A PROOF OF THE ARITHMETIC MEAN-GEOMETRIC MEAN-HARMONIC MEAN INEQUALITIES

Da-Feng Xia, Sen-Lin Xu and Feng Qi

Abstract. In the note, using Cauchy-Schwartz-Buniakowski’s inequality, the authors give a new proof of the arithmetic mean-geometric mean-harmonic mean inequalities.

1 Introduction

The simplest and most classical mean values are the arithmetic, the geometric, and the harmonic mean values. For a positive sequence \(a = (a_1, a_2, \ldots, a_n)\), these mean values are defined respectively by

\[
A_n(a) = \frac{1}{n} \sum_{i=1}^{n} a_i, \quad G_n(a) = \sqrt[n]{\prod_{i=1}^{n} a_i}, \quad H_n(a) = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}.
\]

For a positive integrable function \(f\) defined on \([x, y]\), their integral analogues of (1.1) are given by

\[
A(f) = \frac{1}{y-x} \int_{x}^{y} f(t) \, dt, \quad G(f) = \exp \left( \frac{1}{y-x} \int_{x}^{y} \ln f(t) \, dt \right), \quad H(f) = \frac{y-x}{\int_{x}^{y} f(t) \, dt}.
\]

It is well-known that

\[
A_n(a) \geq G_n(a) \geq H_n(a), \quad A(f) \geq G(f) \geq H(f)
\]

are called the arithmetic mean-geometric mean-harmonic mean inequalities.

For the sake of brevity, the inequality between the arithmetic and geometric means will be called A-G inequality, while the inequality between the geometric and harmonic means will be called G-H inequality.

The A-G inequality has found much interest among many mathematicians, and there are numerous new proofs, extensions, refinements, and variants of it. The study of the A-G inequality has a rich literature, for details, please refer to [2, 3, 4], and the like. Recently, H. Alzer [1] and J. Pecarić and S. Veparosanic [6] gave two new proofs of the A-G inequality.

The concepts of mean values have been generalized, extended in many directions. A recent development concerning the mean values has simply been introduced in [5, 7, 8, 9].

In this note, using Cauchy-Schwartz-Buniakowski’s inequality, we give a new proof of the A-G-H inequalities.

2 A New Proof of the A-G-H Inequalities

For a continuous function \(f\), define

\[
\psi(r) = \left( \frac{1}{y-x} \int_{x}^{y} f(t)^r \, dt \right) \frac{1}{r}, \quad r \neq 0;
\]

\[
\psi(0) = G(f).
\]

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For a positive sequence \( a = (a_1, a_2, \ldots, a_n) \), define
\[
\varphi(r) = \left( \frac{1}{n} \sum_{i=1}^{n} a_i^r \right)^{1/r}, \quad r \neq 0;
\]
\[
\varphi(0) = G_n(a).
\]

**Theorem.** The functions \( \psi(r) \) and \( \varphi(r) \) are increasing with \( r \in \mathbb{R} \), respectively.

**Proof.** Simple calculation yields
\[
\ln \psi(r) = \frac{\ln \int_{x}^{y} f^r(t) \, dt}{r} - \ln(y - x)
\]
\[
= \frac{\ln \int_{x}^{y} f^r(t) \, dt - \ln \int_{x}^{y} f^0(t) \, dt}{r}
\]
\[
= \frac{1}{r} \int_{0}^{y} \frac{f^r(t) \ln f(t) \, dt}{f^r(t) \, dt} \, ds.
\]

The lemma 1 in [10] states that, if \( f \) is a differentiable and increasing function on a given interval \( I \), then the arithmetic mean \( \psi(r, s) \) of \( f \) defined as
\[
\psi(r, s) = \frac{1}{s - r} \int_{r}^{s} f(t) \, dt, \quad r \neq s \neq 0,
\]
\[
\psi(r, r) = f(r)
\]
is also increasing with both \( r \) and \( s \) on \( I \).

Therefore, it is sufficient to verify that
\[
\mathcal{F}(s) \triangleq \frac{\int_{x}^{y} f^r(t) \ln f(t) \, dt}{\int_{x}^{y} f^r(t) \, dt}
\]
is increasing in \( s \in \mathbb{R} \).

Let \( g(s) = \int_{x}^{y} f^r(t) \, dt, \ s \in \mathbb{R} \). Then \( \mathcal{F}(s) \) increases with \( s \) if and only if \( g''(s)g(s) - \left[ g'(s) \right]^2 \geq 0 \), that is,
\[
\left( \int_{x}^{y} f^r(t) \ln f(t) \, dt \right)^2 \leq \int_{x}^{y} f^r(t) \, dt \int_{x}^{y} f^r(t) \left[ \ln f(t) \right]^2 \, dt.
\]

Since
\[
\int_{x}^{y} f^r(t) \ln f(t) \, dt = \int_{x}^{y} f^{r/2}(t) \left[ f^{r/2}(s) \ln f(t) \right] \, dt,
\]
from Cauchy-Schwartz-Buniakowski's integral inequality in integral form, the inequality (2.4) follows. The function \( \psi(r) \) is increasing with \( r \).

From straightforward computation, we have
\[
\ln \varphi(r) = \frac{1}{r} \left( \ln \sum_{i=1}^{n} a_i^r - \ln n \right)
\]
\[
= \frac{1}{r} \left( \ln \sum_{i=1}^{n} a_i^r - \ln \sum_{i=1}^{n} a_i^0 \right)
\]
\[
= \frac{1}{r} \int_{0}^{y} \left( \sum_{i=1}^{n} a_i^r \ln a_i \left/ \sum_{i=1}^{n} a_i^0 \right. \right) \, ds.
\]

Using Cauchy-Schwartz-Buniakowski’s inequality in discrete form, by the similar arguments as proving the monotonicity of \( \psi(r) \), we can easily obtain that the function \( \varphi(r) \) increases with \( r \). The proof of Theorem follows.

**Corollary.** For a positive continuous function \( f \) or a positive sequence \( a = (a_1, a_2, \ldots, a_n) \), we have the following A-G-H inequalities:
\[
A(f) \geq G(f) \geq H(f), \quad A_n(a) \geq G_n(a) \geq H_n(a).
\]

**Proof.** It is easy to see that \( \psi(1) = A(f) \), \( \psi(-1) = H(f) \), \( \varphi(1) = A_n(a) \) and \( \varphi(-1) = H_n(a) \). Thus, the A-G-H inequalities in integral form follows from the monotonicity of \( \psi(r) \), the A-G-H inequalities in discrete form follows from the monotonicity of \( \varphi(r) \). The proof is complete.
REFERENCES


(Da-Feng Xia) Department of Mathematics, Fuyang Normal College, Fuyang City, Anhui Province, The People’s Republic of China

(Sen-Lin Xu) Department of Mathematics, University of Science and Technology of China, Hefei City, Anhui 230026, The People’s Republic of China

(Feng Qi) Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People’s Republic of China

E-mail address: Feng Qi qifeng@jzit.edu.cn