RGMIA Research Report Collection, Vol. 2, No. 1, 1999
http://sci.vu.edu.au/~rgmia

AN ALGEBRAIC INEQUALITY

Feng Qi

Abstract. In the short note, an algebraic inequality is presented by using analytic arguments and Cauchy's mean-value theorem.

1 Results

In this note, we present the following algebraic inequality

Theorem 1.1. *Let $b > a > 0$ and $\delta > 0$ be real numbers, then for any given positive $r \in \mathbb{R}$, we have*

$$\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} > \frac{b}{b+\delta}. \tag{1.1}$$

The lower bound in (1.1) *is best possible.*

Proof. The inequality (1.1) is equivalent to

$$\frac{b^{r+1}-a^{r+1}}{b-a}\bigg/\frac{(b+\delta)^{r+1}-a^{r+1}}{b+\delta-a} > \left(\frac{b}{b+\delta}\right)^r,$$

that is,

$$\frac{b^{r+1}-a^{r+1}}{b^r(b-a)} > \frac{(b+\delta)^{r+1}-a^{r+1}}{(b+\delta)^r(b+\delta-a)}. \tag{1.2}$$

Therefore, it is sufficient to prove that the function $(s^{r+1}-a^{r+1})/s^r(s-a)$ is decreasing with $s > a$. By direct computation, we have

$$\left(\frac{s^{r+1}-a^{r+1}}{s^r(s-a)}\right)_s' = \frac{(r+1)(s-a)s^{2r}-s^{r-1}(s^{r+1}-a^{r+1})[(r+1)s-ra]}{[s^r(s-a)]^2}.$$

So, it also suffices to prove

$$(r+1)(s-a)s^{r+1}-[(r+1)s-ra](s^{r+1}-a^{r+1}) \leqslant 0. \tag{1.3}$$

By straightforwardly calculating and easily simplifying, the inequality (1.3) is reduced to

$$\frac{s^r-a^r}{r(s-a)} > \frac{a^r}{s}. \tag{1.4}$$

¿From Cauchy's mean-value theorem, there exists one point $\xi \in (a, s)$ such that

$$\frac{s^r-a^r}{r(s-a)} = \xi^{r-1} = \frac{\xi^r}{\xi} > \frac{a^r}{\xi} > \frac{a^r}{s}.$$

Hence, the inequality (1.4) holds.

Using L'Hospital principle yields

$$\lim_{r\to+\infty}\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} = \frac{b}{b+\delta}, \tag{1.5}$$

thus, the lower bound in (1.1) is best possible. The proof is complete. ∎

1991 Mathematics Subject Classification. Primary 26D15

Key words and phrases. algebraic inequality, Cauchy's mean-value theorem, Alzer's inequality, analogy

*The author was supported in part by NSF of Henan Province, The People's Republic of China

Remark 1.1. *Note that the inequality* (1.1) *can be rewritten as*

$$\left(\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x \Big/ \frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,\mathrm{d}x\right)^{1/r} > \frac{b}{b+\delta}. \tag{1.6}$$

It is easy to see that inequality (1.6) *is indeed an integral analogy of the following*

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{1.7}$$

where r is a given positive real number, n and m are natural numbers, k is a nonnegative integer. The lower bound in (1.7) *is best possible.*

The inequality (1.7) *was presented in [2] by the author using the Cauchy's mean-value theorem and the mathematical induction, which generalized the so-called Alzer's inequality in [1].*

Using the same method as in [2], the author [3] further generalized the Alzer's inequality and got that, if $a = (a_1, a_2, \dots)$ is a positive and increasing sequence satisfying

$$\frac{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}{(k+1)a_{k+1}^r - k a_k^r} \geqslant \left(\frac{a_{k+2}}{a_{k+1}}\right)^r, \quad k \in \mathbb{N} \tag{1.8}$$

for any positive number r, then we have

$$\frac{a_n}{a_{n+m}} \leqslant \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r}, \tag{1.9}$$

where n and m are natural numbers. The lower bound in (1.9) *is best possible.*

Remark 1.2. *Using L'Hospital principle once again yields*

$$\lim_{r\to 0^+}\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} = \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}, \tag{1.10}$$

hence, we proposed the following

Corollary 1.2. *Let $b > a > 0$ and $\delta > 0$ be real numbers, then for any positive $r \in \mathbb{R}$, we have*

$$\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} \leqslant \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{1.11}$$

The upper bound in (1.11) *is best possible.*

Remark 1.3. *In fact, these inequalities in this paper have some close relationships with the monotonicity of the ratios or differences of mean values.*

References

[1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.

[2] Feng Qi, *Generalization of H. Alzer's inequality*, to appear.

[3] Feng Qi, *Further generalization of H. Alzer's inequality*, to appear.

Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China

E-mail address: qifeng@jzit.edu.cn