SOME INEQUALITIES IN $2-$INNER PRODUCT SPACES

Y.J. CHO, S.S. DRAGOMIR, A. WHITE, AND S.S. KIM

Abstract. In this paper we extend some results on the refinement of Cauchy-Buniakowski-Schwarz's inequality and Aćzel's inequality in inner product spaces to $2-$inner product spaces.

1. Introduction

Let X be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

(N_1) $\|x, y\| = 0$ if and only if x and y are linearly dependent;

(N_2) $\|x, y\| = \|y, x\|$;

(N_3) $\|\alpha x, y\| = |\alpha| \|x, y\|$ for any real number α;

(N_4) $\|x, y + z\| \le \|x, y\| + \|x, z\|$.

$\|\cdot,\cdot\|$ is called a *2−norm* on X and $(X, \|\cdot,\cdot\|)$ *a linear 2–normed space* cf. [10]. Some of the basic properties of the 2-norms are that they are nonnegative, and $\|x, y + \alpha x\| = \|x, y\|$ for every x, y in X and every real number α.

For any non-zero $x_1, x_2, ..., x_n$ in X, let $V(x_1, x_2, ..., x_n)$ denote the subspace of X generated by $x_1, x_2, ..., x_n$. Whenever the notation $V(x_1, x_2, ..., x_n)$ is used, we will understand that $x_1, x_2, ..., x_n$ are linearly independent.

A concept which is closely related to linear 2-normed space is that of 2 inner product spaces. For a linear space X of dimension greater than 1 let $(\cdot,\cdot \mid \cdot)$ be a real-valued function on $X \times X \times X$ which satisfies the following conditions:

(I_1) $(x, x \mid z) \ge 0; (x, x \mid z) = 0$ if and only if x and z are linearly dependent;

(I_2) $(x, x \mid z) = (z, z \mid x)$;

(I_3) $(x, y \mid z) = (y, x \mid z)$;

(I_4) $(\alpha x, y \mid z) = \alpha (x, y \mid z)$ for any real number α;

(I_5) $(x + x', y \mid z) = (x, y \mid z) + (x', y \mid z)$.

$(\cdot,\cdot \mid \cdot)$ is called a *2-inner* product and $(X, (\cdot,\cdot \mid \cdot))$ a *2-inner product space* ([3]).

These spaces are studied extensively in [1], [2], [4]-[6] and [11]. In [3] it is shown that $\|x, z\| = (x, x \mid z)^{\frac{1}{2}}$ is a $2-$norm on $(X, \|\cdot,\cdot\|)$. Every $2-$inner product space will be considered to be a linear $2-$normed space with the $2-$norm

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$\|x,z\| = (x,x \mid z)^{\frac{1}{2}}$. R. Ehret, [9], has shown that for any $2-$inner product space $(X, (\cdot,\cdot \mid \cdot))$, $\|x,z\| = (x,x \mid z)^{\frac{1}{2}}$ defines a 2-norm for which

$$(x,y \mid z) = \frac{1}{4}\left(\|x+y,z\|^2 - \|x-y,z\|^2\right), \tag{1.1}$$

$$\|x+y,z\|^2 + \|x-y,z\|^2 = 2\left(\|x,z\|^2 + \|y,z\|^2\right). \tag{1.2}$$

Besides, if $(X, \|\cdot,\cdot\|)$ is a linear $2-$normed space in which condition (1.2), being a 2-dimensional analogue of the parallelogram law, is satisfied for every $x,y,z \in X$, then a 2-inner product on X is defined on by (1.1).

For a 2-inner product space $(X, (\cdot,\cdot \mid \cdot))$ Cauchy-Schwarz's inequality

$$|(x,y \mid z)| \leq (x,x \mid z)^{\frac{1}{2}} (y,y \mid z)^{\frac{1}{2}} = \|x,z\| \, \|y,z\|,$$

a $2-$dimensional analogue of Cauchy-Buniakowski-Schwarz's inequality, holds (cf. [3]).

2. Refinements of Cauchy-Schwarz's Inequality

Throughout this paper, let $(X, (\cdot,\cdot \mid \cdot))$ denote a $2-$inner product space with $\|x,z\| = (x,x \mid z)^{\frac{1}{2}}$, $\mathbf{R}$ the set of real numbers and $\mathbf{N}$ the set of natural numbers.

Theorem 2.1. *Let $x,y,z,u,v \in X$ with $z \notin V(x,y,u,v)$ be such that*

$$\|u,z\|^2 \leq 2(x,u \mid z), \qquad \|v,z\|^2 \leq 2(y,v \mid z). \tag{2.1}$$

Then, we have the inequality

$$\begin{aligned} &\left(2(x,u \mid z) - \|u,z\|^2\right)^{\frac{1}{2}} \left(2(y,v \mid z) - \|v,z\|^2\right)^{\frac{1}{2}} \\ &+ |(x,y \mid z) - (x,v \mid z) - (u,y \mid z) + (u,v \mid z)| \leq \|x,z\| \, \|y,z\|. \end{aligned} \tag{2.2}$$

Proof. Note that

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \leq (mp - nq)^2 \tag{2.3}$$

for every $m,n,p,q \in \mathbf{R}$. Since

$$\begin{aligned} &|(x,y \mid z) - (x,v \mid z) - (u,y \mid z) + (u,v \mid z)|^2 \\ &= |(x-u, y-v \mid z)|^2 \leq \|x-u,z\|^2 \, \|y-v,z\|^2 \\ &= \left(\|x,z\|^2 + \|u,z\|^2 - 2(x,u \mid z)\right)\left(\|y,z\|^2 + \|v,z\|^2 - 2(y,v \mid z)\right), \end{aligned}$$

by (2.3), we have

$$\begin{aligned} &|(x,y \mid z) - (x,v \mid z) - (u,y \mid z) + (u,v \mid z)|^2 \\ &\leq \left\{\|x,z\| \, \|y,z\| - \left(2(x,u \mid z) - \|u,z\|^2\right)^{\frac{1}{2}} \left(2(y,v \mid z) - \|v,z\|^2\right)^{\frac{1}{2}}\right\}^2. \end{aligned} \tag{2.4}$$

On the other hand

$$0 \leq \left(2(x,u \mid z) - \|u,z\|^2\right)^{\frac{1}{2}} \leq \|x,z\|,$$

$$0 \leq \left(2(y,v \mid z) - \|v,z\|^2\right)^{\frac{1}{2}} \leq \|y,z\|,$$

which imply

$$\left(2\left(x,u\mid z\right)-\left\|u,z\right\|^{2}\right)^{\frac{1}{2}}\left(2\left(y,v\mid z\right)-\left\|v,z\right\|^{2}\right)^{\frac{1}{2}}\leq\left\|x,z\right\|\left\|y,z\right\|.$$

Therefore, from (2.4), we have the inequality (2.2). This completes the proof. ∎

Corollary 2.2. *Let $x,y,z,e\in X$ be such that $\|e,z\|=1$ and $z\notin V\left(x,y,e\right)$. Then*

$$|(x,y\mid z)|\leq|(x,y\mid z)-(x,e\mid z)(e,y\mid z)| \tag{2.5}$$

$$+\left|(x,e\mid z)(e,u\mid z)\right|\leq\left\|x,z\right\|\left\|y,z\right\|.$$

Proof. If we put $u=(x,e\mid z)\,e$ and $v=(y,e\mid z)\,e$, then the conditions (2.1) hold. In fact,

$$\begin{aligned}2\left(x,u\mid z\right)-\left\|u,z\right\|^{2}&=2\left(x,\left(x,e\mid z\right)e\mid z\right)-\left\|\left(x,e\mid z\right)e,z\right\|^{2}\\&=2\left(x,e\mid z\right)\left(x,e\mid z\right)-\left(x,e\mid z\right)^{2}=\left(x,e\mid z\right)\left(x,e\mid z\right)\geq 0.\end{aligned}$$

And similarly for the second condition in (2.1).

Moreover,

$$\begin{aligned}&|(x,y\mid z)-(x,v\mid z)-(u,y\mid z)+(u,v\mid z)|\\&=|(x,y\mid z)-(x,e\mid z)(y,e\mid z)-(x,e\mid z)(e,y\mid z)+(x,e\mid z)(y,e\mid z)|\\&=|(x,y\mid z)-(x,e\mid z)(e,y\mid z)|\end{aligned}$$

so, by Theorem 2.1, we have (2.5). ∎

Corollary 2.3. *Let $x,y,z\in X$ be such that $\|x,z\|^{2}\leq 2,\|y,z\|^{2}\leq 2$ and $z\notin V\left(x,y\right)$. Then*

$$|(x,y\mid z)|^{2}\left(2-\|x,z\|^{2}\right)^{\frac{1}{2}}\left(2-\|y,z\|^{2}\right)^{\frac{1}{2}} \tag{2.6}$$

$$+\left|(x,y\mid z)\right|\left|1-\|x,z\|^{2}-\|y,z\|^{2}+(x,y\mid z)\right|^{2}\leq\|x,z\|\,\|y,z\|.$$

Proof. If we put $u=(x,y\mid z)\,y$ and $v=(y,x\mid z)\,x$, then the inequality (2.3) holds. Moreover, we have

$$\begin{aligned}&\left(2\left(x,u\mid z\right)-\left\|u,z\right\|^{2}\right)^{\frac{1}{2}}\left(2\left(y,v\mid z\right)-\left\|v,z\right\|^{2}\right)^{\frac{1}{2}}\\&=|(x,y\mid z)|^{2}\left(2-\|x,z\|^{2}\right)^{\frac{1}{2}}\left(2-\|y,z\|^{2}\right)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}&|(x,y\mid z)-(x,v\mid z)-(u,y\mid z)+(u,v\mid z)|\\&=|(x,y\mid z)|\left|1-\|x,z\|^{2}-\|y,z\|^{2}+|(x,y\mid z)|^{2}\right|.\end{aligned}$$

Therefore, by Theorem 2.1, we have the inequality (2.6). ∎

Theorem 2.4. *Let $x,y,z,e\in X$ be such that $\|e,z\|=1$ and $z\notin V\left(x,y,e\right)$. Then*

$$|(x,y\mid z)-(x,e\mid z)(e,y\mid z)|^{2} \tag{2.7}$$

$$\leq\left(\|x,z\|^{2}-|(x,e\mid z)|^{2}\right)\left(\|y,z\|^{2}-|(y,e\mid z)|^{2}\right).$$

Proof. Consider a mapping $P:X\times X\times X\rightarrow\mathbf{R}$ defined by $P\left(x,y,z\right)=(x,y\mid z)-(x,e\mid z)\,(e,y\mid z)$ for every $x,y,z,e\in X$, having the properties:

(i) $P(x,x,z) \geq 0,$
(ii) $P(\alpha x + \beta x', y, z) = P(x,y,z) + \beta P(x', y, z),$
(iii) $P(x,y,z) = P(y,x,z).$

Then Cauchy-Schwarz's inequality

$$|P(x,y,z)|^2 \leq P(x,x,z)\, P(y,y,z) \tag{2.8}$$

holds.

Indeed, we observe that

$$0 \leq P(x + \alpha P(x,y,z)\, y, x + \alpha P(x,y,z)\, y, z)$$

$$= P(x,x,z) + 2\alpha P(x,y,z)^2 + \alpha^2 P(x,y,z)^2 P(y,y,z) \quad (\forall)\, \alpha \in \mathbf{R}.$$

It is well known that if $a \geq 0$ and

$$a\alpha^2 + \beta\alpha + c \geq 0 \quad \text{for all } \alpha \in \mathbf{R},$$

then $\Delta = b^2 - 4ac \leq 0.$

Then by the above inequality we deduce

$$P(x,y,z)^4 \leq P(x,x,z)\, P(y,y,z)\, P(x,y,z)^2. \tag{2.9}$$

If $P(x,y,z) = 0$ then (2.8) holds.

If $P(x,y,z) \neq 0$ then we can devide in (2.9) by $P(x,y,z)$ and obtain (2.8).

The theorem is thus proved. ∎

Remark 2.1. *By the inequalities* (2.3) *and* (2.7), *we have*

$$|(x,y \mid z) - (x,e \mid z)(e,y \mid z)|^2$$

$$\leq \left(\|x,z\|^2 - |(x,e \mid z)|^2\right)\left(\|y,z\|^2 - |(y,e \mid z)|^2\right)$$

$$\leq (\|x,z\|\,\|y,z\| - |(x,e \mid z)(e,y \mid z)|)^2.$$

Since $\|x,z\|\,\|y,z\| \geq |(x,e \mid z)(e,y \mid z)|,$ we get

$$|(x,y \mid z) - (x,e \mid z)(e,y \mid z)| \leq \|x,z\|\,\|y,z\| - |(x,e \mid z)(e,y \mid z)|,$$

which yields the inequality (2.5).

Corollary 2.5. *Let $x,y,z,e \in X$ be such that $\|e,z\| = 1$ and $z \notin V(x,y,e)$. Then*

$$\left(\|x+y,z\|^2 - |(x+y,e \mid z)|^2\right)^{\frac{1}{2}} \tag{2.10}$$

$$\leq \left(\|x,z\|^2 - |(x,e \mid z)|^2\right)^{\frac{1}{2}} + \left(\|y,z\|^2 - |(y,e \mid z)|^2\right)^{\frac{1}{2}}.$$

Proof. If we define $S : X \times X \to \mathbf{R}$ by $S(x,z) = P(x,x,z)^{\frac{1}{2}}$ for every $x,y \in X$ and use the triangle inequality for $S(x,z)$, then we have (2.10). ∎

Corollary 2.6. *For every non-zero $x,y,z,u \in X$, with $z \notin V(x,y,u)$, we have*

$$\left|\frac{(x,y \mid z)}{\|x,z\|\,\|y,z\|}\right|^2 + \left|\frac{(y,u \mid z)}{\|y,z\|\,\|u,z\|}\right|^2 + \left|\frac{(u,x \mid z)}{\|u,z\|\,\|x,z\|}\right|^2 \tag{2.11}$$

$$\leq 1 + 2\left|\frac{(x,y \mid z)(y,u \mid z)(u,x \mid z)}{\|x,z\|^2\,\|y,z\|^2\,\|u,z\|^2}\right|.$$

For the proof of next theorem, we need the following lemma:

Lemma 2.7. *For every non-zero $x, y, z \in X$ with $z \notin V(x, y)$, we have*

$$(\|x, z\| + \|y, z\|) \left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\| \leq 2 \|x - y, z\| . \tag{2.12}$$

Proof. Since

$$\frac{\|x, z\|}{\|y, z\|} + \frac{\|y, z\|}{\|x, z\|} \geq 2,$$

we have the inequality

$$\begin{aligned}(\|x, z\| + \|y, z\|)^2 - (x, y \mid z) \left(\frac{\|x, z\|}{\|y, z\|} + \frac{\|y, z\|}{\|x, z\|} \right) - 2 (x, y \mid z) \\ \leq 2 \|x, z\|^2 + \|y, z\|^2 - 4 (x, y \mid z)\end{aligned}$$

which implies (2.12). ∎

Theorem 2.8. *For every non-zero $x, y, z \in X$ with $z \notin V(x, y)$ we have*

$$\begin{aligned}(\|x, z\| + \|y, z\|)^2 \left(\left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\|^2 + \left\| \frac{x}{\|x, z\|} + \frac{y}{\|y, z\|}, z \right\|^2 \right) \\ \leq 8 \left(\|x, z\|^2 + \|y, z\|^2 \right).\end{aligned} \tag{2.13}$$

Proof. By (2.12) we have

$$\begin{aligned}&(\|x, z\| + \|y, z\|)^2 \left(\left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\|^2 + \left\| \frac{x}{\|x, z\|} + \frac{y}{\|y, z\|}, z \right\|^2 \right) \\ \leq \quad & 4 \left(\|x - y, z\|^2 + \|x + y, z\|^2 \right)\end{aligned}$$

and, by a 2−dimensional analogue of the parallelogram law, we get (2.13). ∎

Remark 2.2. *For some similar results in inner product spaces, see* [7].

3. Aćzel's Inequality

In this section, we shall point out some results in 2−inner product spaces in connection to Aćzel's inequality [12]. For some other similar results in inner products, see [8]. We note that the results obtained here, in 2−inner product spaces used different techniques as those in [8].

Theorem 3.1. *Let $(X, (\cdot, \cdot \mid \cdot))$ be a 2−inner product space, $M_1, M_2 \in \mathbf{R}$ and $x, y, z \in X$ such that*

$$\|x, z\| \leq |M_1|, \qquad \|y, z\| \leq |M_2|,$$

then

$$\left(M_1^2 - \|x, z\|^2 \right) \left(M_2^2 - \|y, z\|^2 \right) \leq (|M_1 M_2| - (x, y \mid z))^2 . \tag{3.1}$$

Proof. Using the elementary inequality (2.3), we get

$$0 \leq \left(M_1^2 - \|x,z\|^2\right)\left(M_2^2 - \|y,z\|^2\right) \leq \left(|M_1 M_2| - \|x,z\| \|y,z\|\right)^2,$$

and by Cauchy-Schwarz's inequality,

$$0 \leq |M_1 M_2| - \|x,z\| \|y,z\| \leq |M_1 M_2| - (x, y \mid z)$$

implying (3.1). ∎

Corollary 3.2. *If $x, y, z \in X$, are such that $\|x,z\|, \|y,z\| \leq M, M > 0$, then we have the inequality*

$$0 \leq \|x,z\|^2 \|y,z\|^2 - (x, y \mid z)^2 \leq M^2 \|x - y, z\|^2 \tag{3.2}$$

which is a counterpart of Cauchy-Schwarz's inequality.

Another similar results to the generalization (3.1) of Aćzel's inequality is the following one

Theorem 3.3. *Let $(X, (\cdot, \cdot \mid \cdot))$ be a $2-$inner product space, and $M_1, M_2 \in \mathbf{R}$ and $x, y, z \in X$ such that $\|x,z\| \leq |M_1|, \quad \|y,z\| \leq |M_2|$. Then*

$$(|M_1| - \|x,z\|)^{\frac{1}{2}} (|M_2| - \|y,z\|)^{\frac{1}{2}} \leq |M_1 M_2|^{\frac{1}{2}} - |(x, y \mid z)|^{\frac{1}{2}}. \tag{3.3}$$

Proof. Applying (2.3) for $m = \sqrt{|M_1|}, \quad p = \sqrt{|M_2|}, \quad n = \sqrt{\|x,z\|}, \quad q = \sqrt{\|y,z\|}$ and using Cauchy-Schwarz's inequality for $2-$inner products we deduce (3.3). ∎

Corollary 3.4. *Suppose that $x, y, z \in X$ and $M > 0$ are such that $\|x,z\|, \|y,z\| \leq M$. Then we have the following converse of Cauchy-Schwarz's inequality*

$$\begin{aligned} 0 &\leq \|x,z\| \|y,z\| - |(x, y \mid z)| \\ &\leq M \left(\|x,z\| + \|y,z\| - 2 |(x, y \mid z)|^{1/2}\right). \end{aligned} \tag{3.4}$$

Theorem 3.5. *Let $(\cdot, \cdot \mid \cdot)$ be a $2-$inner product and $\{(\cdot, \cdot \mid \cdot)_i\}_{i \in \mathbf{N}}$ a sequence of $2-$inner products satisfying*

$$\|x,z\|^2 > \sum_{i=0}^{\infty} \|x,z\|_i^2 \tag{3.5}$$

for all x, z, being linearly independent. Then we have the following refinemenet of Cauchy-Schwarz's inequality

$$\|x,z\| \|y,z\| - |(x, y \mid z)| \tag{3.6}$$

$$\geq \left[\sum_{i=0}^{\infty} \|x,z\|_i \sum_{i=0}^{\infty} \|y,z\|_i - |(x, y \mid z)|\right] \geq 0 \tag{3.7}$$

for all $x, y, z \in X$.

Proof. Let $n \in \mathbf{N}$ and $n \geq 1$. Define the mapping

$$(x, y \mid z)_n = (x, y \mid z) - \sum_{i=0}^{n} (x, y \mid z)_i, \qquad x, y, z \in X.$$

We observe, by (3.5), that the mapping $(\cdot, \cdot \mid \cdot)_n$ satisfies the properties

(i) $(x, x \mid z)_n \geq 0$,
(ii) $(\alpha x + \beta x', y \mid z)_n = \alpha (x, y \mid z)_n + \beta (x', y \mid z)_n$,

(iii) $(x, y \mid z)_n = (y, x \mid z)_n$
for every $x, x', y, z \in X$ and $\alpha, \alpha' \in \mathbf{R}$.

By a similar proof to that in Theorem 2.4, we can state Cauchy-Schwarz's inequality

$$(x, x \mid z)_n (y, y \mid z)_n \geq |(x, y \mid z)_n|^2, \quad x, y, z \in X,$$

that is

$$\left(\|x,z\|^2 - \sum_{i=0}^{n} \|x,z\|_i^2\right)\left(\|y,z\|^2 - \sum_{i=0}^{n} \|y,z\|_i^2\right) \geq \left((x, y \mid z) - \sum_{i=0}^{n} (x, y \mid z)_i\right)^2. \tag{3.8}$$

Using Aćzel's inequality [12]

$$\left(a^2 - \sum_{i=0}^{m} a_i^2\right)\left(b^2 - \sum_{i=0}^{m} b_i^2\right) \leq \left(ab - \sum_{i=0}^{m} a_i b_i\right)^2,$$

where $a, b, a_i, b_i \in \mathbf{R}$ for $i = 0, ..., m$; we can prove that

$$\left(\|x,z\| \|y,z\| - \sum_{i=0}^{n} \|x,z\|_i \|y,z\|_i\right)^2 \geq \left(\|x,z\|^2 - \sum_{i=0}^{n} \|x,z\|_i^2\right)\left(\|y,z\|^2 - \sum_{i=0}^{n} \|y,z\|_i^2\right) \tag{3.9}$$

for all $x, y, z \in X$. Since, by Cauchy-Buniakowski-Schwarz's inequality

$$\|x,z\| \|y,z\| \geq \left(\sum_{i=0}^{n} \|x,z\|_i^2 \sum_{i=0}^{n} \|y,z\|_i^2\right)^{1/2} \geq \sum_{i=0}^{n} \|x,z\|_i \|y,z\|_i,$$

then by (3.8) and (3.9) we deduce

$$\|x,z\| \|y,z\| - \sum_{i=0}^{n} \|x,z\|_i \|y,z\|_i$$

$$= \left|\|x,z\| \|y,z\| - \sum_{i=0}^{n} \|x,z\|_i \|y,z\|_i\right| \geq |(x, y \mid z)| - \sum_{i=0}^{n} |(x, y \mid z)_i|$$

which implies (3.6), by using the inequality

$$\|x,z\|_i \|y,z\|_i - |(x, y \mid z)_i| \geq 0.$$

The theorem is thus proved. ∎

The following corollaries are interesting as refinements of the triangle inequality for 2-norms generated by 2-inner products.

Corollary 3.6. *With the assumptions from Theorem, we have the following refinement of the triangle inequality*

$$(\|x,z\| + \|y,z\|)^2 - \|x+y,z\|^2$$

$$\geq \sum_{i=0}^{\infty} \left[(\|x,z\|_i + \|y,z\|_i)^2 - \|x+y,z\|_i^2 \right] \geq 0, x, y, z \in X.$$

Corollary 3.7. *Let $(.,. \mid .)_1, (.,. \mid .)_2$ be two 2-inner products such that*

$$\|x,z\|_2 > \|x,z\|_1$$

for all x, z being linearly independent in X. Then

$$\|x,z\|_2 \|y,z\|_2 - |(x,y \mid z)_2|$$
$$\geq \|x,z\|_1 \|y,z\|_1 - |(x,y \mid z)_1| \geq 0, x, y, z \in X.$$

Corollary 3.8. *Let $(.,. \mid .)_1, (.,. \mid .)_2$ be as above. Then*

$$(\|x,z\|_2 + \|y,z\|_2)^2 - \|x+y,z\|_2^2$$

$$\geq (\|x,z\|_1 + \|y,z\|_1)^2 - \|x+y,z\|_1^2 \geq 0, x, y, z \in X.$$

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Department of Mathematics, Gyeonsang National University, Chinju 660-701, Korea
E-mail address, Y.J. Cho: `yjcho@nongae.gsnu.ac.kr`

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address, S.S. Dragomir: `sever@matilda.vut.edu.au`

Department of Mathematics, St. Bonaventure University, St. Bonaventure, NY 14778, U.S.A.

Department of Mathematics, Dongeui University, Pusan 614-714