A FRAMEWORK FOR PROVING HILBERT'S DOUBLE INTEGRAL INEQUALITY AND RELATED RESULTS

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Abstract. The classical Hilbert and Hardy integral inequalities are derived within a functional analytic framework. That framework is further used to generate a new inequality, sharper than Hilbert's.

1. Introduction

The inequality of our title is the following celebrated result.

Theorem 1. *Suppose* $p > 1,\ q = \frac{p}{(p-1)},\ f \in L_p(0,\infty)$ *and* $g \in L_q(0,\infty)$. *Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dxdy \ \le\ \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\,\|f\|_p\,\|g\|_q\,. \tag{1.1}$$

The above inequality is the integral analogue of Hilbert's double series theorem. It was first proved by Schur [7] for the case $p = 2$, and as given here by Hardy [2].

We are also concerned with the integral version of Hardy's inequality.

Theorem 2. *Suppose* $p > 1,\ f \in L_p(0,\infty),\ F(x) = \int_0^x f(t)\,dt$. *Then*

$$\left\|x^{-1}F(x)\right\|_p \ \le\ \frac{p}{p-1}\,\|f\|_p\,. \tag{1.2}$$

In [3], Hardy, Littlewood and Polya show how these two inequalities are implied by their Theorem 319. The relevant part of that theorem is:

Theorem 3. *Suppose* $p > 1, q = \frac{p}{p-1}$, *that* $K(x,y)$ *is non-negative and homogeneous of degree -1, and that*

$$\int_0^\infty K(x,1)\,x^{-\frac{1}{p}}\,dx \ =\ \int_0^\infty K(1,y)\,y^{-\frac{1}{q}}\,dy \ =\ C.$$

Then

$$\int_0^\infty\int_0^\infty K(x,y)\,f(x)\,g(y)\,dxdy \ \le\ C\left(\int_0^\infty |f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty |g(y)|^q\,dy\right)^{\frac{1}{q}} \tag{1.3}$$

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and

$$\int_0^\infty dy \left(\int_0^\infty K(x,y)\, f(x)\, dx \right)^p \leq\ C^p \int_0^\infty |f(x)|^p\, dx. \tag{1.4}$$

The aim of this paper is to show how results essentially equivalent to this theorem may be derived in a natural way from considerations of a certain inner product space.

The classical results above have been greatly generalized in recent decades [1], [6]. The approach presented here may not be capable of proving such generalizations. It does, however, seem conducive to generating some novel results. For example, an inequality that implies (1.1), but that is sharper, will be derived.

2. Framework for the Proofs

The general functions considered in this work are assumed to be real-valued and measurable on a domain $(0,\infty)$. For two such functions, u and v, we define the convolution

$$u * v(x)\ =\ \int_0^\infty u\left(\frac{x}{t}\right) v(t)\, \frac{dt}{t}. \tag{2.1}$$

(The integral here is Lebesgue, as are all the following ones.) It has long been realized, [1], that in terms of this convolution, (1.4) is essentially $\|g * f\|_p \leq\ \|g\|_1 \|f\|_p$.

The convolution (2.1) satisfies the commutative property

$$u * v(x) = v * u(x) \tag{2.2}$$

in the sense that, when one side exists, so does the other, and the two sides agree. Likewise, for the associative property

$$[u * (v * w)](x) = [(u * v) * w](x), \tag{2.3}$$

which will be considered in more detail later.

We also employ an inner product

$$\langle u, v \rangle\ =\ \int_0^\infty u(x)\, v(x)\, \frac{dx}{x}. \tag{2.4}$$

This can be related to the convolution (2.1) by

$$\langle u, v \rangle = [(Ru) * v](1), \tag{2.5}$$

using the operator R, defined by

$$Ru(x) = u\left(\frac{1}{x}\right). \tag{2.6}$$

We will also have occasion to use the operator M_a defined for $a > 0$ by

$$M_a u(x) = u(ax). \tag{2.7}$$

Associated with the inner product (2.4) is the norm

$$\nparallel u \nparallel_p = \left[\int_0^\infty |u(x)|^p \frac{dx}{x} \right]^{\frac{1}{p}} . \tag{2.8}$$

Further, $u \in \text{Ł}_p$ means that $\nparallel u \nparallel_p < \infty$. Note the unusual notation. This is used to avoid confusion with the more common norm $\|f\|$ which lacks the $\frac{1}{x}$ factor in the integrand. The following relations are simply proved:

$$\nparallel Ru \nparallel_p = \nparallel u \nparallel_p \tag{2.9}$$

and

$$\nparallel M_a u \nparallel_p = \nparallel u \nparallel_p . \tag{2.10}$$

3. Results

This section establishes some relations connecting the definitions of Section 2.

Theorem 4. *If $p > 1$, $q = \frac{p}{p-1}$, $u \in \text{Ł}_p$, $v \in \text{Ł}_q$, then $u * v(x)$ exists for all $x > 0$ and*

$$|u * v(x)| \leq \nparallel u \nparallel_p \nparallel v \nparallel_q . \tag{3.1}$$

Proof. By Hölder's inequality

$$\begin{aligned}
\int_0^\infty \left| u\left(\frac{x}{t}\right) \right| |v(t)| \frac{dt}{t} &= \int_0^\infty \left| u\left(\frac{x}{t}\right) t^{-\frac{1}{p}} \right| \left| v(t)\, t^{-\frac{1}{q}} \right| dt \\
&\leq \left[\int_0^\infty \left| u\left(\frac{x}{t}\right) \right|^p \frac{dt}{t} \right]^{\frac{1}{p}} \left[\int_0^\infty |v(t)|^q \frac{dt}{t} \right]^{\frac{1}{q}} \\
&= \nparallel RM_x u \nparallel_p \nparallel v \nparallel_q \\
&= \nparallel u \nparallel_p \nparallel v \nparallel_q ,
\end{aligned}$$

where the last step uses (2.9) and (2.10). Existence of $u * v(x)$ and (3.1) follow. ∎

Next, we consider a Hölder type inequality for the inner product and a partial converse.

Theorem 5. *Suppose $p > 1$, and $q = \frac{p}{p-1}$. Then*

(*i*)

$$|\langle u, v \rangle| \leq \nparallel u \nparallel_p \nparallel v \nparallel_q , \tag{3.2}$$

(*ii*) *if* $\langle |u|, |v| \rangle \leq U \nparallel v \nparallel_q$ *for all* $v \in \text{Ł}_q$, *then* $\nparallel u \nparallel_p \leq U$.

Proof. Part (i) can be deduced from the previous theorem using (2.5). Alternatively, defining $\phi(x) = x^{-\frac{1}{p}} u(x)$ and $\psi(x) = x^{-\frac{1}{q}} v(x)$, then $\nparallel u \nparallel_p = \|\phi\|_p$, $\nparallel v \nparallel_q = \|\psi\|_q$ and

$$\int_0^\infty u(x)\, v(x) \frac{dx}{x} = \int_0^\infty \phi(x)\, \psi(x)\, dx.$$

Then both (i) and (ii) follow from similar results for ϕ and ψ in conventional Lebesgue spaces. See [3], Theorem 191 for a statement of the required theory. ∎

We are now in a position to prove a result equivalent to the first part of Theorem 3. This shows that, for the inner product (2.4), the operator $(Ru)*$ is adjoint to $u*$.

Theorem 6. *Suppose that* $p>1$, $q=\frac{p}{p-1}$, $u\in Ł_p$, $v\in Ł_q$ *and* $k\in Ł_1$. *Then* $\langle k*u,v\rangle$ *exists with*

$$\langle k*u,v\rangle=\langle k,(Ru)*v\rangle=\langle u,(Rk)*v\rangle \tag{3.3}$$

and

$$|\langle k*u,v\rangle| \;\le\; \| k \|_1 \| u \|_p \| v \|_q \,. \tag{3.4}$$

Proof. Consider

$$\begin{aligned}
\langle |k|*|u|\,,|v|\rangle &= \int_0^\infty |v(x)|\,\frac{dx}{x}\int_0^\infty \left|k\left(\frac{x}{t}\right)\right| |u(t)|\,\frac{dt}{t}\\
&= \int_0^\infty |v(x)|\,\frac{dx}{x}\int_0^\infty |k(s)| \left|u\left(\frac{x}{s}\right)\right| \frac{ds}{s}\\
&= \int_0^\infty |k(s)|\,\frac{ds}{s}\int_0^\infty \left|(Ru)\left(\frac{s}{x}\right)\right| |v(x)|\,\frac{dx}{x}\\
&= \langle |k|\,,|Ru|*|v|\rangle\,.
\end{aligned}$$

However, by (2.9), $Ru\in Ł_p$, so by Theorem 4 with u replaced by Ru,

$$|Ru|*|v| \;\le\; \| Ru \|_p \| v \|_q = \| u \|_p \| v \|_q$$

using (2.9) again. Thus

$$\begin{aligned}
\langle |k|*|u|\,,|v|\rangle &\le \langle |k|\,,\| u \|_p \| v \|_q\rangle\\
&= \| u \|_p \| v \|_q \,\langle |k|\,,1\rangle\\
&= \| k \|_1 \| u \|_p \| v \|_q < \infty.
\end{aligned}$$

This shows the existence of $\langle k*u,v\rangle$ and gives (3.4). Now, Fubini's theorem allows us to repeat the argument above, changing the order of integration, but with $|k|$, $|u|$, and $|v|$ replaced by k, u and v respectively. This gives $\langle k*u,v\rangle=\langle k,(Ru)*v\rangle$, the first part of (3.3). The remaining part can be shown by changing the order of integration without the change of variable. ∎

Theorem 7. *If* $p>1$, $u\in Ł_p$ *and* $k\in Ł_1$, *then* $k*u\in Ł_p$ *and*

$$\| k*u \|_p \le \| k \|_1 \| u \|_p \,. \tag{3.5}$$

Proof. Introduce $q=\frac{p}{p-1}$. Theorem 6 ensures that for all $v\in Ł_q$,

$$\langle |k|*|u|\,,|v|\rangle \;\le\; \| k \|_1 \| u \|_p \| v \|_q \,.$$

Then, Theorem 5 Part (ii) with $|u|$ replaced by $|k|*|u|$ gives $|k|*|u|\in Ł_p$, with $\| \,|k|*|u|\, \|_p \le \| k \|_1 \| u \|_p$. The required results follow. ∎

4. Applications

In Theorem 6, if we replace $u(x)$ by $x^{\frac{1}{p}}f(x)$ and $v(x)$ by $x^{\frac{1}{q}}g(x)$, then $\nparallel u \nparallel_p = \|f\|_p$ and $\nparallel v \nparallel_q = \|g\|_q$. If we also replace $k(x)$ by $\frac{x^{\frac{1}{p}}}{(x+1)}$, we have

$$\nparallel k \nparallel_1 = \int_0^\infty \frac{x^{\frac{1}{p}}}{x+1}\frac{dx}{x} = \int_0^1 s^{\frac{1}{p}-1}(x-1)^{-\frac{1}{p}}\,ds,$$

which is $B\left(\frac{1}{p},1-\frac{1}{p}\right)$ or $\pi \operatorname{cosec}\left(\frac{\pi}{p}\right)$. Further,

$$\langle k*u,v\rangle = \int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dxdy. \tag{4.1}$$

Then (3.4) becomes the Hilbert inequality (1.1).

To obtain the Hardy inequality we use Theorem 7. There we substitute $x^{\frac{1}{p}}f(x)$ for $u(x)$, as before. In addition, we take $k(x)$ to be

$$h_\mu(x) = \begin{cases} \Gamma^{-1}(\mu)\,x^{\frac{1}{p}-\mu}(x-1)^{\mu-1} & \text{for } x>1, \\ 0 \text{ for } 0<x\le 1, \end{cases} \tag{4.2}$$

where μ is any positive constant. Note that $k \in Ł_1$, in fact,

$$\begin{aligned} \nparallel k \nparallel_1 &= \int_1^\infty \frac{1}{\Gamma(\mu)}x^{\frac{1}{p}-\mu-1}(x-1)^{\mu-1}\,dx \\ &= \frac{1}{\Gamma(\mu)}\int_0^1 s^{-\frac{1}{p}}(1-s)^{\mu-1}\,ds \\ &= \frac{1}{\Gamma(\mu)}B\left(1-\frac{1}{p},\mu\right) \\ &= \frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(\mu+1-\frac{1}{p}\right)}. \end{aligned}$$

Moreover,

$$k*u(x) = \int_0^x \frac{1}{\Gamma(\mu)}\left(\frac{x}{t}-1\right)^{\mu-1}\left(\frac{x}{t}\right)^{\frac{1}{p}-\mu} t^{\frac{1}{p}}f(t)\,\frac{dt}{t} = x^{\frac{1}{p}-\mu}I^\mu f(x),$$

where

$$I^\mu f(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}f(t)\,dt,$$

the Riemann-Liouville fractional integral of order μ. Then Theorem 7 gives the following result.

Theorem 8. *If $p>1$, $\mu>0$ and $f\in L_p(0,\infty)$, then*

$$\left\|x^{-\mu}I^\mu f(x)\right\|_p \le \frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(\mu+1-\frac{1}{p}\right)}\|f\|_p.$$

This is a generalization of Theorem 1.2, due to Knopp [4]; the case $\mu = 1$ is the original Hardy inequality.

Alternatively, substituting $u(x) = x^{\mu + \frac{1}{p}} f(x)$ and

$$k(x) = \begin{cases} \Gamma^{-1}(\mu)\, x^{\frac{1}{p}} (1-x)^{\mu-1} & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad x \geq 1, \end{cases}$$

gives a theorem involving the Weyl fractional integral

$$J^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^{\infty} (t-x)^{\mu-1} f(t)\, dt,$$

namely,

Theorem 9. *If $p > 1$, $\mu > 0$ and $f \in L_p(0,\infty)$, then*

$$\|J^{\mu} f(x)\|_p \leq \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\mu + \frac{1}{p}\right)} \|x^{\mu} f(x)\|_p .$$

This is also a well-known result. See [3], Theorem 329.

5. Extensions

We begin with sufficient conditions for the relation (2.3).

Theorem 10. *If $p > 1$, $w \in \text{Ł}_p$ and $u, v \in \text{Ł}_1$, then*

$$[(u * v) * w](x) = [u * (v * w)](x) \tag{5.1}$$

for almost all $x > 0$ and

$$\nparallel u * (v * w) \nparallel_p \leq \nparallel u \nparallel_1 \ \nparallel v \nparallel_1 \ \nparallel w \nparallel_p . \tag{5.2}$$

Proof. By Theorem 7, $v * w \in \text{Ł}_p$. Then another application of that theorem, with k and u replaced by u and $v * w$ respectively, gives

$$\nparallel u * (v * w) \nparallel_p \leq \nparallel u \nparallel_1 \ \nparallel v * w \nparallel_p$$

which cannot exceed $\nparallel u \nparallel_1 \nparallel v \nparallel_1 \nparallel w \nparallel_p \leq \infty$.
This proves (5.2) and ensures the existence of $[u * (v * w)](x)$ for almost all $x > 0$. For those x,

$$\int_0^{\infty} \left| u\left(\frac{x}{t}\right) \right| \frac{dt}{t} \int_0^{\infty} \left| v\left(\frac{t}{s}\right) \right| |w(s)| \frac{ds}{s} \ < \ \infty.$$

Fubini's theorem now permits the second step in

$$\begin{aligned} [u * (v * w)](x) &= \int_0^{\infty} u\left(\frac{x}{t}\right) \frac{dt}{t} \int_0^{\infty} v\left(\frac{t}{s}\right) w(s) \frac{ds}{s} \\ &= \int_0^{\infty} w(s) \frac{ds}{s} \int_0^{\infty} u\left(\frac{x}{t}\right) v\left(\frac{t}{s}\right) \frac{dt}{t} \\ &= \int_0^{\infty} w(s) \frac{ds}{s} \int_0^{\infty} u\left(\frac{x}{ys}\right) v(y) \frac{dy}{y} \\ &= [(u * v) * w](x), \end{aligned}$$

giving (5.1). ∎

Versions of the index laws for fractional integration can be deduced from Theorem 10. Let $p>1$, μ, $\nu>0$. Replace $w(x)$ *by* $x^{\frac{1}{p}}f(x)$ so $f\in L_p$. Replace v by h_μ as defined in (4.2) and replace $u(x)$ by $x^{-\mu}h_\nu(x)$. Then $u*v(x)=h_{\nu+\mu}(x)$. Theorem 10 then gives:

Theorem 11. *If $p>1$, $f\in L_p$ then*

$$x^{-\mu-\nu}I^\nu I^\mu f(x)=x^{-\mu-\nu}I^{\nu+\mu}f(x)\in L_p.$$

If, instead, we swap the definitions of u and v, then we obtain

Theorem 12. *Under the conditions of Theorem 11:*

$$x^{-\mu}I^\mu x^{-\mu-\nu}I^\nu x^\mu f(x)=x^{-\mu-\nu}I^{\nu+\mu}f(x).$$

This is a version of the "second index law" due to Love [5]. Use of the adjoints $(Ru)*$ and $(Rv)*$ give similar results for the Weyl fractional integral.

Returning to more general considerations, we have

Theorem 13. *Suppose $p>1$, $q=\frac{p}{p-1}$, $u\in Ł_p$, $v\in Ł_q$ and $k_1,k_2\in Ł_1$. Then*

$$\langle (k_1*k_2)*u,v\rangle=\langle k_2*u,(Rk_1)*v\rangle \tag{5.3}$$

and

$$|\langle (k_1*k_2)*u,v\rangle|\ \le \nparallel k_2*u\nparallel_p \nparallel (Rk_1)*v\nparallel_q . \tag{5.4}$$

Proof. By Theorem 10, $(k_1*k_2)*u=k_1*(k_2*u)$ in $Ł_p$.
So

$$\langle (k_1*k_2)*u,v\rangle=\langle k_1*(k_2*u),v\rangle. \tag{5.5}$$

Since, by Theorem 7, $k_2*u\in Ł_p$, Theorem 6 applies to the right hand side and this yields (5.3).

By (2.9), $Rk_1\in Ł_1$ so that $(Rk_1*v)\in Ł_q$, again using Theorem 7. Hence, by Theorem 5

$$\begin{aligned}|\langle (k_1*k_2)*u,v\rangle| &= |\langle k_2*u,(Rk_1)*v\rangle| \\ &\le \nparallel k_2*u\nparallel_p \nparallel (Rk_1)*v\nparallel_q,\end{aligned}$$

which completes the proof. ∎

As an application of this result, take $k_1(x)=x^{\frac{1}{p}}e^{-x}$, $k_2(x)=x^{-\frac{1}{q}}e^{-\frac{1}{x}}$, $u(x)=x^{\frac{1}{p}}f(x)$ and $v(x)=x^{\frac{1}{q}}g(x)$. Then

$$(Rk_1)*v(x)=x^{-\frac{1}{p}}\int_0^\infty e^{-\frac{t}{x}}g(t)\,dt=x^{-\frac{1}{p}}G\left(\frac{1}{x}\right),$$

where G is the (one-sided) Laplace transform of g. Similarly, $k_2 * u\,(x) = x^{-\frac{1}{q}} F\left(\frac{1}{x}\right)$, F being the Laplace transform of f. In addition,

$$k_1 * k_2\,(x) = \frac{x^{\frac{1}{p}}}{x+1} = k\,(x)$$

say. So the left side of (5.3) is $\langle k * u, v \rangle$ which we have seen in (4.1) is equal to Hilbert's double integral.

The right side of (5.3) is

$$\begin{aligned} \langle k_2 * u, (Rk_1) * v \rangle &= \int_0^\infty x^{-\frac{1}{q}} F\left(\frac{1}{x}\right) x^{-\frac{1}{p}} G\left(\frac{1}{x}\right) \frac{dx}{x} \\ &= \int_0^\infty F\,(s)\, G\,(s)\, ds. \end{aligned}$$

Thus, in this case (5.3) gives

Theorem 14. *For $f \in L_p$, $g \in L_q$ where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\int_0^\infty \int_0^\infty \frac{f\,(x)\, g\,(y)}{x+y}\, dxdy = \int_0^\infty F\,(s)\, G\,(s)\, ds, \tag{5.6}$$

where F and G are the Laplace transforms of f and g.

Further, since $\nparallel k_2 * u \nparallel_p = \|x^{1-\frac{2}{p}} F\,(x)\,\|_p$ and $\nparallel (Rk_1) * v \nparallel_q = \|x^{1-\frac{2}{q}} G\,(x)\,\|_q$, then (5.4) gives the following.

Theorem 15. *Under the conditions of Theorem 14*

$$\left| \int_0^\infty \int_0^\infty \frac{f\,(x)\, g\,(y)}{x+y}\, dxdy \right| \leq \left\| x^{1-\frac{2}{p}} F\,(x) \right\|_p \left\| x^{1-\frac{2}{q}} G\,(x) \right\|_q. \tag{5.7}$$

This inequality implies Hilbert's. For

$$\begin{aligned} \left\| x^{1-\frac{2}{p}} F\,(x) \right\|_p &= \nparallel k_2 * u \nparallel_p \\ &\leq \nparallel k_2 \nparallel_1 \nparallel u \nparallel_p \\ &= \Gamma\left(\frac{1}{q}\right) \|f\|_p \end{aligned}$$

and similarly

$$\left\| x^{1-\frac{2}{q}} G\,(x) \right\|_q \leq \Gamma\left(\frac{1}{p}\right) \|g\|_q.$$

As

$$\Gamma\left(\frac{1}{q}\right) \Gamma\left(\frac{1}{p}\right) = \pi \operatorname{cosec}\left(\frac{\pi}{p}\right),$$

(5.7) implies (1.1).

To show that (5.7) is a sharper inequality than (1.1), consider the case $f\,(x) = g\,(x) = e^{-x}$, $p = q = 2$. Then $F\,(x) = G\,(x) = \frac{1}{x+1}$ and

$$\left\| x^{1-\frac{2}{p}} F\,(x) \right\|_p = \left\| x^{1-\frac{2}{q}} G\,(x) \right\|_q = 1$$

while

$$\|f\|_p = \|g\|_q = p^{-\frac{1}{p}} = \frac{1}{\sqrt{2}}.$$

The right side of (5.7) is thus equal to 1 while that of (1.1) is $\frac{\pi}{2}$.

Note that (5.6) can be obtained directly by changing the order of integration on the right side, using Hilbert's inequality to justify the process. However, we can find no mention of formula (5.6) in the literature.

References

[1] Erdély; A., An extension of a Hardy-Littlewood-Pólya inequality, *Proc. Edinburgh Mathematical Society*, **21**, (1978), pp. 11-15.

[2] Hardy G.H., Note on a theorem of Hilbert concerning series of positive terms, *Proc. London Mathematical Society (2)*, **23**, (1925), Records of Proceedings, xlv-xlvi.

[3] Hardy G.H., J.E. Littlewood and G. Polya, *Inequalities,* (Second Edition) CUP, 1952.

[4] Knopp K., Über Reihen mit positiven Gliedern (2te Mitteilung), *Journal of the London Mathematical Society,* **5**, (1930), pp. 13-21.

[5] Love E. R., Two index laws for fractional integrals and derivatives, *Journal of the Australian Mathematical Society,* **14**, (1972), pp. 385-410

[6] Opic B. and A. Kufner, *Hardy-type inequalities,* Longman, Harlow, 1990.

[7] Schur I., Bemerkungen zur Theorie der beschränken Bilinearformen mit unendlich vielen Veränderlichen, *Journal f. Math,* **140**, (1911), pp. 1-28.

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