac{1}{\int_a^b f(x) g(x) h(x) \, dx} \right| \]

\[= \left( \Phi - \frac{1}{\int_a^b f(x) h(x) \, dx} \right) \cdot \left( \frac{1}{\int_a^b f(x) h(x) \, dx} \right).\]
\[
\times \left( \Gamma - \frac{1}{b-a} \int_a^b f(x) h(x) dx \right) \cdot \left( \frac{1}{b-a} \int_a^b g(x) h(x) dx - \gamma \right).
\]

Using the elementary inequality for real numbers:
\[4pq \leq (p + q)^2, \quad p, q \in \mathbb{R}\]
we can state
\[
(1.8) \quad 4 \left( \Phi - \frac{1}{b-a} \int_a^b f(x) h(x) dx \right) \cdot \left( \frac{1}{b-a} \int_a^b f(x) h(x) dx - \varphi \right)
\leq (\Phi - \varphi)^2
\]
and
\[
(1.9) \quad 4 \left( \Gamma - \frac{1}{b-a} \int_a^b g(x) h(x) dx \right) \cdot \left( \frac{1}{b-a} \int_a^b g(x) h(x) dx - \gamma \right)
\leq (\Gamma - \gamma)^2.
\]

Now, combining (1.7) with (1.8) and (1.9) we deduce the desired inequality (1.2).
To prove the sharpness of (1.2), let choose \( h(x) = 1 \), \( f(x) = g(x) = \text{sgn} \left( x - \frac{a+b}{2} \right) \) for all \( x \in [a, b] \). Then
\[
\frac{1}{b-a} \int_a^b f(x) dx = 1,
\]
\[
\frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{b-a} \int_a^b g(x) dx = 0,
\]
\[
\Phi - \varphi = \Gamma - \gamma = 2
\]
and the equality in (1.2) is realized.

For other inequalities of Grüss type see the book [1], where many other references are given.
We omit the details.
2 The Case When Both Mappings Are Lipschitzian

The following inequality of Grüss’ type for Lipschitzian mappings holds:

**Theorem 2.1.** Let \( f, g : [a, b] \to \mathbb{R} \) be two Lipschitzian mappings with the constants \( L_1 > 0 \) and \( L_2 > 0 \), i.e.,

\[
|f(x) - f(y)| \leq L_1 |x - y|, \quad |g(x) - g(y)| \leq L_2 |x - y|
\]

for all \( x, y \in [a, b] \). If \( p : [a, b] \to [0, \infty) \) is integrable, then

\[
\left| \int_a^b p(x) f(x) g(x) \, dx \right| \leq L_1 L_2 \left[ \int_a^b p(x) \, dx \int_a^b x^2 \, dx - \left( \int_a^b p(x) \, dx \right)^2 \right]
\]

and the inequality is sharp.

**Proof.** By (2.1) we have that

\[
|f(x) - f(y)| (g(x) - g(y)) \leq L_1 L_2 (x - y)^2
\]

for all \( x, y \in [a, b] \).

Multiplying by \( p(x) p(y) \geq 0 \) and integrating on \([a, b]^2\), we get

\[
\int_a^b \int_a^b p(x) p(y) f(x) (f(y) - f(y)) (g(x) - g(y)) \, dx \, dy
\]

\[
\leq \int_a^b \int_a^b p(x) p(y) |(f(x) - f(y)) (g(x) - g(y))| \, dx \, dy
\]

\[
\leq L_1 L_2 \int_a^b \int_a^b p(x) p(y) (x - y)^2 \, dx \, dy.
\]

As it is easy to see that

\[
\frac{1}{2} \int_a^b \int_a^b (f(x) - f(y)) (g(x) - g(y)) p(x) p(y) \, dx \, dy
\]

\[
= \int_a^b p(x) \, dx \int_a^b p(x) f(x) g(x) \, dx - \int_a^b p(x) f(x) \, dx \int_a^b p(x) g(x) \, dx
\]

and

\[
\frac{1}{2} \int_a^b \int_a^b p(x) p(y) (x - y)^2 \, dx \, dy = \int_a^b p(x) \, dx \int_a^b p(x) x^2 \, dx - \left( \int_a^b p(x) \, dx \right)^2
\]

the inequality (2.2) is thus obtained.

Now, if we chose \( f(x) = L_1 x \), \( g(x) = L_2 x \), then \( f \) is \( L_1 \)-Lipschitzian, \( g \) is \( L_2 \)-Lipschitzian and the equality in (2.2) is realized for any \( p \) as above.
Corollary 2.2. Under the above assumptions, we have

\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \frac{L_1 L_2 (b-a)^2}{12}.
\]

The constant \(\frac{1}{12}\) is the best possible.

We note that the above corollary is a natural generalization of a well-known result by Čebyshev (see for example [1, p. 297]):

Corollary 2.3. Let \(f, g: [a, b] \to \mathbb{R}\) be two differentiable mappings whose derivatives are bounded on \((a, b)\). Denote \(\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty\). Then we have the inequality:

\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \frac{\|f'\|_\infty \|g'\|_\infty}{12} (b-a)^2.
\]

The constant \(\frac{1}{12}\) is the best possible.

3 The Case When \(f\) Is Lipschitzian

We are able now to prove another inequality of Grüss type assuming that only one mapping is Lipschitzian as follows:

Theorem 3.1. Let \(f: [a, b] \to \mathbb{R}\) be a \(M\)-Lipschitzian mapping on \([a, b]\). Then we have the inequality:

\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \begin{cases} 
M \|g\|_1 & \text{provided that } g \in L_1[a, b] \\
M \left[ \frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \|g\|_q & \text{provided that } g \in L_q[a, b] \\
M \frac{(b-a)^3}{3} \|g\|_\infty & \text{provided that } g \in L_\infty[a, b].
\end{cases}
\]

Proof. We have that

\[
|f(x)g(y) - f(y)g(y)| \leq M |x-y| |g(y)|
\]

for all \(x, y \in [a, b]\), from where, by integration on \([a, b]^2\), we get that

\[
\left| \int_a^b \int_a^b (f(x)g(y) - f(y)g(y)) \, dxdy \right| \leq M \int_a^b \int_a^b |x-y| |g(y)| \, dxdy.
\]
But
\[
\int_a^b \int_a^b (f(x)g(y) - f(y)g(y)) \, dx \, dy = \int_a^b f(x) \, dx \int_a^b g(x) \, dx - (b-a) \int_a^b f(x) \, g(x) \, dx.
\]

Now, if \( g \in L_1 [a,b] \), then
\[
\int_a^b \int_a^b |x-y| |g(y)| \, dx \, dy \leq (b-a) \max_{(x,y) \in [a,b]^2} \int_a^b |g(y)| \, dy = (b-a)^2 \| g \|_1.
\]

Now, assume that \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), \( g \in L_q [a, b] \). Then by Hölder’s integral inequality we have:
\[
\int_a^b \int_a^b |x-y| |g(y)| \, dx \, dy \leq \left( \int_a^b \int_a^b |x-y|^p \, dx \, dy \right)^{\frac{1}{p}} \left( \int_a^b \int_a^b |g(y)|^q \, dx \, dy \right)^{\frac{1}{q}} = K \frac{1}{p} (b-a)^{\frac{1}{p}} \| g \|_q
\]
where
\[
K := \int_a^b \int_a^b |x-y|^p \, dx \, dy = \int_a^b \left( \int_a^b |y-x|^p \, dy \right) \, dx
\]
\[
= \int_a^b \left( \int_a^x |x-y|^p \, dy + \int_x^b |y-x|^p \, dy \right) \, dx
\]
\[
= \int_a^b \left[ (x-a)^{p+1} + (b-x)^{p+1} \right] \frac{1}{p+1} \, dx = \frac{2 (b-a)^{p+2} }{p+1(p+2)}
\]
and then we get
\[
\int_a^b \int_a^b |x-y| |g(y)| \, dx \, dy \leq \left[ \frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}+\frac{1}{q}} \| g \|_q.
\]

Finally, assuming that \( g \in L_\infty [a, b] \), we have that
\[
\int_a^b \int_a^b |x-y| |g(y)| \, dx \, dy \leq \| g \|_\infty \int_a^b \int_a^b |x-y| \, dx \, dy = \frac{(b-a)^3}{3} \| g \|_\infty.
\]

The theorem is thus proved. \( \square \)

The following corollary is important in applications.

**Corollary 3.2.** Let \( f : [a,b] \to \mathbb{R} \) be a differentiable mapping whose derivative is bounded on \((a, b)\). Then we have the inequality:

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \tag{3.2}
\]
\[ \frac{|f(x) - f(y)||g(x) - g(y)|}{|p(x) - p(y)|} \leq \frac{1}{2} \left( \frac{b - a}{p} \right) \left( \frac{b - a}{q} \right) \frac{1}{\sqrt{p}} \|f\|_\infty \|g\|_q \quad \text{provided that } g \in L_q[a, b] \]

\[ \|f\|_\infty \|g\|_1 \quad \text{provided that } g \in L_1[a, b] \]

\[ \frac{1}{3} \left( \frac{b - a}{p} \right) \left( \frac{b - a}{q} \right) \|f\|_\infty \|g\|_\infty \quad \text{provided that } g \in L_\infty[a, b]. \]

4 The Case When \( f \) Is \( M - g \)-Lipschitzian

Another generalization of Grüss’ integral inequality is embodied in the following theorem:

**Theorem 4.1.** Let \( f, g : [a, b] \to \mathbb{R} \) be two integrable mappings on \([a, b]\) such that

\[
|f(x) - f(y)| \leq M |g(x) - g(y)| \quad \text{for all } x, y \in [a, b].
\]

Then we have the inequality:

\[
\left| \int_a^b f(x) g(x) \, dx - \int_a^b f(y) g(y) \, dx \right| \leq \frac{1}{2} \int_a^b \left( \frac{b - a}{p} \right)^2 \|f\|_\infty \left( \int_a^b g^2(x) \, dx \right) \, dx
\]

\[
\leq \frac{1}{2} \int_a^b \left( \int_a^b p(x) \, dx \right)^2 \left( \int_a^b p(x) \, dx \right)^2 \left( \int_a^b g(x) \, dx \right)^2 \, dx
\]

where \( p : [a, b] \to [0, \infty) \) is an arbitrary integrable function on \([a, b]\). The inequality (4.2) is sharp.

**Proof.** By condition (4.1) we have

\[
|f(x) - f(y)(g(x) - g(y))| \leq M (g(x) - g(y))^2 \quad \text{for all } x, y \in [a, b].
\]

Multiplying by \( p(x)p(y) \geq 0 \) and integrating on \([a, b]^2\) we get

\[
\left| \int_a^b \int_a^b p(x)p(y)f(x)f(y) g(x) - g(y) \, dx \, dy \right|
\]

\[
\leq \int_a^b \int_a^b p(x)p(y) |f(x) - f(y)(g(x) - g(y))| \, dx \, dy
\]

\[
\leq M \int_a^b \int_a^b p(x)p(y) (g(x) - g(y))^2 \, dx \, dy
\]

which is clearly equivalent to (4.2).

Now, if we choose \( f(x) = Mx, g(x) = x \), then the equality in the above inequality is realized for any \( p \) as above. \( \blacksquare \)

The following corollary is important for applications.
Corollary 4.2. Let \(f, g : [a, b] \to \mathbb{R}\) be two differentiable mappings with \(g'(x) \neq 0\) on \((a, b)\) and there exists a constant \(M > 0\) so that:

\[
|\frac{f'(x)}{g'(x)}| \leq M \quad \text{for all } x \in (a, b).
\]

Then we have the inequality (4.2). The inequality is sharp.

Proof. Use the Cauchy’s mean value theorem, i.e., for every \(x, y \in [a, b]\) with \(x \neq y\), there exists a \(c\) between \(x\) and \(y\) so that

\[
\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)}.
\]

Consequently, for each \(x, y \in [a, b]\) we have

\[
|f(x) - f(y)| \leq M |g(x) - g(y)|
\]

i.e., (4.1) holds. Applying Theorem 4.1, we get (4.3).

Remark 4.1. Under the assumption of Corollary 4.2 we can choose

\[
M = \sup_{x \in (a, b)} \left|\frac{f'(x)}{g'(x)}\right| = \left\|\frac{f'}{g'}\right\|_{\infty},
\]

assuming that the norm is finite.

Remark 4.2. If \(f, g\) are as in the above theorem, then we have the inequality

\[
\left|\frac{1}{b-a} \int_{a}^{b} f(x) g(x) \, dx - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \cdot \frac{1}{b-a} \int_{a}^{b} g(x) \, dx\right|
\]

\[
\leq M \left[\frac{1}{b-a} \int_{a}^{b} g^2(x) \, dx - \left(\frac{1}{b-a} \int_{a}^{b} g(x) \, dx\right)^2\right]
\]

and the inequality is sharp.

2. If \(f, g\) are as in Corollary 4.2, then we have the inequality

\[
\left|\frac{1}{b-a} \int_{a}^{b} f(x) g(x) \, dx - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \cdot \frac{1}{b-a} \int_{a}^{b} g(x) \, dx\right|
\]

\[
\leq \left\|\frac{f'}{g'}\right\|_{\infty} \left[\frac{1}{b-a} \int_{a}^{b} g^2(x) \, dx - \left(\frac{1}{b-a} \int_{a}^{b} g(x) \, dx\right)^2\right]
\]

and the inequality is sharp.

5 The Case When Both Mappings Are of Hölder Type

In this section we point out a Grüss’ type inequality for mappings satisfying the condition of Hölder as follows:
Theorem 5.1. Suppose that $f$ is of $r$–Hölder type and $g$ is of $s$–Hölder, i.e.,

$$|f(x) - f(y)| \leq H_1 |x - y|^r \quad \text{and} \quad |g(x) - g(y)| \leq H_2 |x - y|^s$$

for all $x, y \in [a, b]$, where $H_1, H_2 > 0$ and $r, s \in (0, 1]$ are fixed. Then we have the inequality:

$$\left| \frac{1}{b-a} \int_a^b f(x) \, g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right|$$

$$\leq \frac{H_1 H_2 (b - a)^{r+s}}{(r + s + 1)(r + s + 2)}.$$  

Proof. By the assumption (5.1) we have

$$|(f(x) - f(y))(g(x) - g(y))| \leq H_1 H_2 |x - y|^{r+s}$$

for all $x, y \in [a, b]$.

Integrating on $[a, b]^2$ we get

$$\left| \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) \, dxdy \right|$$

$$\leq \int_a^b \int_a^b (|f(x) - f(y)| |g(x) - g(y)|) \, dxdy \leq H_1 H_2 \int_a^b \int_a^b |x - y|^{r+s} \, dxdy.$$  

Now, we observe that:

$$\int_a^b \int_a^b |x - y|^{r+s} \, dxdy = \int_a^b \left( \int_a^b |y - x|^{r+s} \, dy \right) \, dx$$

$$= \int_a^b \left( \int_a^x (x - y)^{r+s} \, dy + \int_x^b (y - x)^{r+s} \, dy \right) \, dx$$

$$= \int_a^b \left[ \frac{(x-a)^{r+s+1} + (b-x)^{r+s+1}}{r + s + 1} \right] \, dx$$

$$= \frac{2(b-a)^{r+s+1}}{(r + s + 1)(r + s + 2)}$$

and as

$$\frac{1}{2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) \, dxdy$$

$$= (b-a) \int_a^b f(x) \, g(x) \, dx - \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx$$

we get the desired inequality (5.2).
6 The Case When $f'$ and $g'$ Belong to Some $L_p$-Spaces

In this section we point out some inequalities of Grüss’ type for differentiable mappings whose derivatives belong firstly to $L_\infty(a,b)$, then to $L_p(a,b)$ ($p > 1$) and finally to $L_1(a,b)$.

**Theorem 6.1.** Let $f, g : [a, b] \to \mathbb{R}$ be two differentiable mappings on $(a, b)$ and $p : [a, b] \to [0, \infty)$ is integrable on $[a, b]$. If $f', g' \in L_\infty(a,b)$, then we have the inequality

\[
\left| \int_a^b p(x) dx \cdot \int_a^b p(x) f(x) g(x) dx - \int_a^b p(x) f(x) dx \cdot \int_a^b p(x) g(x) dx \right| \\
\leq \frac{1}{2} \int_a^b \int_a^b p(x) p(y) \left( \int_x^y |f'(t)| dt \right) \left( \int_x^y |g'(z)| dz \right) dxdy \\
\leq \|f'||_\infty \|g'||_\infty \left[ \int_a^b p(x) dx \int_a^b p(x) x^2 dx - \left( \int_a^b p(x) x dx \right)^2 \right].
\]

Moreover, the inequality (6.1) is sharp.

**Proof.** Let observe that for any $x, y \in [a, b]$ we have that

\[
(f(x) - f(y)) (g(x) - g(y)) = \int_x^y f'(t) g'(z) dtdz.
\]

As $f', g' \in L_\infty(a,b)$, then we have

\[
p(x) p(y) |(f(x) - f(y)) (g(x) - g(y))|
\]

\[
\leq \int_x^y |f'(t)| dt \int_x^y |g'(z)| dz p(x) p(y) \leq \|f'||_\infty \|g'||_\infty (x - y)^2 p(x) p(y)
\]

for all $x, y \in [a, b]$.

By the properties of the modulus, we have

\[
\left| \int_a^b \int_a^b p(x) p(y) (f(x) - f(y)) (g(x) - g(y)) dxdy \right|
\]

\[
\leq \int_a^b \int_a^b p(x) p(y) \left( \int_x^y |f'(t)| dt \right) \left( \int_x^y |g'(z)| dz \right) dxdy \leq \|f'||_\infty \|g'||_\infty \int_a^b \int_a^b (x - y)^2 p(x) p(y) dxdy,
\]

from where we get the desired inequality (6.1).
To prove the sharpness of (6.1), let consider the mappings \( f(x) = \alpha x + \beta, g(x) = \gamma x + \delta \) \((\alpha, \gamma > 0, \beta, \delta \in \mathbb{R})\) on \([a, b]\). A simple calculation gives

\[
\begin{aligned}
\int_{a}^{b} p(x) \, dx \cdot \int_{a}^{b} p(x) f(x) \, dx - \int_{a}^{b} p(x) f(x) \, dx \cdot \int_{a}^{b} p(x) g(x) \, dx \\
= \frac{1}{2} \int_{a}^{b} \int_{a}^{b} p(x) p(y) \left| \int_{x}^{y} f'(t) \, dt \right| \left| \int_{x}^{y} g'(z) \, dz \right| \, dxdy \\
= \|f'\|_{\infty} \|g'\|_{\infty} \left[ \int_{a}^{b} p(x) \, dx \int_{a}^{b} p(x) x^2 \, dx - \left( \int_{a}^{b} p(x) \, dx \right)^2 \right] \\
= \frac{\alpha \gamma}{2} \int_{a}^{b} \int_{a}^{b} (x - y)^2 p(x) p(y) \, dxdy 
\end{aligned}
\]

which proves that we can have equality in all inequalities in (6.1).

The following corollary holds.

**Corollary 6.2.** With the above assumptions on the mappings \( f, g \), we have:

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) g(x) \, dx - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \cdot \frac{1}{b-a} \int_{a}^{b} g(x) \, dx \right| \\
\leq \frac{1}{2} \int_{a}^{b} \int_{a}^{b} \left| f'(t) \right| \, dt \left| \int_{x}^{y} g'(z) \, dz \right| \, dxdy \leq \frac{\|f'\|_{\infty} \|g'\|_{\infty}}{12} (b-a)^2.
\]

The constants \( \frac{1}{2} \) and \( \frac{1}{12} \), respectively, are the best possible.

**Remark 6.1.** We shall show that some time the estimation given by classical Grüss’ inequality for the difference

\[
\frac{1}{b-a} \int_{a}^{b} f(x) g(x) \, dx - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \cdot \frac{1}{b-a} \int_{a}^{b} g(x) \, dx
\]

is better than the estimation (6.3) and some other time the other way around.

Let \( f, g : [0, 1] \to [0, \infty) \) given by \( f(x) = x^p, \quad g(x) = x^q, \quad p, q > 1 \). Then

\[
\varphi = \inf_{x \in [0,1]} f(x) = 0, \quad \Phi = \sup_{x \in [0,1]} f(x) = 1; \\
\gamma = \inf_{x \in [0,1]} g(x) = 0, \quad \Gamma = \sup_{x \in [0,1]} g(x) = 1.
\]

Also we have

\[
f'(x) = px^{p-1}, \quad g'(x) = qx^{q-1}, \quad x \in [0, 1]
\]
and obviously \( \|f'|\|_\infty = p, \|g'|\|_\infty = q. \)

Now, we observe that
\[
\frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma) = \frac{1}{4}
\]
and
\[
\frac{\|f'\|_\infty \|g'\|_\infty}{12} (b - a)^2 = \frac{pq}{12}.
\]

Consequently, if \( pq > 3 \), then the bound provided by Gr"uss’ inequality is better than the bound provided by (6.3). If \( pq < 3 \) (\( p, q > 1 \)) then (6.3) is better than (1.1).

**Remark 6.2.** The inequality (6.3) is also a refinement of Čebyšev’s inequality embodied in Corollary 2.2.

The following theorem also holds

**Theorem 6.3.** Let \( f, g : [a, b] \to \mathbb{R} \) be two differentiable mappings on \( (a, b) \) and \( p : [a, b] \to [0, \infty) \) is integrable on \( [a, b] \). If \( f' \in L_\alpha(a, b) \), \( g' \in L_\beta(a, b) \) with \( \alpha > 1 \) and \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \), then we have the inequality
\[
\left| \int_a^b p(x) dx \int_a^b p(x) f(x) g(x) dx - \int_a^b p(x) f(x) dx \int_a^b p(x) g(x) dx \right| \leq \frac{1}{2} \left( \int_a^b p(x) p(y) |x - y| \left( \int_x^y |f'(t)|^\alpha dt \right)^{\frac{1}{\alpha}} \right)
\]
\[
\times \left( \int_a^b p(x) p(y) |x - y| \left( \int_x^y |g'(t)|^\beta dt \right)^{\frac{1}{\beta}} \right)
\]
\[
\leq \frac{1}{2} \|f'\|_\alpha \|g'\|_\beta \int_a^b \int_a^b |x - y| p(x) p(y) dy dx.
\]

**Note that, the first inequality in (6.4) is sharp.**

**Proof.** Using Hölder’s inequality for double integrals, we have
\[
\left| \int_x^y \int_x^y |f'(t) g'(z)| dt dz \right|
\]
\[
\leq \left( \int_x^y \int_x^y |f'(t)|^\alpha dt dz \right)^{\frac{1}{\alpha}} \left( \int_x^y \int_x^y |g'(z)|^\beta dt dz \right)^{\frac{1}{\beta}}
\]
\[
= |x - y|^{\frac{1}{\alpha}} \left( \int_x^y |f'(t)|^\alpha dt \right) \left( \int_x^y |g'(z)|^\beta dz \right)^{\frac{1}{\beta}}
\]
\[
= |x - y|^{\frac{1}{\alpha}} \left( \int_x^y |f'(t)|^\alpha dt \right) \left( \int_x^y |g'(t)|^\beta dt \right)^{\frac{1}{\beta}}.
\]
Now, as in the proof of Theorem 6.1, we have:

\[
\left| \int_a^b \int_a^b p(x) p(y) (f(x) - f(y))(g(x) - g(y)) \, dx \, dy \right|
\]

\[
\leq \int_a^b \int_a^b p(x) p(y) \left| \frac{y}{x} \int_x^y |f'(t)|^\alpha \, dt \right| \left| \frac{y}{x} \int_x^y |g'(z)|^\beta \, dz \right| \, dx \, dy.
\]

Using again Hölder's inequality for double integrals, we have

\[
(6.5) \quad \int_a^b \int_a^b p(x) p(y) |x - y| \left| \frac{y}{x} \int_x^y |f'(t)|^\alpha \, dt \right| \left| \frac{y}{x} \int_x^y |g'(z)|^\beta \, dz \right| \, dx \, dy
\]

\[
\leq \left( \int_a^b \int_a^b p(x) p(y) |x - y| \left| \frac{y}{x} \int_x^y |f'(t)|^\alpha \, dt \right| \, dx \, dy \right)^{\frac{1}{\alpha}}
\]

\[
\times \left( \int_a^b \int_a^b p(x) p(y) |x - y| \left| \frac{y}{x} \int_x^y |g'(z)|^\beta \, dz \right| \, dx \, dy \right)^{\frac{1}{\beta}}
\]

and, as

\[
(6.6) \quad \int_a^b \int_a^b p(x) p(y) (f(x) - f(y))(g(x) - g(y)) \, dx \, dy
\]

\[
= 2 \left[ \int_a^b p(x) \, dx \int_a^b p(x) f(x) g(x) \, dx - \int_a^b p(x) f(x) \, dx \int_a^b p(x) g(x) \, dx \right]
\]

the inequality (6.5) and (6.6) provide the first inequality in (6.4).

Now, let observe that

\[
\left| \int_x^y |f'(t)|^\alpha \, dt \right| \leq \|f'\|_\alpha^\alpha, \quad \left| \int_x^y |g'(z)|^\beta \, dz \right| \leq \|g'\|_\beta^\beta
\]

for all \(x, y \in [a, b]\), and then

\[
\left( \int_a^b \int_a^b p(x) p(y) |x - y| \left| \frac{y}{x} \int_x^y |f'(t)|^\alpha \, dt \right| \, dx \, dy \right)^{\frac{1}{\alpha}}
\]
\[
\times \left( \int_a^b \int_a^b p(x) p(y) |x - y| \left| \int_z^y |g'(z)|^\beta \, dz \right| \, dx \, dy \right)^{\frac{1}{\beta}}
\]

\[
\leq \|f'\|_\alpha \left( \int_a^b \int_a^b p(x) p(y) |x - y| \, dx \, dy \right)^{\frac{1}{\alpha}} \times \|g'\|_\beta \left( \int_a^b \int_a^b p(x) p(y) |x - y| \, dx \, dy \right)^{\frac{1}{\beta}}
\]

\[
= \|f'\|_\alpha \|g'\|_\beta \left( \int_a^b \int_a^b p(x) p(y) |x - y| \, dx \, dy \right)^{\frac{1}{\alpha}} \times \left( \int_a^b \int_a^b p(x) p(y) |x - y| \, dx \, dy \right)^{\frac{1}{\beta}}
\]

and the second inequality in (6.4) is also proved.

For the sharpness of the first inequality in (6.4), let consider the mappings \(f, g : [a, b] \to \mathbb{R}\), \(f(x) = mx + n, g(x) = sx + z\) with \(m, t > 0\). Then, obviously

\[
\int_a^b p(x) \, dx \int_a^b p(x) f(x) g(x) \, dx - \int_a^b \int_a^b p(x) f(x) \, dx \cdot \int_a^b p(x) g(x) \, dx
\]

\[
= \frac{1}{2} ms \int_a^b \int_a^b p(x) p(y) (x - y)^2 \, dx \, dy
\]

and

\[
\left| \int_x^y |f'(t)|^\alpha \, dt \right| = m^\alpha |x - y|, \quad \left| \int_x^y |g'(z)|^\beta \, dz \right| = s^\beta |x - y|
\]

then

\[
\left( \int_a^b \int_a^b p(x) p(y) |x - y| \left| \int_x^y |f'(t)|^\alpha \, dt \right| \, dx \, dy \right)^{\frac{1}{\alpha}}
\]

\[
\times \left( \int_a^b \int_a^b p(x) p(y) |x - y| \left| \int_x^y |g'(z)|^\beta \, dz \right| \, dx \, dy \right)^{\frac{1}{\beta}}
\]

\[
= ms \left( \int_a^b \int_a^b p(x) p(y) |x - y|^2 \, dx \, dy \right)^{\frac{1}{2}} \times \left( \int_a^b \int_a^b p(x) p(y) |x - y|^2 \, dx \, dy \right)^{\frac{1}{2}}
\]

\[
= ms \int_a^b \int_a^b p(x) p(y) (x - y)^2 \, dx \, dy
\]

and the equality is realized in the first inequality in (6.4).

The following corollary holds.
Corollary 6.4. Let \( f, g \) be as above. Then we have the inequality

\[
\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| 
\]

\[
\leq \frac{1}{2} \left( \frac{1}{(b-a)^2} \int_a^b \int_a^b |x-y| |f'(t)| \, dt \, dx \right)^{\frac{1}{2}} 
\]

\[
\times \left( \frac{1}{(b-a)^2} \int_a^b \int_a^b |x-y| |g'(t)| \, dt \, dx \right)^{\frac{1}{2}} 
\]

\[
\leq \frac{1}{6} \| f' \|_\infty \| g' \|_1 (b-a). 
\]

The first inequality in (6.7) is sharp.

In a similar way we can prove the following theorem:

Theorem 6.5. Let \( f, g : [a, b] \to \mathbb{R} \) be two differentiable mappings on \((a, b)\). If \( f' \in L_\infty(a, b) \) and \( g' \in L_1(a, b) \) then we have the inequalities:

\[
\left| \frac{1}{p(x)} \int_a^b p(x) f(x) g(x) \, dx - \frac{1}{p(x)} \int_a^b f(x) \, dx \cdot \frac{1}{p(x)} \int_a^b g(x) \, dx \right| 
\]

\[
\leq \frac{1}{2} \int_a^b \int_a^b p(x) p(y) |x-y| \sup_{t \in [x,y]} |f'(t)| \int_x^y |g'(z)| \, dz \, dx \, dy 
\]

\[
\leq \frac{1}{2} \| f' \|_\infty \| g' \|_1 \int_a^b \int_a^b p(x) p(y) |x-y| \, dx \, dy. 
\]

The first inequality in (6.8) is sharp.

The following corollary also holds.

Corollary 6.6. Under the above assumptions for the mappings \( f \) and \( g \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| 
\]

\[
\leq \frac{1}{2} \left( \frac{1}{(b-a)^2} \int_a^b \int_a^b |x-y| |f'(t)| \, dt \, dx \right)^{\frac{1}{2}} 
\]

\[
\times \left( \frac{1}{(b-a)^2} \int_a^b \int_a^b |x-y| |g'(t)| \, dt \, dx \right)^{\frac{1}{2}} 
\]

\[
\leq \frac{1}{6} \| f' \|_\infty \| g' \|_1 (b-a). 
\]

The first inequality in (6.9) is sharp.
Remark 6.3. We note that some time the upper bound provided by (6.4) is better than the upper bound given by (6.8) and other time, the other way around.

Indeed, choosing \( f, g : [0, 1] \rightarrow \mathbb{R} \), \( f(x) = x^p \), \( g(x) = x^q \) \((p, q > 1)\) we have

\[
f'(x) = px^{p-1}, \quad g'(x) = qx^{q-1}, \quad \|f'\|_{\infty} = p, \quad \|g'\|_1 = 1,
\]

\[
\|f'\|_\alpha = \frac{p}{\left[\alpha (p - 1) + 1\right]^\frac{1}{\alpha}}
\]

and

\[
\|g'\|_\beta = \frac{q}{\left[\beta (q - 1) + 1\right]^\frac{1}{\beta}}
\]

where \( \alpha, \beta > 1 \) and \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \).

Also, let

\[
A := \frac{1}{6} \|f'\|_{\infty} \|g'\|_1 (b - a) = \frac{p}{6}
\]

and

\[
B := \frac{1}{6} \|f'\|_\alpha \|g'\|_\beta (b - a) = \frac{pq}{6 \left[\alpha (p - 1) + 1\right]^\frac{1}{\alpha} \left[\beta (q - 1) + 1\right]^\frac{1}{\beta}}.
\]

If we choose \( \alpha = \beta = 2 \), we get

\[
\frac{A}{B} = \left(\frac{(2p + 1)(2q + 1)}{q}\right)^\frac{1}{2}
\]

which can be greater or less than 1 for different values of \( p, q > 1 \).

References