A GENERALIZATION OF OSTROWSKI INTEGRAL INEQUALITY FOR MAPPINGS WHOSE DERIVATIVES BELONG TO $L_1[a,b]$ AND APPLICATIONS IN NUMERICAL INTEGRATION

S.S. DRAGOMIR

Abstract. A generalization of Ostrowski integral inequality for mappings whose derivatives belong to $L_1[a,b]$, and applications for general quadrature formulae are given.

1. Introduction

In 1938, A. Ostrowski proved the following integral inequality [5, p. 468]

Theorem 1. *Let $f:[a,b]\to\mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbf{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then we the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

For some generalizations and related results see the book [5, p. 468-484].

In paper [1], S.S. Dragomir and S. Wang pointed out the following inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{2} + \frac{\left|x-\frac{a+b}{2}\right|}{b-a} \right](b-a)\|f'\|_1$$

for all $x\in[a,b]$, provided f is continuous on $[a,b]$ and differentiable on (a,b) and the derivative $f'\in L_1(a,b)$.

Note that this result also can be obtained from Fink's theorem (Theorem 1, p. 471, [5]) for $n=1$ and appropriate computations.

Some applications of the above results in Numerical Integration and for special means have been given in [1]-[4].

In this paper we point out a new generalization of Ostrowski's inequality for absolutely continuous mappings and apply it for quadrature formulae in Numerical Analysis. Some connections with the rectangle, the midpoint and Simpson's rule are also established.

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2. Some Integral Inequalities

We start with the following theorem

Theorem 2. *Let $I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and α_i $(i = 0, ..., k+1)$ be "k+2" points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., k)$ and $\alpha_{k+1} = b$. If $f : [a,b] \to \mathbf{R}$ is absolutely continuous on $[a,b]$, then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right|$$

$$\leq \left[\frac{1}{2}\nu(h) + \max\left\{ \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|, i = 0, ..., k-1 \right\} \right] \|f'\|_1$$

$$\leq \nu(h) \|f'\|_1 \tag{2.1}$$

where $\nu(h) := \max\{h_i | i = 0, ..., k-1\}$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., k-1)$ and $\|f'\|_1 := \int_a^b |f'(t)|\,dt$, is the usual $L_1[a,b]-$norm.

Proof. Define the mapping $K : [a,b] \to \mathbf{R}$ given by (see also [6])

$$K(t) := \begin{cases} t - \alpha_1, t \in [a, x_1) \\ t - \alpha_2, t \in [x_1, x_2) \\ \dots\dots\dots\dots\dots\dots \\ t - \alpha_{k-1}, t \in [x_{k-2}, x_{k-1}) \\ t - \alpha_k, t \in [x_{k-1}, b]. \end{cases}$$

Integrating by parts, we have successively

$$\int_a^b K(t) f'(t)\,dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(t) f'(t)\,dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (t - \alpha_{i+1}) f'(t)\,dt$$

$$= \sum_{i=0}^{k-1} \left[(t - \alpha_{i+1}) f(t) \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} f(t)\,dt \right]$$

$$= \sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i) f(x_i) + (x_{i+1} - \alpha_{i+1}) f(x_{i+1}) \right] - \int_a^b f(t)\,dt$$

$$= (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - x_i) f(x_i) + \sum_{i=0}^{k-2} (x_{i+1} - \alpha_{i+1}) f(x_{i+1})$$

$$+ (b - \alpha_n) f(b) - \int_a^b f(t)\,dt$$

$$= (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - x_i) f(x_i) + \sum_{i=1}^{k-1} (x_i - \alpha_i) f(x_i)$$

$$+\left(b-\alpha_n\right) f(b)-\int_a^b f(t)\, dt$$

$$=\left(\alpha_1-a\right) f(a)+\sum_{i=1}^{k-1}\left(\alpha_{i+1}-\alpha_i\right) f\left(x_i\right)+\left(b-\alpha_n\right) f(b)-\int_a^b f(t)\, dt$$

$$=\sum_{i=0}^{k}\left(\alpha_{i+1}-\alpha_i\right) f\left(x_{i+1}\right)-\int_a^b f(t)\, dt$$

and then we have the integral equality (see also [6])

$$\int_a^b f(t)\, dt=\sum_{i=0}^{k}\left(\alpha_{i+1}-\alpha_i\right) f\left(x_i\right)-\int_a^b K(t) f'(t)\, dt. \tag{2.2}$$

On the other hand, we have

$$\left|\int_a^b K(t) f'(t)\, dt\right|=\left|\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} K(t) f'(t)\, dt\right|$$

$$\leq \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}}|K(t)|\left|f'(t)\right| dt=\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}}\left|t-\alpha_{i+1}\right|\left|f'(t)\right| dt=: T.$$

But

$$\int_{x_i}^{x_{i+1}}\left|t-\alpha_{i+1}\right|\left|f'(t)\right| dt \leq \sup_{t \in\left[x_i, x_{i+1}\right]}\left|t-\alpha_{i+1}\right| \int_{x_i}^{x_{i+1}}\left|f'(t)\right| dt$$

$$=\max \left\{\alpha_{i+1}-x_i, x_{i+1}-\alpha_{i+1}\right\} \int_{x_i}^{x_{i+1}}\left|f'(t)\right| dt$$

$$=\left[\frac{1}{2}\left(x_{i+1}-x_i\right)+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right] \int_{x_i}^{x_{i+1}}\left|f'(t)\right| dt.$$

Then

$$T \leq \sum_{i=0}^{k-1}\left[\frac{1}{2} h_i+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right] \int_{x_i}^{x_{i+1}}\left|f'(t)\right| dt$$

$$\leq \max_{i=0, \ldots, k-1}\left[\frac{1}{2} h_i+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right] \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}}\left|f'(t)\right| dt$$

$$\leq\left[\frac{1}{2} \nu(h)+\max \left\{\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|, i=0, \ldots, k-1\right\}\right]\left\|f'\right\|_1=: V$$

Now, as

$$\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\leq\frac{1}{2}h_i,$$

then

$$\max\left\{\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|,i=0,...,k-1\right\}\leq\frac{1}{2}\nu\left(h\right)$$

and, consequently,

$$V\leq\nu\left(h\right)\left\|f'\right\|_1.$$

The theorem is completely proved. ∎

Now, if we assume that the points of the division I_k are given, then the best inequality we can obtain from Theorem 2 is embodied in the following corollary:

Corollary 1. *Let f and I_k. Then we have the inequality:*

$$\left|\int_a^b f\left(x\right)dx-\frac{1}{2}\left[\left(x_1-a\right)f\left(a\right)+\sum_{i=1}^{k-1}\left(x_{i+1}-x_{i-1}\right)f\left(x_i\right)+\left(b-x_{k-1}\right)f\left(b\right)\right]\right|$$

$$\leq\frac{1}{2}\nu\left(h\right)\left\|f'\right\|_1 \tag{2.3}$$

Proof. We choose in Theorem 2,

$$\begin{aligned}\alpha_0&=a,\alpha_1=\frac{a+x_1}{2},\alpha_2=\frac{x_1+x_2}{2},...,\\ \alpha_{k-1}&=\frac{x_{k-2}+x_{k-1}}{2},\alpha_k=\frac{x_{k-1}+x_k}{2}\text{ and }\alpha_{k+1}=b.\end{aligned}$$

In this case we get

$$\sum_{i=0}^{k}\left(\alpha_{i+1}-\alpha_i\right)f\left(x_i\right)$$

$$=\left(\alpha_1-\alpha_0\right)f\left(a\right)+\left(\alpha_2-\alpha_1\right)f\left(x_1\right)+...+\left(\alpha_k-\alpha_{k-1}\right)f\left(x_{k-1}\right)+\left(b-\alpha_k\right)f\left(b\right)$$

$$=\left(\frac{a+x_1}{2}-a\right)f\left(a\right)+\left(\frac{x_1+x_2}{2}-\frac{a+x_1}{2}\right)f\left(x_1\right)$$

$$+...+\left(\frac{x_{k-1}+b}{2}-\frac{x_{k-2}+x_{k-1}}{2}\right)f\left(x_{k-1}\right)+\left(b-\frac{x_{k-1}+b}{2}\right)f\left(b\right)$$

$$=\frac{1}{2}\left[\left(x_1-a\right)f\left(a\right)+\sum_{i=1}^{k-1}\left(x_{i+1}-x_{i-1}\right)f\left(x_i\right)+\left(b-x_{k-1}\right)f\left(b\right)\right].$$

Now, applying the inequality (2.1), we get (2.3). ∎

The following corollary for equidistant partitioning also holds.

Corollary 2. *Let*

$$I_k : x_i := a + (b-a)\frac{i}{k}\,(i = 0, ..., k)$$

be an equidistant partitioning of $[a,b]$*. If* f *is as above, then we have the inequality:*

$$\left|\int_a^b f(x)\,dx - \left[\frac{1}{k}\cdot\frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)}{k}\sum_{i=1}^{k-1} f\left[\frac{(k-i)\,a+ib}{k}\right]\right]\right|$$

$$\leq \frac{1}{2k}(b-a)\left\|f'\right\|_1. \tag{2.4}$$

3. The Convergence of a General Quadrature Formula

Let $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of division of $[a,b]$ and consider the sequence of numerical integration formulae

$$I_n(f,\Delta_n,w_n) := \sum_{j=0}^{n} w_j^{(n)} f\left(x_j^{(n)}\right)$$

where $w_j^{(n)}$ $(j = 0, ..., n)$ are the quadrature weights.

The following theorem provides a sufficient condition for the weights $w_j^{(n)}$ so that $I_n(f,\Delta_n,w_n)$ approximates the integral $\int_a^b f(x)\,dx$.

Theorem 3. *Let* $f : [a,b] \to \mathbf{R}$ *be an absolutely continuous mapping on* $[a,b]$*. If the quadrature weights* $w_j^{(n)}$ *satisfy the condition*

$$x_i^{(n)} - a \leq \sum_{j=0}^{i} w_j^{(n)} \leq x_{i+1}^{(n)} - a \text{ for all } i = 0, ..., n-1, \tag{3.1}$$

then we have the estimate

$$\left|I_n(f,\Delta_n,w_n) - \int_a^b f(x)\,dx\right|$$

$$\leq \left[\frac{1}{2}\nu\left(h^{(n)}\right) + \max\left\{\left|a + \sum_{j=0}^{i} w_j^{(n)} - \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2}\right|, i = 0, ..., n-1\right\}\right]\left\|f'\right\|_1$$

$$\leq \nu\left(h^{(n)}\right)\left\|f'\right\|_1 \tag{3.2}$$

where $\nu\left(h^{(n)}\right) := \max\left\{h_i^{(n)} | i = 0, ..., n-1\right\}$ *and* $h_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)}$*. Particularly,*

$$\lim_{\nu\left(h^{(n)}\right)\to 0} I_n(f,\Delta_n,w_n) = \int_a^b f(x)\,dx \tag{3.3}$$

uniformly by rapport of the w_n*.*

Proof. Define the sequence of real numbers

$$\alpha_{i+1}^{(n)} := a + \sum_{j=0}^{i} w_j^{(n)}, i = 0, ..., n.$$

Note that

$$\alpha_{n+1}^{(n)} = a + \sum_{j=0}^{n} w_j^{(n)} = a + b - a = b,$$

and observe also that $\alpha_{i+1}^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$.

Define $\alpha_0^{(n)} := a$ and compute

$$\alpha_1^{(n)} - \alpha_0^{(n)} = a,$$

$$\alpha_{i+1}^{(n)} - \alpha_i^{(n)} = a + \sum_{j=0}^{i} w_j^{(n)} - a - \sum_{j=0}^{i-1} w_j^{(n)} = w_i^{(n)} \left(i = 1, ..., n-1\right),$$

$$\alpha_{n+1}^{(n)} - \alpha_n^{(n)} = b - \left(a + \sum_{j=0}^{n-1} w_j^{(n)}\right) = w_n^{(n)}.$$

Then

$$\sum_{i=0}^{n} \left(\alpha_{i+1}^{(n)} - \alpha_i^{(n)}\right) f\left(x_i^{(n)}\right) = \sum_{i=0}^{n} w_i^{(n)} f\left(x_i^{(n)}\right) = I_n\left(f, \Delta_n, w_n\right).$$

Applying the inequality (2.1), we get the estimate (3.2).

The uniform convergence by rapport of quadrature weights $w_j^{(n)}$ is obvious by the last inequality. ∎

Now, consider the equidistant partitioning of $[a, b]$ given by

$$E_n : x_i^{(n)} := a + \frac{i}{n}\left(b - a\right) \quad \left(i = 0, ..., n\right)$$

and define the sequence of numerical quadrature formulae

$$I_n\left(f, w_n\right) := \sum_{i=0}^{n} w_i^{(n)} f\left[a + \frac{i}{n}\left(b - a\right)\right].$$

The following corollary which can be more useful in practice holds:

Corollary 3. *Let f be as above. If the quadrature weight $w_j^{(n)}$ satisfy the condition:*

$$\frac{i}{n} \le \frac{1}{b-a} \sum_{j=0}^{i} w_j^{(n)} \le \frac{i+1}{n}, i = 0, ..., n-1; \tag{3.4}$$

then we have:

$$\left| I_n\left(f, w_n\right) - \int_a^b f\left(x\right) dx \right|$$

$$\leq \left[\frac{b-a}{2n} + \max\left\{\left|a + \sum_{j=0}^{i} w_j^{(n)} - \frac{2i+1}{2} \cdot \frac{(b-a)}{n}\right|, i = 0, ..., n-1\right\}\right] \|f'\|_1$$

$$\leq \frac{(b-a)}{n} \|f'\|_1 . \tag{3.5}$$

Particularly, we have the limit

$$\lim_{n\to\infty} I_n(f, w_n) = \int_a^b f(x)\,dx,$$

uniformly by rapport of w_n.

4. Some Particular Integral Inequalities

The following proposition holds

Proposition 1. *Let* $f : [a, b] \to \mathbf{R}$ *be an absolutely continuous mapping on* $[a, b]$. *Then we have the inequality:*

$$\left|\int_a^b f(x)\,dx - [(\alpha - a) f(a) + (b - \alpha) f(b)]\right|$$

$$\leq \left[\frac{1}{2}(b-a) + \left|\alpha - \frac{a+b}{2}\right|\right] \|f'\|_1 \tag{4.1}$$

for all $\alpha \in [a, b]$.

The proof follows by Theorem 2 choosing $x_0 = a, x_1 = b, \alpha_0 = a, \alpha_1 = \alpha \in [a, b]$ and $\alpha_2 = b$.

Remark 1. *a) If in* (4.1) *we put* $\alpha = b$, *then we get the "*left rectangle inequality*"*

$$\left|\int_a^b f(x)\,dx - (b-a) f(a)\right| \leq (b-a) \|f'\|_1 ; \tag{4.2}$$

b) If $\alpha = a$, *then by* (4.1) *we get the "*right rectangle inequality*"*

$$\left|\int_a^b f(x)\,dx - (b-a) f(b)\right| \leq (b-a) \|f'\|_1 ; \tag{4.3}$$

c) It is easly to see that the best inequality we can get from (4.1) *is for* $\alpha = \frac{a+b}{2}$ *obtaining the "*trapezoid inequality*"*

$$\left|\int_a^b f(x)\,dx - \frac{f(a) + f(b)}{2}(b-a)\right| \leq \frac{1}{2}(b-a) \|f'\|_1 . \tag{4.4}$$

Another proposition with many interesting particular cases is the following one:

Proposition 2. *Let f be as above and $a \le x_1 \le b$, $a \le \alpha_1 \le x_1 \le \alpha_2 \le b$. Then we have*

$$\left| \int_a^b f(x)\,dx - \left[(\alpha_1 - a) f(a) + (\alpha_2 - \alpha_1) f(x_1) + (b - \alpha_2) f(b) \right] \right|$$

$$\le \frac{1}{2}\left[\frac{1}{2}(b-a) + \left| x_1 - \frac{a+b}{2} \right| + \left| \alpha_1 - \frac{a+x_1}{2} \right| \right.$$
$$\left. + \left| \alpha_2 - \frac{x_1+b}{2} \right| + \left| \left| \alpha_1 - \frac{a+x_1}{2} \right| - \left| \alpha_2 - \frac{x_1+b}{2} \right| \right| \right] \|f'\|_1$$

$$\le \left[\frac{(b-a)}{2} + \left| x_1 - \frac{a+b}{2} \right| \right] \|f'\|_1 \le (b-a)\|f'\|_1 . \tag{4.5}$$

Proof. Consider the division $a = x_0 \le x_1 \le x_2 \le b$ and the numbers $\alpha_0 = a, \alpha_1 \in [a, x_1], \alpha_2 \in [x_1, b]$ and $\alpha_3 = b$. Now, applying Theorem 2, we get

$$\left| \int_a^b f(x)\,dx - \left[(\alpha_1 - a) f(a) + (\alpha_2 - \alpha_1) f(x_1) + (b - \alpha_2) f(b) \right] \right|$$

$$\le \frac{1}{2}\left[\max\{x_1 - a, b - x_1\} + \max\left\{ \left| \alpha_1 - \frac{a+x_1}{2} \right|, \left| \alpha_2 - \frac{x_1+b}{2} \right| \right\} \right] \|f'\|_1$$

$$= \left[\frac{1}{4}(b-a) + \frac{1}{2}\left| x_1 - \frac{a+b}{2} \right| + \frac{1}{2}\left| \alpha_1 - \frac{a+x_1}{2} \right| \right.$$

$$\left. + \frac{1}{2}\left| \alpha_2 - \frac{x_1+b}{2} \right| + \frac{1}{2}\left| \left| \alpha_1 - \frac{a+x_1}{2} \right| - \left| \alpha_2 - \frac{x_1+b}{2} \right| \right| \right] \|f'\|_1$$

and the first inequality in (4.5) is proved.

Now, let observe that

$$\left| \alpha_1 - \frac{a+x_1}{2} \right| \le \frac{x_1 - a}{2}, \left| \alpha_2 - \frac{x_1+b}{2} \right| \le \frac{b - x_1}{2}.$$

Consequently,

$$\max\left\{ \left| \alpha_1 - \frac{a+x_1}{2} \right|, \left| \alpha_2 - \frac{x_1+b}{2} \right| \right\} \le \frac{1}{2}\max\{x_1 - a, b - x_1\}$$

and the second inequality in (4.5) is proved.

The last inequality is obvious. ∎

Remark 2. *If we choose above $\alpha_1 = a, \alpha_2 = b$, then we get the following Ostrowski's type inequality obtained by Dragomir-Wang in the recent paper* [1]*:*

$$\left| \int_a^b f(x)\,dx - (b-a) f(x_1) \right| \le \left[\frac{1}{2}(b-a) + \left| x_1 - \frac{a+b}{2} \right| \right] \|f'\|_1 \tag{4.6}$$

for all $x_1 \in [a, b]$.

We note that the best inequality we can get in (4.6) is for $x_1 = \frac{a+b}{2}$ obtaining the "*midpoint inequality*"

$$\left| \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \leq \frac{1}{2}(b-a)\|f'\|_1 \tag{4.7}$$

b) If we choose in (4.5) $\alpha_1 = \frac{5a+b}{6}, \alpha_2 = \frac{a+5b}{6}$ and $x_1 \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$, then we get

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(x_1)\right] \right|$$

$$\leq \frac{1}{2}\left[\frac{1}{2}\cdot(b-a) + \left|x_1 - \frac{a+b}{2}\right|\right.$$

$$\left. + \max\left\{\left|x_1 - \frac{2a+b}{3}\right|, \left|\frac{a+2b}{3} - x_1\right|\right\}\right]. \tag{4.8}$$

Particularly, if we choose in (4.8), $x_1 = \frac{a+b}{2}$, then we get the following "*Simpson's inequality*"

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right|$$

$$\leq \frac{1}{3}(b-a)\|f'\|_1. \tag{4.9}$$

5. Some composite quadrature formulae

Let us consider the partitioning of the interval $[a,b]$ given by $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and put $h_i := x_{i+1} - x_i \; (i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$.

The following theorem holds:

Theorem 4. *Let $f : [a,b] \to \mathbf{R}$ be absolutely continuous on $[a,b]$ and $k \geq 1$. Then we have the composite quadrature formula*

$$\int_a^b f(x)\, dx = A_k(\Delta_n, f) + R_k(\Delta_n, f) \tag{5.1}$$

where

$$A_k(\Delta_n, f) := \frac{1}{k}\left[T(\Delta_n, f) + \sum_{i=0}^{n}\sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right] h_i\right] \tag{5.2}$$

and

$$T(\Delta_n, f) := \frac{1}{2}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})]h_i \tag{5.3}$$

is the trapezoid quadrature formula.

The remainder $R_k(\Delta_n, f)$ satisfies the estimate

$$|R_k(\Delta_n, f)| \leq \frac{1}{2k}\nu(h)\|f'\|_1 . \tag{5.4}$$

Proof. Applying Corollary 2 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ we get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[\frac{1}{k}\frac{f(x_i)+f(x_{i+1})}{2}h_i + \frac{h_i}{k}\sum_{j=1}^{k} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right]\right|$$
$$\leq \frac{1}{2k}h_i \int_{x_i}^{x_{i+1}} |f'(t)|\,dt.$$

Now, using the generalized triangle inequality, we get:

$$|R_k(\Delta_n, f)|$$
$$\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[\frac{1}{k}\cdot\frac{f(x_i)+f(x_{i+1})}{2}h_i + \frac{h_i}{k}\sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right]\right|$$
$$\leq \frac{1}{2k}\sum_{i=0}^{n-1} h_i \int_{x_i}^{x_{i+1}} |f'(t)|\,dt \leq \frac{\nu(h)}{2k}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} |f'(t)|\,dt = \frac{\nu(h)}{2k}\|f'\|_1$$

and the theorem is proved. ∎

The following corollaries hold:

Corollary 4. *Let f be as above. Then we have the formula:*

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[T_n(\Delta_n, f) + M_n(\Delta_n, f)\right] + R_2(\Delta_n, f) \tag{5.5}$$

where $M_n(\Delta_n, f)$ is the midpoint quadrature formula,

$$M_n(\Delta_n, f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i$$

and the remainder $R_2(\Delta_n, f)$ satisfies the inequality:

$$|R_2(\Delta_n, f)| \leq \frac{1}{4}\nu(h)\|f'\|_1 . \tag{5.6}$$

Corollary 5. *Under the above assumptions we have*

$$\int_a^b f(x)\,dx$$
$$= \frac{1}{3}\left[T_n(\Delta_n, f) + \sum_{i=0}^{n-1} f\left(\frac{2x_i + x_{i+1}}{3}\right)h_i + \sum_{i=0}^{n-1} f\left(\frac{x_i + 2x_{i+1}}{3}\right)h_i\right]$$
$$+R_3(\Delta_n, f) . \tag{5.7}$$

The remainder $R_3(\Delta_n, f)$ *satisfies the bound:*

$$|R_3(\Delta_n, f)| \le \frac{1}{6}\nu(h)\|f'\|_1. \tag{5.8}$$

The following theorem holds:

Theorem 5. *Let* f *and* Δ_n *be as above and* $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. *Then we have the quadrature formula:*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right] + R(\xi, \Delta_n, f). \tag{5.9}$$

The remainder $R(\xi, \Delta_n, f)$ *satisfies the estimation:*

$$\begin{aligned} |R(\xi, \Delta_n, f)| &\le \left[\frac{1}{2}\nu(h) + \max\left\{\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|, i = 0, ..., n-1\right\}\right]\|f'\|_1 \\ &\le \nu(h)\|f'\|_1. \end{aligned} \tag{5.10}$$

for all ξ_i *as above.*

Proof. Apply Proposition 1 on the interval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to get

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right]\right| \\ &\le \left[\frac{1}{2}h_i + \max\left\{\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right\}\right]\int_{x_i}^{x_{i+1}} |f'(t)|\,dt. \end{aligned}$$

Summing over i from 0 to $n-1$, using the generalized triangle inequality and the properties of the maximum mapping, we get (5.10). ∎

Corollary 6. *Let* f *and* Δ_n *be as above. Then we have*

1) the "left rectangle rule"

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i) h_i + R_l(\Delta_n, f); \tag{5.11}$$

2) the "right rectangle rule"

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_{i+1}) h_i + R_r(\Delta_n, f); \tag{5.12}$$

3) the "trapezoid rule"

$$\int_a^b f(x)\,dx = T(\Delta_n, f) + R_T(\Delta_n, f) \tag{5.13}$$

where

$$|R_l(\Delta_n, f)|\,|R_r(\Delta_n, f)| \le \nu(h)\|f'\|_1$$

and

$$|R_T(\Delta_n, f)| \leq \frac{1}{2}\nu(h)\|f'\|.$$

The following theorem also holds.

Theorem 6. *Let f and Δ_n be as above and $\xi_i \in [x_i, x_{i+1}]$, $x_i \leq \alpha_i^{(1)} \leq \xi_i \leq \alpha_i^{(2)} \leq x_{i+1}$, then we have the quadrature formula:*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\left(\alpha_i^{(1)} - x_i\right) f(x_i) + \sum_{i=0}^{n-1}\left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f(\xi_i)$$

$$+\sum_{i=0}^{n-1}\left(x_{i+1} - \alpha_i^{(2)}\right) f(x_{i+1}) + R\left(\xi, \alpha_i^{(1)}, \alpha_i^{(2)}, \Delta_n, f\right). \tag{5.14}$$

The remainder $R\left(\xi, \alpha_i^{(1)}, \alpha_i^{(2)}, \Delta_n, f\right)$ satisfies the estimation

$$\left|R\left(\xi, \alpha_i^{(1)}, \alpha_i^{(2)}, \Delta_n, f\right)\right|$$

$$\leq \left\{\frac{1}{2}\left[\frac{1}{2}\nu(h) + \max_{i=0,\dots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\right.$$

$$\left.+\max\left\{\max_{i=0,\dots,n-1}\left|\alpha_i^{(1)} - \frac{x_i + \xi_i}{2}\right|, \max_{i=0,\dots,n-1}\left|\alpha_i^{(2)} - \frac{\xi_i + x_{i+1}}{2}\right|\right\}\right\}\|f'\|_1$$

$$\leq \left[\frac{1}{2}\nu(h) + \max_{i=0,\dots n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\|f'\|_1 \leq \nu(h)\|f'\|_1. \tag{5.15}$$

Proof. Apply Proposition 2 on the interval $[x_i, x_{i+1}]$ to obtain

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \left[\left(\alpha_i^{(1)} - x_i\right) f(x_i) + \left(\alpha_i^{(2)} - \alpha_i^{(1)}\right) f(\xi_i) + (x_{i+1} - \xi_i) f(x_{i+1})\right]\right|$$

$$\leq \frac{1}{2}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right.$$

$$\left.+\max\left\{\left|\alpha_i^{(1)} - \frac{x_i + \xi_i}{2}\right|, \left|\alpha_i^{(2)} - \frac{\xi_i + x_{i+1}}{2}\right|\right\}\right]\int_{x_i}^{x_{i+1}} |f'(t)|\,dt.$$

Summing over i from 0 to $n-1$ and using the properties of modulus and maximum, we get the desired inequality.

We shall omit the details. ∎

The following corollary is the result of Dragomir-Wang from the recent paper [1]

Corollary 7. *Under the above assumptions, we have the Riemann's quadrature formula:*

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(\xi_i)\,h_i + R_R(\xi,\Delta_n,f). \tag{5.16}$$

The remainder $R_R(\xi,\Delta_n,f)$ *satisfies the bound*

$$\begin{aligned} &|R_R(\xi,\Delta_n,f)| \\ &\le \left[\frac{1}{2}\nu(h) + \max\left\{\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|, i=0,...,n-1\right\}\right]\|f'\|_1 \\ &\le \nu(h)\,\|f'\|_1\,. \end{aligned} \tag{5.17}$$

for all $\xi_i \in [x_i,x_{i+1}]$ $(i=0,...,n)$.

Finally, the following corollary which generalizes Simpson's quadrature formula holds

Corollary 8. *Under the above assumptions and if* $\xi_i \in \left[\frac{x_{i+1}+5x_i}{6}, \frac{x_i+5x_{i+1}}{6}\right]$ $(i=0,$ $...,n-1)$, *then we have the formula:*

$$\begin{aligned} \int_a^b f(x)\,dx &= \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i \\ &+\frac{2}{3}\sum_{i=0}^{n-1} f(\xi_i)\,h_i + S(f,\Delta_n,\xi). \end{aligned} \tag{5.18}$$

The remainder $S(f,\Delta_n,\xi)$ *satisfies the estimate:*

$$\begin{aligned} &|S(f,\Delta_n,\xi)| \\ &\le \left\{\frac{1}{2}\left[\frac{\nu(h)}{2} + \max_{i=0,\dots,n-1}\left\{\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right\}\right.\right. \\ &\left.\left.+\max\left\{\max_{i=0,\dots,n-1}\left|\xi_i - \frac{2x_i+x_{i+1}}{3}\right|, \max_{i=0,\dots,n-1}\left|\frac{x_i+2x_{i+1}}{3}-\xi_i\right|\right\}\right]\right\}\|f'\|_1\,. \end{aligned} \tag{5.19}$$

The proof follows by the inequality (4.8) and we omit the details.

Remark 3. *Now, if we choose in* (5.18), $\xi_i = \frac{x_i+x_{i+1}}{2}$, *then we get* "Simpson's quadrature formula*"*

$$\begin{aligned} \int_a^b f(x)\,dx &= \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i \\ &+\frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i + S(f,\Delta_n) \end{aligned} \tag{5.20}$$

where the remainder term $S(f,\Delta_n)$ satisfies the bound:

$$|S(f,\Delta_n)| \leq \frac{1}{3}\nu(h)\|f'\|_1 . \tag{5.21}$$

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School of Communication and Informatics, Victoria University of Technology, PO Box 14428, MCMC Melbourne City, Victoria, Australia.

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://matilda.vut.edu.au/~rgmia/dragomirweb.html`