SOME REMARKS ON THE NOISELESS CODING THEOREM

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Abstract. An improvement of the Noiseless Coding Theorem for certain probability distributions is given.

1. Introduction

The following analytic inequality for the log (·) map is well known in the literature (see for example [1, Lemma 1.2.2, p. 22]):

Lemma 1. Let $P = (p_1, \ldots, p_n)$ be a probability distribution that is, $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1$. Let $Q = (q_1, \ldots, q_n)$ have the property that $0 \leq q_i \leq 1$ and $\sum_{i=1}^{n} q_i \leq 1$, then

$$\sum_{i=1}^{n} p_i \log_b \frac{1}{p_i} \leq \sum_{i=1}^{n} p_i \log_b \frac{1}{q_i} \quad (b > 1)$$

(1.1)

where $0 \log_b \frac{1}{0} = 0$ and $p \log_b \frac{1}{p} = +\infty$ for $p > 0$. Furthermore, the equality holds if and only if $q_i = p_i$ for all $i$.

Note that the proof of this result in [1] uses the elementary inequality:

$$\ln x \leq x - 1 \quad \text{for all } x > 0.$$

We give here an alternative proof based on the concavity of the mapping $\log_r (\cdot)$.

As the mapping $f(x) = \log_r (x)$ ($r > 1$) is a strictly concave mapping on $(0, \infty)$, we have

$$f(x) - f(y) \geq f'(x)(x-y)$$

for all $x, y > 0$, i.e., as $f'(x) = \frac{1}{\ln r} \cdot \frac{1}{x}$ for $x > 0$,

$$\log_r x - \log_r y \geq \frac{1}{\ln r} \left( \frac{x-y}{x} \right)$$

(1.2)

for all $x, y > 0$.

Choosing $x = \frac{1}{q_i}, y = \frac{1}{p_i}$, in (1.2) gives

$$\log_r \frac{1}{q_i} - \log_r \frac{1}{p_i} \geq \frac{1}{\ln r} \left( \frac{p_i - q_i}{p_i} \right)$$

(1.3)

for all $i \in \{1, \ldots, n\}$.

Multiplying this inequality by $p_i > 0$ ($i = 1, \ldots, n$) we get

$$p_i \log_r \frac{1}{q_i} - p_i \log_r \frac{1}{p_i} \geq \frac{1}{\ln r} (p_i - q_i)$$

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for all \(i \in \{1, \ldots, n\}\).

Summing over \(i\) from 1 to \(n\), gives

\[
\sum_{i=1}^{n} p_i \log_r \frac{1}{q_i} = \sum_{i=1}^{n} p_i \log_r \frac{1}{p_i} \geq \frac{1}{\ln r} \left( \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i \right)
\]

\[
= \frac{1}{\ln r} \left( 1 - \sum_{i=1}^{n} q_i \right) \geq 0
\]

and the inequality (1.1) is obtained.

The case of equality follows by the strict concavity of the mapping \(\log_r\).

In this paper, by use of (1.1), we point out an improvement to the Noiseless Coding Theorem.

2. The Results

Consider an encoding scheme \((c_1, \ldots, c_n)\) for a probability distribution \((p_1, \ldots, p_n)\). The average codeword length of an encoding scheme \((c_1, \ldots, c_n)\) for \((p_1, \ldots, p_n)\) is

\[
\text{AveLen}(c_1, \ldots, c_n) = \sum_{i=1}^{n} p_i \text{len}(c_i).
\]

We denote the length \(\text{len}(c_i)\) by \(l_i\).

The \(r\)-ary entropy of a probability distribution is given by

\[
H_r(c_1, \ldots, c_n) = \sum_{i=1}^{n} p_i \log_r \left( \frac{1}{p_i} \right).
\]

The following theorem is well known in the literature (see for example [1, Theorem 2.3.1, p. 62]):

**Theorem 2.** Let \(C = (c_1, \ldots, c_n)\) be an instantaneous (or uniquely decipherable) encoding scheme for \(P = (p_1, \ldots, p_n)\), then,

\[
H_r(p_1, \ldots, p_n) \leq \text{AveLen}(c_1, \ldots, c_n)
\]

with equality if and only if \(l_i = \log_r \left( \frac{1}{p_i} \right)\) for all \(i = 1, \ldots, n\).

The following result, providing a counterpart inequality, holds.

**Theorem 3.** Let \(P = (p_1, \ldots, p_n)\) be a given probability distribution and \(r \in \mathbb{N}, r \geq 2\). If \(\varepsilon > 0\) is fixed and there exists natural numbers \(l_1, \ldots, l_n\) such that:

\[
(2.1) \quad \log_r \left( \frac{1}{p_i} \right) \leq l_i \leq \log_r \left( \frac{r^\varepsilon}{p_i} \right)
\]

for all \(i \in \{1, \ldots, n\}\), then there exists an instantaneous \(r\)-ary code \(C = (c_1, \ldots, c_n)\) with codeword length \(\text{len}(c_i) = l_i\) such that

\[
(2.2) \quad H_r(p_1, \ldots, p_n) \leq \text{AveLen}(c_1, \ldots, c_n) \leq H_r(p_1, \ldots, p_n) + \varepsilon.
\]

**Proof.** Note that (2.1) is equivalent to

\[
(2.3) \quad \frac{1}{p_i} \leq r^{l_i} \leq \frac{r^\varepsilon}{p_i} \quad \text{for all} \quad i \in \{1, \ldots, n\}.
\]
Now, since $\frac{1}{r^i} \leq p_i$ $(i = 1, \ldots, n)$, it follows that
\[
\sum_{i=1}^{n} \frac{1}{r^i} \leq \sum_{i=1}^{n} p_i = 1
\]
and by Kraft’s theorem (see for example [1, Theorem 2.1.2, p. 44]), there exists an instantaneous $r$–ary code $C = (c_1, \ldots, c_n)$ such that $\text{len}(c_i) = l_i$.

Obviously, by Theorem 2, the first inequality in (2.2) holds. We have:
\[
\text{AveLen}(c_1, \ldots, c_n) = \sum_{i=1}^{n} p_i l_i = \sum_{i=1}^{n} p_i \log_r 1 = \sum_{i=1}^{n} p_i \log_r \frac{1}{q_i}
\]
choosing $q_i = \frac{1}{r^i} \in [0, 1]$. Also, by Kraft’s theorem, $\sum_{i=1}^{n} q_i \leq 1$.

By Lemma 1, we have,
\[
0 \leq \sum_{i=1}^{n} p_i \log_r \frac{1}{q_i} - \sum_{i=1}^{n} p_i \log_r \frac{1}{p_i} = \text{AveLen}(c_1, \ldots, c_n) - H_r (p_1, \ldots, p_n)
\]
\[
= \sum_{i=1}^{n} p_i \left( \log_r r^{l_i} - \log_r \frac{1}{p_i} \right) = \left| \sum_{i=1}^{n} p_i \left( \log_r r^{l_i} - \log_r \frac{1}{p_i} \right) \right|
\]
\[
\leq \sum_{i=1}^{n} p_i \left| l_i - \log_r \left( \frac{1}{p_i} \right) \right| \leq \varepsilon \sum_{i=1}^{n} p_i = \varepsilon
\]
since, by (2.1), $0 \leq l_i - \log_r \frac{1}{p_i} \leq \log_r r^\varepsilon = \varepsilon$. □

We shall use the notation:
\[
\text{MinAveLen}_r (p_1, \ldots, p_n)
\]
to denote the minimum average codeword length among all $r$–ary instantaneous encoding schemes for the probability distribution $P = (p_1, \ldots, p_n)$.

The following Noiseless Coding Theorem is well known in the literature (see for example [1, Theorem 2.3.2, p. 64]):

**Theorem 4.** For any probability distribution $P = (p_1, \ldots, p_n)$ we have:
\[
H_r (p_1, \ldots, p_n) \leq \text{MinAveLen}_r (p_1, \ldots, p_n) < H_r (p_1, \ldots, p_n) + 1.
\]

The following question is then a natural one to pose.

**Question:** Is it possible to replace the constant 1 in the above inequality by a smaller one $\varepsilon \in (0, 1)$ and, if so, under what conditions for the probability distribution $P = (p_1, \ldots, p_n)$?

The following is a partial answer to this question:

**Theorem 5.** Let $r$ be a given natural number and $\varepsilon \in (0, 1)$. If a probability distribution $P = (p_1, \ldots, p_n)$ satisfies the condition that every closed interval of real numbers
\[
I_i = \left[ \log_r \left( \frac{1}{p_i} \right), \log_r \left( \frac{r^\varepsilon}{p_i} \right) \right], \quad i \in \{1, \ldots, n\},
\]

...
contains one natural number, then, for that probability distribution $P$, we have:

$$H_r (p_1, ..., p_n) \leq \min \text{AveLen}_r (p_1, ..., p_n) \leq H_r (p_1, ..., p_n) + \varepsilon.$$ (2.5)

**Proof.** Suppose that $l_i \in I_i (i = 1, ..., n)$ are these natural numbers, then, as above,

$$\sum_{i=1}^{n} \frac{1}{p_i} \leq \sum_{i=1}^{n} p_i = 1$$

and by Kraft’s theorem there exists an instantaneous code $C = (c_1, ..., c_n)$ such that $\text{len} (c_i) = l_i$. For this code we have (2.1) and, by Theorem 3, the inequality (2.2) for $C$. Taking the infimum in this inequality over all $r$–ary instantaneous codes, gives (2.5). ☐

**Remark 1.** The lengths of the intervals $I_i$ are,

$$\text{len} (I_i) = \log_r \left( \frac{p_i^r}{p_i} \right) - \log_r \frac{1}{p_i} = \varepsilon \in (0, 1), \quad i = 0, ..., n$$

but we cannot be sure that $I_i$ always contains a natural number. Also, $I_i$ could contain at most one natural number.

The following result can be useful in practice.

**Practical Criterion.** Let $a_i$ be $n$ natural numbers, $i = 1, ..., n$. If $p_i (i = 1, ..., n)$ are such that

$$\frac{1}{p_i} \leq p_i \leq \frac{r^\varepsilon}{p_i}$$

and $\sum_{i=1}^{n} p_i = 1$, then there exists an instantaneous code $C = (c_1, ..., c_n)$ with $\text{len} (c_i) = a_i (i = 1, ..., n)$ such that (2.2) holds for the probability distribution $P = (p_1, ..., p_n)$.

For other recent results in the applications of Theory of Inequalities in Information Theory and Coding, see the following references.

**References**


