SOME INEQUALITIES IN 2–INNER PRODUCT SPACES

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Abstract. In this paper we extend some results on the refinement of Cauchy-Buniakowski-Schwarz’s inequality and Aćzel’s inequality in inner product spaces to 2–inner product spaces.

1. Introduction

Let $X$ be a real linear space of dimension greater than 1 and let $\|\cdot, \cdot\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

$(N_1)$ $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent;
$(N_2)$ $\|x, y\| = \|y, x\|$;
$(N_3)$ $\|\alpha x, y\| = |\alpha| \|x, y\|$ for any real number $\alpha$;
$(N_4)$ $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

$\|\cdot, \cdot\|$ is called a 2–norm on $X$ and $(X, \|\cdot, \cdot\|)$ a linear 2–normed space cf. [10].

Some of the basic properties of the 2-norms are that they are nonnegative, and $\|x, y + \alpha x\| = \|x, y\|$ for every $x, y$ in $X$ and every real number $\alpha$.

For any non-zero $x_1, x_2, ..., x_n$ in $X$, let $V(x_1, x_2, ..., x_n)$ denote the subspace of $X$ generated by $x_1, x_2, ..., x_n$. Whenever the notation $V(x_1, x_2, ..., x_n)$ is used, we will understand that $x_1, x_2, ..., x_n$ are linearly independent.

A concept which is closely related to linear 2-normed space is that of 2 inner product spaces. For a linear space $X$ of dimension greater than 1 let $(\cdot, \cdot | \cdot)$ be a real-valued function on $X \times X \times X$ which satisfies the following conditions:

$(I_1)$ $(x, x | z) \geq 0; (x, x | z) = 0$ if and only if $x$ and $z$ are linearly dependent;
$(I_2)$ $(x, x | z) = (z, z | x)$;
$(I_3)$ $(x, y | z) = (y, x | z)$;
$(I_4)$ $(\alpha x, y | z) = \alpha (x, y | z)$ for any real number $\alpha$;
$(I_5)$ $(x + x', y | z) = (x, y | z) + (x', y | z)$.

$(\cdot, \cdot | \cdot)$ is called a 2–inner product and $(X, (\cdot, \cdot | \cdot))$ a 2–inner product space ([3]).

These spaces are studied extensively in [1], [2], [4]–[6] and [11]. In [3] it is shown that $\|x, z\| = (x, x | z)^2$ is a 2–norm on $(X, (\cdot, \cdot | \cdot))$. Every 2–inner product space will be considered to be a linear 2–normed space with the 2–norm

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\[ \|x, z\| = (x, x \mid z)^{\frac{1}{2}}. \]  
R. Ehret, [9], has shown that for any 2-inner product space \( (X, \langle \cdot, \cdot \rangle) \), \( \|x, z\| = (x, x \mid z)^{\frac{1}{2}} \) defines a 2-norm for which

\[
(x, y \mid z) = \frac{1}{4} \left( \|x + y, z\|^2 - \|x - y, z\|^2 \right),
\]

(1.1)

\[
\|x + y, z\|^2 + \|x - y, z\|^2 = 2 \left( \|x, z\|^2 + \|y, z\|^2 \right).
\]

(1.2)

Besides, if \( (X, \| \cdot \|) \) is a linear 2-normed space in which condition (1.2), being a 2-dimensional analogue of the parallelogram law, is satisfied for every \( x, y, z \in X \), then a 2-inner product on \( X \) is defined on by (1.1).

For a 2-inner product space \( (X, \langle \cdot, \cdot \rangle) \) Cauchy-Schwarz’s inequality

\[
|\langle x, y \mid z \rangle| \leq (x, x \mid z)^{\frac{1}{2}} (y, y \mid z)^{\frac{1}{2}} = \|x, z\| \|y, z\|,
\]

a 2-dimensional analogue of Cauchy-Buniakowski-Schwarz’s inequality, holds (cf. [3]).

2. Refinements of Cauchy-Schwarz’s Inequality

Throughout this paper, let \( (X, \langle \cdot, \cdot \rangle) \) denote a 2-inner product space with \( \|x, z\| = (x, x \mid z)^{\frac{1}{2}}, \mathbb{R} \) the set of real numbers and \( \mathbb{N} \) the set of natural numbers.

**Theorem 2.1.** Let \( x, y, z, u, v \in X \) with \( z \notin V(x, y, u, v) \) be such that

\[
\|u, z\|^2 \leq 2 (x, u \mid z), \quad \|v, z\|^2 \leq 2 (y, v \mid z).
\]

Then, we have the inequality

\[
n\left( 2 (x, u \mid z) - \|u, z\|^2 \right)^{\frac{1}{2}} \left( 2 (y, v \mid z) - \|v, z\|^2 \right)^{\frac{1}{2}}
\]

\[
+ |\langle x, y \mid z \rangle - (x, v \mid z) - (u, y \mid z) + (u, v \mid z)| \leq \|x, z\| \|y, z\|.
\]

**Proof.** Note that

\[
(m^2 - n^2) (p^2 - q^2) \leq (mp - nq)^2
\]

for every \( m, n, p, q \in \mathbb{R} \). Since

\[
|\langle x, y \mid z \rangle - (x, v \mid z) - (u, y \mid z) + (u, v \mid z)|^2
\]

\[
= |(x - u, y - v \mid z)|^2 \leq \|x - u, z\|^2 \|y - v, z\|^2
\]

\[
= \left( \|x, z\|^2 + \|u, z\|^2 - 2 (x, u \mid z) \right) \left( \|y, z\|^2 + \|v, z\|^2 - 2 (y, v \mid z) \right),
\]

by (2.3), we have

\[
|\langle x, y \mid z \rangle - (x, v \mid z) - (u, y \mid z) + (u, v \mid z)|^2
\]

\[
\leq \left\{ \|x, z\| \|y, z\| - \left( 2 (x, u \mid z) - \|u, z\|^2 \right)^{\frac{1}{2}} \left( 2 (y, v \mid z) - \|v, z\|^2 \right)^{\frac{1}{2}} \right\}^2.
\]

On the other hand

\[
0 \leq \left( 2 (x, u \mid z) - \|u, z\|^2 \right)^{\frac{1}{2}} \leq \|x, z\|,
\]

\[
0 \leq \left( 2 (y, v \mid z) - \|v, z\|^2 \right)^{\frac{1}{2}} \leq \|y, z\|,
\]
which imply
\[
\left(2(x,u \mid z) - \|u,z\|^2\right)^\frac{1}{2} \left(2(y,v \mid z) - \|v,z\|^2\right)^\frac{1}{2} \leq \|x,z\| \|y,z\|.
\]

Therefore, from (2.4), we have the inequality (2.2). This completes the proof. \[\blacksquare\]

**Corollary 2.2.** Let \(x, y, z, e \in X\) be such that \(\|e, z\| = 1\) and \(z \notin V(x, y, e)\). Then
\[
(x,y \mid z) \leq (x,y \mid z) - (x,e \mid z)(e,y \mid z)
\]

and similarly for the second condition in (2.1).

**Proof.** If we put \(u = (x,e \mid z)e\) and \(v = (y,e \mid z)e\), then the conditions (2.1) hold. In fact,
\[
2(x,u \mid z) - \|u,z\|^2 = 2(x,e \mid z)e - \|(x,e \mid z)e,z\|^2
\]
\[
= 2(x,e \mid z)(x,e \mid z) - (x,e \mid z)^2 = (x,e \mid z)(x,e \mid z) \geq 0.
\]

And similarly for the second condition in (2.1).

Moreover,
\[
(x,y \mid z) - (x,v \mid z) - (u,y \mid z) + (u,v \mid z)
\]
\[
= x,y \mid z) - (x,e \mid z)(y,e \mid z) - (x,e \mid z)(e,y \mid z) + (x,e \mid z)(y,e \mid z)
\]
\[
= x,y \mid z) - (x,e \mid z)(e,y \mid z)
\]

so, by Theorem 2.1, we have (2.5). \[\blacksquare\]

**Corollary 2.3.** Let \(x, y, z \in X\) be such that \(\|x\|^2 \leq 2, \|y, z\|^2 \leq 2\) and \(z \notin V(x, y)\). Then
\[
\left|\left(2 - \|y, z\|^2\right)^{\frac{1}{2}} \left(2 - \|y, z\|^2\right)^{\frac{1}{2}}\right|
\]
\[
\|x,y \mid z\|\left[1 - \|x,y \mid z\|^2 - \|y, z\|^2 + (x, y \mid z)\right] \leq \|x, z\| \|y, z\|.
\]

**Proof.** If we put \(u = (y,e \mid z)y\) and \(v = (x, y \mid z)x\), then the inequality (2.3) holds. Moreover, we have
\[
\left(2(x,u \mid z) - \|u,z\|^2\right)^\frac{1}{2} \left(2(y,v \mid z) - \|v,z\|^2\right)^\frac{1}{2}
\]
\[
= (x,y \mid z)^2 \left(2 - \|x, z\|^2\right)^\frac{1}{2} \left(2 - \|y, z\|^2\right)^\frac{1}{2},
\]
\[
|(x,y \mid z) - (x,v \mid z) - (u,y \mid z) + (u,v \mid z)|
\]
\[
= (x,y \mid z)\left[1 - \|x, z\|^2 - \|y, z\|^2 + (x, y \mid z)\right].
\]

Therefore, by Theorem 2.1, we have the inequality (2.6). \[\blacksquare\]

**Theorem 2.4.** Let \(x, y, z, e \in X\) be such that \(\|e, z\| = 1\) and \(z \notin V(x, y, e)\). Then
\[
\left|x,y \mid z\right| - (x,e \mid z)(e,y \mid z)
\]
\[
\leq \left(\|x, z\|^2 - \|x,e \mid z\|^2\right) \left(\|y, z\|^2 - \|y,e \mid z\|^2\right).
\]

**Proof.** Consider a mapping \(P : X \times X \times X \to \mathbb{R}\) defined by \(P(x,y,z) = (x,y \mid z) - (x,e \mid z)(e,y \mid z)\) for every \(x, y, z, e \in X\), having the properties:
Corollary 2.5. Let \( P(x, y, z) \geq 0 \),
(ii) \( P(\alpha x + \beta x', y, z) = P(x, y, z) + \beta P(x', y, z) \),
(iii) \( P(x, y, z) = P(y, x, z) \).

Then Cauchy-Schwarz's inequality
\[
|P(x, y, z)|^2 \leq P(x, x, z) P(y, y, z)
\]
holds.

Indeed, we observe that
\[
0 \leq P(x + \alpha P(x, y, z) y, x + \alpha P(x, y, z) y, z)
= P(x, x, z) + 2\alpha P(x, y, z)^2 + \alpha^2 P(x, y, z)^2 \quad (\forall \alpha \in \mathbb{R}.
\]
It is well known that if \( a \geq 0 \) and
\[
aa^2 + \beta a + c \geq 0 \quad \text{for all } a \in \mathbb{R},
\]
then \( \Delta = \beta^2 - 4ac \leq 0 \).

Then by the above inequality we deduce
\[
P(x, y, z)^4 \leq P(x, x, z) P(y, y, z) P(x, y, z)^2.
\]
If \( P(x, y, z) = 0 \) then (2.8) holds.

If \( P(x, y, z) \neq 0 \) then we can divide in (2.9) by \( P(x, y, z) \) and obtain (2.8).

The theorem is thus proved.

Remark 2.1. By the inequalities (2.3) and (2.7), we have
\[
\|(x, y | z) - (x, e | z) (e, y | z)\|^2
\leq \left(\|x, z\|^2 - |(x, e | z)|^2\right) \left(\|y, z\|^2 - |(y, e | z)|^2\right)
\leq (\|x, z\|\|y, z\| - |(x, e | z) (e, y | z)|)^2.
\]

Since \( \|x, z\|\|y, z\| \geq |(x, e | z) (e, y | z)| \), we get
\[
|(x, y | z) - (x, e | z) (e, y | z)| \leq \|x, z\|\|y, z\| - |(x, e | z) (e, y | z)|,
\]
which yields the inequality (2.5).

Corollary 2.5. Let \( x, y, z, e \in X \) be such that \( \|e, z\| = 1 \) and \( z \notin V(x, y, e) \). Then
\[
(\|x + y, z\|^2 - |(x, e | z)|^2)^{\frac{1}{2}}
\leq \left(\|x, z\|^2 - |(x, e | z)|^2\right)^{\frac{1}{2}} + \left(\|y, z\|^2 - |(y, e | z)|^2\right)^{\frac{1}{2}}.
\]

Proof. If we define \( S : X \times X \to \mathbb{R} \) by \( S(x, z) = P(x, x, z)^{\frac{1}{2}} \) for every \( x, y \in X \)
and use the triangle inequality for \( S(x, z) \), then we have (2.10).

Corollary 2.6. For every non-zero \( x, y, z, u \in X \), with \( z \notin V(x, y, u) \), we have
\[
\left(\|x, y | z\|^2 \|y, z\|^2 \|y, z\|^2 + \|x, z\|^2 \|y, z\|^2 \|u, z\|^2 + \|u, z\|^2 \|x, z\|^2 \|y, z\|^2 \|u, z\|^2 \|x, z\|^2 \right)^{\frac{1}{2}}
\leq 1 + 2 \left(\|x, y | z\| \|y, z\| \|u, x | z\| \|u, z\|^2 \|y, z\|^2 \|u, z\|^2 \right)^{\frac{1}{2}}.
\]
For the proof of next theorem, we need the following lemma:

Lemma 2.7. For every non-zero $x, y, z \in X$ with $z \notin V(x, y)$, we have

\begin{equation}
(\|x, z\| + \|y, z\|) \left(\frac{x}{\|x, z\|} - \frac{y}{\|y, z\|} \cdot z\right) \leq 2\|x - y, z\|.
\end{equation}

Proof. Since

\[ \frac{\|x, z\|}{\|y, z\|} + \frac{\|y, z\|}{\|x, z\|} \geq 2, \]

we have the inequality

\begin{align*}
(\|x, z\| + \|y, z\|)^2 - (x, y) \left( \frac{\|x, z\|}{\|y, z\|} + \frac{\|y, z\|}{\|x, z\|} \right) - 2(x, y)z \\
\leq 2\|x, z\|^2 + \|y, z\|^2 - 4(x, y)z
\end{align*}

which implies (2.12).

Theorem 2.8. For every non-zero $x, y, z \in X$ with $z \notin V(x, y)$ we have

\begin{equation}
(\|x, z\| + \|y, z\|)^2 \left(\left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|} \cdot z \right\|^2 + \left\| \frac{x}{\|x, z\|} + \frac{y}{\|y, z\|} \cdot z \right\|^2 \right) \\
\leq 8 \left(\|x, z\|^2 + \|y, z\|^2 \right).
\end{equation}

Proof. By (2.12) we have

\begin{align*}
(\|x, z\| + \|y, z\|)^2 \left(\left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|} \cdot z \right\|^2 + \left\| \frac{x}{\|x, z\|} + \frac{y}{\|y, z\|} \cdot z \right\|^2 \right) \\
\leq 4 \left(\|x - y, z\|^2 + \|x + y, z\|^2 \right)
\end{align*}

and, by a 2-dimensional analogue of the parallelogram law, we get (2.13).

Remark 2.2. For some similar results in inner product spaces, see [7].

3. Aczél’s Inequality

In this section, we shall point out some results in 2-inner product spaces in connection to Aczél’s inequality [12]. For some other similar results in inner products, see [8]. We note that the results obtained here, in 2-inner product spaces used different techniques as those in [8].

Theorem 3.1. Let $(X, (\cdot, \cdot | \cdot))$ be a 2-inner product space, $M_1, M_2 \in \mathbb{R}$ and $x, y, z \in X$ such that

\[ \|x, z\| \leq |M_1|, \quad \|y, z\| \leq |M_2|, \]

then

\begin{equation}
(M_1^2 - \|x, z\|^2) \left( M_2^2 - \|y, z\|^2 \right) \leq (|M_1M_2| - (x, y)z)^2.
\end{equation}
Proof. Using the elementary inequality (2.3), we get
\[ 0 \leq \left( M_1^2 - \|x, z\|^2 \right) \left( M_2^2 - \|y, z\|^2 \right) \leq (\|M_1M_2\| - \|x, z\| \|y, z\|)^2, \]
and by Cauchy-Schwarz's inequality,
\[ 0 \leq \|M_1M_2\| - \|x, z\| \|y, z\| \leq \|M_1M_2\| - (x, y | z) \]
implying (3.1).

**Corollary 3.2.** If \( x, y, z \in X \), are such that \( \|x, z\|, \|y, z\| \leq M, M > 0 \), then we have the inequality
\[ 0 \leq \|x, z\|^2 \|y, z\|^2 - (x, y | z)^2 \leq M^2 \|x - y, z\|^2 \]
which is a counterpart of Cauchy-Schwarz's inequality.

Another similar results to the generalization (3.1) of Aćzel’s inequality is the following one

**Theorem 3.3.** Let \((X, (\cdot, \cdot | \cdot))\) be a 2-inner product space, and \( M_1, M_2 \in \mathbb{R} \) and \( x, y, z \in X \) such that \( \|x, z\| \leq |M_1|, \quad \|y, z\| \leq |M_2| \). Then
\[ (|M_1| - \|x, z\|) \left( |M_2| - \|y, z\| \right) \leq |M_1M_2|^{1/2} - (x, y | z) \]

**Proof.** Applying (2.3) for \( m = \sqrt{|M_1|}, \quad p = \sqrt{|M_2|}, \quad n = \sqrt{\|x, z\|}, \quad q = \sqrt{\|y, z\|} \)
and using Cauchy-Schwarz's inequality for 2-inner products we deduce (3.3).

**Corollary 3.4.** Suppose that \( x, y, z \in X \) and \( M > 0 \) are such that \( \|x, z\|, \|y, z\| \leq M \). Then we have the following converse of Cauchy-Schwarz’s inequality
\[ 0 \leq \|x, z\|^2 \|y, z\|^2 - (x, y | z)^2 \leq M^2 \|x - y, z\|^2 \]

**Theorem 3.5.** Let \((\cdot, \cdot | \cdot)\) be a 2-inner product and \( \{(\cdot, \cdot | \cdot)\}_{i \in \mathbb{N}} \) a sequence of 2-inner products satisfying
\[ \|x, z\|^2 > \sum_{i=0}^{\infty} \|x, z\|^2_i \]
for all \( x, z \), being linearly independent. Then we have the following refinement of Cauchy-Schwarz’s inequality
\[ \|x, z\|^2 \|y, z\|^2 - (x, y | z)^2 \geq \sum_{i=0}^{\infty} \|x, z\|^2_i \|y, z\|^2_i - (x, y | z)^2 \]
\[ \geq 0 \]
for all \( x, y, z \in X \).

**Proof.** Let \( n \in \mathbb{N} \) and \( n \geq 1 \). Define the mapping
\[ (x, y | z)_n = (x, y | z) - \sum_{i=0}^{n-1} (x, y | z)_i, \quad x, y, z \in X. \]
We observe, by (3.5), that the mapping \((\cdot, \cdot | \cdot)_n\) satisfies the properties
(i) \( (x, x | z)_n \geq 0 \),
(ii) \( (\alpha x + \beta x', y | z)_n = \alpha (x, y | z)_n + \beta (x', y | z)_n \),
(iii) \((x, y \mid z)_n = (y, x \mid z)_n\)
for every \(x, x', y, z \in X\) and \(\alpha, \alpha' \in \mathbb{R}\).

By a similar proof to that in Theorem 2.4, we can state Cauchy-Schwarz's inequality
\[
(x, x \mid z)_n (y, y \mid z)_n \geq |(x, y \mid z)_n|^2, \quad x, y, z \in X,
\]
that is
\[
\left(\|x, z\| - \sum_{i=0}^{n} \|x, z\|^2_i \right) \left(\|y, z\| - \sum_{i=0}^{n} \|y, z\|^2_i \right) \geq \left( (x, y \mid z) - \sum_{i=0}^{n} (x, y \mid z)_i \right)^2.
\]

Using Ačzel's inequality [12]
\[
\left( a^2 - \sum_{i=0}^{m} a_i^2 \right) \left( b^2 - \sum_{i=0}^{m} b_i^2 \right) \leq \left( ab - \sum_{i=0}^{m} a_i b_i \right)^2,
\]
where \(a, b, a_i, b_i \in \mathbb{R}\) for \(i = 0, \ldots, m\); we can prove that
\[
\left( \|x, z\| \|y, z\| - \sum_{i=0}^{n} \|x, z\|^2_i \|y, z\|_i \right)^2 \
\geq \left( \|x, z\|^2 - \sum_{i=0}^{n} \|x, z\|^2_i \right) \left( \|y, z\|^2 - \sum_{i=0}^{n} \|y, z\|^2_i \right)
\]
for all \(x, y, z \in X\). Since, by Cauchy-Buniakowski-Schwarz's inequality
\[
\|x, z\| \|y, z\| \geq \left( \sum_{i=0}^{n} \|x, z\|^2_i \sum_{i=0}^{n} \|y, z\|^2_i \right)^{1/2} \geq \sum_{i=0}^{n} \|x, z\|_i \|y, z\|_i,
\]
then by (3.8) and (3.9) we deduce
\[
\|x, z\| \|y, z\| - \sum_{i=0}^{n} \|x, z\|_i \|y, z\|_i \
= \left| \|x, z\| \|y, z\| - \sum_{i=0}^{n} \|x, z\|_i \|y, z\|_i \right| \geq \|(x, y \mid z)\| - \sum_{i=0}^{n} |(x, y \mid z)_i| \geq 0.
\]

The theorem is thus proved.

The following corollaries are interesting as refinements of the triangle inequality for 2-norms generated by 2-inner products.

**Corollary 3.6.** With the assumptions from Theorem, we have the following refinement of the triangle inequality
\[
\left( \|x, z\| + \|y, z\| \right)^2 - \|x + y, z\|^2 
\]
\[ \geq \sum_{i=0}^{\infty} \left( \| x, z \|_i + \| y, z \|_i \right)^2 - \| x + y, z \|_i^2 \geq 0, x, y, z \in X. \]

**Corollary 3.7.** Let \((\cdot, \cdot)_1, (\cdot, \cdot)_2\) be two 2-inner products such that

\[ \| x, z \|_2 \geq \| x, z \|_1 \]

for all \(x, z\) being linearly independent in \(X\). Then

\[ \| x, z \|_2 \| y, z \|_2 - |(x, y | z)|_2 \geq \| x, z \|_1 \| y, z \|_1 - |(x, y | z)|_1 \geq 0, x, y, z \in X. \]

**Corollary 3.8.** Let \((\cdot, \cdot)_1, (\cdot, \cdot)_2\) be as above. Then

\[ (\| x, z \|_2 + \| y, z \|_2)^2 - \| x + y, z \|_2^2 \geq (\| x, z \|_1 + \| y, z \|_1)^2 - \| x + y, z \|_1^2 \geq 0, x, y, z \in X. \]

**References**


