AN ESTIMATION FOR $\ln k$

S.S. DRAGOMIR AND A. SOFO

Abstract. In this paper we point out a better estimate for $\ln k$ than Kicey and Goel in their recent paper [1] from American Mathematical Monthly.

1 Introduction

In their recent paper Kicey and Goel [1], established the following series expansion for $\ln k, k = 2, 3, ...$

\[
\ln k = \sum_{i=1}^{\infty} \left[ 1 + k \left( \left\lfloor \frac{i-1}{k} \right\rfloor - \left\lfloor \frac{i}{k} \right\rfloor \right) \right] \cdot \frac{1}{i}
\]

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. Basically, Kicey and Goel proved the following inequality:

\[
\left| \ln k - \sum_{i=1}^{Nk} \left[ 1 + k \left( \left\lfloor \frac{i-1}{k} \right\rfloor - \left\lfloor \frac{i}{k} \right\rfloor \right) \right] \cdot \frac{1}{i} \right| \leq \frac{k-1}{N}
\]

for all $k \geq 2$ and $N \geq 1$. In this paper the authors shall prove that inequality (1.2) can be improved as follows.

2 The Results

The following result holds.

Theorem 2.1. With the above assumptions, we have the inequality:

\[
\left| \ln k - \sum_{i=1}^{Nk} \left[ 1 + k \left( \left\lfloor \frac{i-1}{k} \right\rfloor - \left\lfloor \frac{i}{k} \right\rfloor \right) \right] \cdot \frac{1}{i} \right| \leq \frac{1}{Nk} \min \left\{ k - 1, \left( k - 1 \right)^{1/q} \left( \frac{k^{2p-1} - 1}{2p-1} \right)^{1/p} \right\}
\]

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, for all $k \geq 2$ and $N \geq 1$.

We prove, firstly the following lemma.

Lemma 2.2. Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$. Then we have
the inequality:

\[
\frac{1}{2} \left| \int_a^b f(x) \, dx - (b-a) f(b) \right| \leq \begin{cases} 
\frac{(b-a)^2}{2} \left\| f' \right\|_\infty & \text{if } f' \in L_\infty [a, b]; \\
\frac{(b-a)^{1+1/q}}{(q+1)^{1/q}} \left\| f' \right\|_p & \text{if } f' \in L_p [a, b] \text{ where } p > 1, \text{ and } \frac{1}{p} + \frac{1}{q} = 1; \\
(b-a) \left\| f' \right\|_1 & \text{otherwise.} 
\end{cases}
\]

Proof. Integrating by parts we have

\[
\int_a^b (x-a) f'(x) \, dx = (b-a) f(b) - \int_a^b f(x) \, dx
\]

and from this identity, we may write

\[
\left| \int_a^b f(x) \, dx - (b-a) f(b) \right| \leq \int_a^b \left| (x-a) f'(x) \right| \, dx
\]

\[
\leq \left\| f' \right\|_\infty \int_a^b |x-a| \, dx
\]

\[
= \frac{(b-a)^2}{2} \left\| f' \right\|_\infty
\]

and the first inequality in (2.2) is proved. Now using Hölder’s inequality we obtain

\[
\int_a^b (x-a) \left| f'(x) \right| \, dx \leq \left\| f' \right\|_p \left[ \int_a^b (x-a)^q \, dx \right]^{1/q}
\]

\[
= \frac{(b-a)^{1+1/q}}{(q+1)^{1/q}} \left\| f' \right\|_p
\]

and the second inequality in (2.2) is proved. Finally, we may write

\[
\int_a^b (x-a) \left| f'(x) \right| \, dx \leq (b-a) \left\| f' \right\|_1
\]

and therefore (2.2) is completely proved. □

The following Lemma also holds:

**Lemma 2.3.** Let \( f \) be as above and let \( I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) be a division...
of \([a, b]\). Then we have the inequality:

\[
\left| \int_a^b f(x) \, dx - \sum_{i=0}^{n-1} h_i f(x_{i+1}) \right| \leq \left\{ \begin{array}{l}
\frac{\| f' \|_\infty}{2} \sum_{i=0}^{n-1} h_i^2, \\
\| f' \|_p \left( \frac{n-1}{(q+1)^{1/q}} \sum_{i=0}^{n-1} h_i^{q+1} \right)^{1/q}, \\
\nu(h) \| f' \|_1,
\end{array} \right.
\]

where \(h_i := x_{i+1} - x_i, i = 0, 1, \ldots, n - 1\) and \(\nu(h) := \max_{i=0, n-1} h_i\).

**Proof.** We have

\[
\left| \int_a^b f(x) \, dx - \sum_{i=0}^{n-1} h_i f(x_{i+1}) \right| = \left| \sum_{i=0}^{n-1} \left( \int_{x_i}^{x_{i+1}} f(x) \, dx - h_i f(x_{i+1}) \right) \right|
\]

and using the first inequality in (2.2) we obtain

\[
\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) \, dx - h_i f(x_{i+1}) \right| \leq \frac{\| f' \|_\infty}{2} \sum_{i=0}^{n-1} h_i^2,
\]

so the first inequality in (2.3) is proved. Using the second inequality in (2.2) and Hölder’s discrete inequality, we obtain

\[
\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) \, dx - h_i f(x_{i+1}) \right| \leq \frac{1}{(q+1)^{1/q}} \sum_{i=0}^{n-1} h_i^{q+1} \left( \int_{x_i}^{x_{i+1}} |f'(t)|^p \, dt \right)^{1/p}
\]

\[
\leq \frac{1}{(q+1)^{1/q}} \left( \sum_{i=0}^{n-1} \left( h_i^{q+1} \right)^{1/q} \left( \sum_{i=0}^{n-1} \left( \int_{x_i}^{x_{i+1}} |f'(t)|^p \, dt \right)^{1/p} \right)^{1/q} \left( \sum_{i=0}^{n-1} h_i^{q+1} \right)^{1/q} \right)^{1/p}
\]

\[
= \frac{1}{(q+1)^{1/q}} \left( \sum_{i=0}^{n-1} h_i^{q+1} \right)^{1/q} \left( \sum_{i=0}^{n-1} h_i^{q+1} \right)^{1/q}
\]

and the second inequality in (2.3) is proved. Finally we have:

\[
\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) \, dx - h_i f(x_{i+1}) \right| \leq \sum_{i=0}^{n-1} h_i \int_{x_i}^{x_{i+1}} |f'(t)| \, dt
\]

\[
\leq \nu(h) \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f'(t)| \, dt
\]

\[
= \nu(h) \| f' \|_1
\]

and the Lemma is completely proved. 

\[\square\]
**Corollary 2.4.** If \( I_n : x_i = a + \frac{b-a}{n} i, i = 1, 2, ..., n \), then we have the inequality

\[
\left| \int_a^b f(x) \, dx - \frac{b-a}{n} \sum_{i=1}^n f \left( a + \frac{b-a}{n} i \right) \right| \leq \begin{cases} 
\frac{(b-a)^2}{2n} \| f \|_\infty & \\
\frac{(b-a)^{1+1/q}}{n (q+1)^{1/q}} \| f' \|_p & \text{where } p > 1, \text{ and } \frac{1}{p} + \frac{1}{q} = 1,
\end{cases}
\]

**Proof.** Using (2.4) and noting that \( a = N, b = Nk, n = N(k-1) \) and \( f(x) = \frac{1}{x} \) we have

\[
\left| \int_1^N \frac{1}{x} \, dx - \sum_{i=1}^{N-k-N} \frac{1}{N+i} \right| \leq \begin{cases} 
\frac{N(k-1)}{2} \| f \|_\infty & \\
\left( \frac{Nk-N}{q+1} \right)^{1/q} \| f' \|_p & \\
\| f' \|_1 &
\end{cases}
\]

But we have that

\[
\| f' \|_\infty = \frac{1}{N^2},
\]

\[
\| f' \|_p = \left( \frac{1}{N} \int_1^N \frac{1}{x^2} \, dx \right)^{1/p} = \left( \frac{k^{2p-1} - 1}{(2p-1)(Nk)^{2p-1}} \right)^{1/p}
\]

and

\[
\| f' \|_1 = \frac{k-1}{Nk},
\]

hence from (2.5) we obtain

\[
\left| \ln k - \sum_{i=1}^{N-k-N} \frac{1}{N+i} \right| \leq \begin{cases} 
\frac{k-1}{2N} & \\
\frac{1}{Nk} \left( \frac{k-1}{(q+1)k} \right)^{1/q} \left( \frac{k^{2p-1} - 1}{2p-1} \right)^{1/p} & \text{where } p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1;
\end{cases}
\]

from (2.6), \( \frac{k-1}{2} \geq \frac{k-1}{k} \) for \( k \geq 2 \), hence, by the identity, see [1],

\[
\sum_{i=1}^{N-k-N} \frac{1}{N+i} = \sum_{i=1}^{N-k} \left[ 1 + k \left( \left| \frac{i-1}{k} \right| - \left| \frac{i}{k} \right| \right) \right] \frac{1}{i}.
\]

Theorem 1 is proved. \( \blacksquare \)
Remark 2.1. Clearly, for a minimum of (2.6) we need only investigate the terms $T_1 = k - 1$ and $T_2 = \left( \frac{k_1 - 1}{(k+1)^{1/2}} \right)^{1/2} \left( \frac{k_1^{1/3} - 1}{2k_1^{1/3} - 1} \right)^{1/3}$. 

Using a computer package we may obtain the contour line $\frac{T_2}{T_1} = 1$ as follows. From figure 1, the region on the left of the contour line is described by $T_2 < T_1$ and the region on the right of the contour line is described by $T_2 > T_1$. This demonstrates, clearly, that each of the bounds $T_1$ or $T_2$ may be best under different circumstances.

REFERENCES


School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City, Victoria, Australia

E-mail address: {sewer, sofo}@matilda.vut.edu.au