THREE POINT QUADRATURE RULES INVOLVING, AT MOST, A FIRST DERIVATIVE

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Abstract. A unified treatment of three point quadrature rules is presented in which the classical rules of mid-point, trapezoidal and Simpson type are recaptured as particular cases. Riemann integrals are approximated for the derivative of the integrand belonging to a variety of norms. The Grüss inequality and a number of variants are also presented which provide a variety of inequalities that are suitatable for numerical implementation. Mappings that are of bounded total variation, Lipschitzian and monotonic are also investigated with relation to Riemann-Stieltjes integrals. Explicit *a priori* bounds are provided allowing the determination of the partition required to achieve a prescribed error tolerance.

It is demonstrated that with the above classes of functions, the average of a mid-point and trapezoidal type rule produces the best bounds.

1. Introduction

Three point quadrature rules of Newton-Cotes type have been examined extensively in the literature. In particular, the mid- point, trapezoidal and Simpson rules have been investigated more recently [1]-[13] with the view of obtaining bounds on the quadrature rule in terms of a variety of norms involving, at most, the first derivative. The bounds that have been obtained more recently also depend on the Peano kernel used in obtaining the quadrature rule. The general approach used in the past involves the assumption of bounded derivatives of degree higher than one. The partitioning is halved until the desired accuracy is obtained (see for example Atkinson [14]). The work in papers [1]-[13] aims at obtaining *a priori* estimates of the partition required in order to attain a particular bound on the error.

The current work employs the modern theory of inequalities to obtain bounds for three-point quadrature rules consisting of an interior point and boundary points. The mid-point, trapezoidal and Simpson rules are recaptured as particular instances of the current development. Riemann integrals are approximated for the derivative of the integrand belonging to a variety of norms. An inequality due to Grüss together with a number of extensions and variants is used to obtain perturbed three-point rules which produce tight bounds suitable for numerical quadrature. The approximation of Riemann-Stieltjes integrals is also investigated for which the mappings belong to a variety of classes including: total bounded variation, Lipschitzian and monotonic.

The paper is arranged in the following manner.

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In Section2, an identity is derived that involves a three-point rule whose bound may be obtained in terms of the first derivative, $f' \in L_\infty[a,b]$. Application of the results in numerical integration are presented in Section 3. An Ostrowski-Grüss inequality is developed in Section 4, as is a *premature* Grüss which produces perturbed three-point rules. A further Ostrowski- Grüss inequality is developed in Section 5 which produces bounds that are even sharper than those obtained from the *premature* Grüss results.

Results and numerical implementation of inequalities in which the first derivative $f' \in L_1[a,b]$ are developed in Section 6, while perturbed three-point rules are obtained in Section 7 through the analysis of some new Grüss-type results.

Three-point Lobatto rules are obtained in Section 8 when $f' \in L_p[a,b]$, while perturbed rules through the development of Grüss-type rules are investigated in Section 9.

Section 10 is reserved for functions that are not necessarily differentiable and so inequalities involving Riemann-Stieltjes integrals that are suitable for numerical implementation are investigated in which the functions are assumed to be either of total bounded variation, Lipschitzian or monotonic.

The work repeatedly demonstrates that a Newton-Cotes rule that is equivalent to the average of a mid-point and trapezoidal rule consistently gives tighter bounds than a Simpson-type rule.

Some concluding remarks and discussion are given in Section 11.

2. Inequalities Involving the First Derivative

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)}|f'(t)| < \infty$. Further, let $\alpha:[a,x]\to\mathbb{R}$ and $\beta:(x,b]\to\mathbb{R}$. Then, for all $x\in[a,b]$ we have the inequality*

$$\left|\int_a^b f(t)\,dt - [(\beta(x)-\alpha(x))f(x) + (b-\beta(x))f(b) + (\alpha(x)-a)f(a)]\right| \tag{2.1}$$
$$\leq \|f'\|_\infty \left\{\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right] + \left(\alpha(x)-\frac{a+x}{2}\right)^2 + \left(\beta(x)-\frac{b+x}{2}\right)^2\right\}.$$

Proof. Let

$$K(x,t) = \begin{cases} t-\alpha(x), & t\in[a,x] \\ t-\beta(x), & t\in(x,b]\end{cases}, \tag{2.2}$$

and consider

$$\int_a^b K(x,t)\,f'(t)\,dt.$$

Now, from (2.2),

$$\int_a^b K(x,t)\,f'(t)\,dt = \int_a^x (t-\alpha(x))\,f'(t)\,dt + \int_x^b (t-\beta(x))\,f'(t)\,dt,$$

and integrating by parts produces the identity

$$\begin{aligned}(2.3)\qquad & \int_a^b K(x,t)\,f'(t)\,dt \\ = \; & (\beta(x)-\alpha(x))\,f(x)+(b-\beta(x))\,f(b)+(\alpha(x)-a)\,f(a)-\int_a^b f(t)\,dt.\end{aligned}$$

Thus,

$$\begin{aligned}(2.4)\qquad & \left|\int_a^b f(t)\,dt-[(\beta(x)-\alpha(x))\,f(x)+(b-\beta(x))\,f(b)+(\alpha(x)-a)\,f(a)]\right| \\ \leq \; & \|f'\|_\infty \int_a^b |K(x,t)|\,dt.\end{aligned}$$

Let $Q(x)=\int_a^b |K(x,t)|\,dt$ and so

$$\begin{aligned}Q(x) &= -\int_a^{\alpha(x)}(t-\alpha(x))\,dt+\int_{\alpha(x)}^{x}(t-\alpha(x))\,dt \\ &\quad -\int_x^{\beta(x)}(t-\beta(x))\,dt+\int_{\beta(x)}^{b}(t-\beta(x))\,dt \\ &= \frac{1}{2}\left\{(a-\alpha(x))^2+(x-\alpha(x))^2+(x-\beta(x))^2+(b-\beta(x))^2\right\}.\end{aligned}$$

If we use the identity

$$(2.5)\qquad \frac{X^2+Y^2}{2}=\left(\frac{X+Y}{2}\right)^2+\left(\frac{X-Y}{2}\right)^2$$

we may write $Q(x)$ as

$$(2.6)\quad Q(x)=\left(\frac{x-a}{2}\right)^2+\left(\alpha(x)-\frac{a+x}{2}\right)^2+\left(\frac{b-x}{2}\right)^2+\left(\beta(x)-\frac{b+x}{2}\right)^2.$$

The reutilizing of identity (2.5) on $\frac{1}{2}\left[(x-a)^2+(b-x)^2\right]$ in (2.6) and substitution into (2.4) will produce the result (2.1) and thus the theorem is proved. ∎

Corollary 1. *Let f satisfy the conditions of Theorem 1. Then $\alpha(x)=\frac{a+x}{2}$ and $\beta(x)=\frac{b+x}{2}$ give the best bound for any $x\in[a,b]$ and so*

$$\begin{aligned}(2.7)\qquad & \left|\int_a^b f(t)\,dt-\frac{b-a}{2}\left[f(x)+\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)\right]\right| \\ \leq \; & \frac{\|f'\|_\infty}{2}\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right].\end{aligned}$$

Proof. The proof is trivial since (2.1) is a sum of squares and the minimum occurs when each of the terms are zero. ∎

Remark 1. *Result* (2.7) *is similar to that obtained by Milanović and Pečarić* [14, p. 470]*, although their bound relies on the second derivative being bounded. This is not always possible so that the weaker assumption of the first derivative being bounded as in* (2.4) *may prove to be useful.*

Remark 2. *An even more accurate quadrature formula is obtained when* $x = \frac{a+b}{2}$, *giving from* (2.7) :

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \right| \le \frac{\|f'\|_\infty}{2}\left(\frac{b-a}{2}\right)^2. \tag{2.8}$$

This is equivalent to approximating an integral as the average of a mid-point and trapezoidal rule. The bound in (2.7) *however, only requires the first derivative of the function* f *to be bounded.*

Motivated by the results of Theorem 1 and Corollary 1, we consider a kernel of the form (2.2) where $\alpha(x)$ and $\beta(x)$ are convex combinations of the end points.

Theorem 2. *Let* f *satisfy the conditions as stated in Theorem 1. Then the following inequality holds for any* $\gamma \in [0,1]$ *and* $x \in [a,b]$:

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} \right| \\ &\le 2\|f'\|_\infty \left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2\right]\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \qquad (2.9) \\ &\le \frac{(b-a)^2}{2}\|f'\|_\infty . \qquad (2.10) \end{aligned}$$

Proof. Let

$$\alpha(x) = \gamma x + (1-\gamma) a \text{ and } \beta(x) = \gamma x + (1-\gamma) b. \tag{2.11}$$

Thus, from Theorem 1 and its proof utilizing (2.3) where

$$Q(x) = \int_a^b |K(x,t)|\,dt \tag{2.12}$$

we have from (2.5) , on substituting for $\alpha(x)$ and $\beta(x)$ from (2.11) , that

$$Q(x) = \left(\frac{x-a}{2}\right)^2 + \left[\left(\gamma - \frac{1}{2}\right)(x-a)\right]^2 + \left(\frac{b-x}{2}\right)^2 + \left[\left(\gamma - \frac{1}{2}\right)(b-x)\right]^2.$$

Thus,

$$\begin{aligned} Q(x) &= \left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2\right]\left[(x-a)^2 + (b-x)^2\right] \qquad (2.13) \\ &= 2\left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2\right]\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right], \end{aligned}$$

upon using the identity (2.5) . Now, utilizing (2.12) and (2.13) in (2.4) will give the first part of the theorem, namely, equation (2.8) . Inequality (2.10) can easily be ascertained since (2.9) attains its maximum at $\gamma = 0$ or 1, and at $x = a$ or b. ∎

Remark 3. *Corollary 1 may be recovered of γ is set at its optimal value of $\frac{1}{2}$ in Theorem 2.*

Remark 4. $\gamma = 0$ *in* (2.9) *reproduces Ostrowski's inequality* [14, p. 468] *whose bound is sharpest when* $x = \frac{a+b}{2}$, *giving the midpoint rule.*

Remark 5. $\gamma = 1$ *produces the generalized trapezoidal rule for which again the best bound occurs when* $x = \frac{a+b}{2}$ *giving the standard trapezoidal type rule.*

Corollary 2. *Let the conditions on f be as in Theorem 1. Then, the following inequality holds*

$$\left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \gamma \left[\frac{f(a)+f(b)}{2}\right]\right\}\right| \leq \frac{\|f'\|_\infty}{2}(b-a)^2\left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2\right]. \tag{2.14}$$

Proof. Placing the optimal value of $x = \frac{a+b}{2}$ in (2.8) produces the result (2.14). ∎

Remark 6. *Result* (2.14) *gives a linear combination between a mid-point and a trapezoidal rule. The optimal result is obtained by taking* $\gamma = \frac{1}{2}$ *in* (2.14), *giving the optimal bound when only the assumption of a bounded first derivative is used. This gives the result from* (2.14)

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \leq \frac{(b-a)^2}{8}\|f'\|_\infty . \tag{2.15}$$

Which is equivalent to (2.8).
It should be noted that taking $\gamma = \frac{1}{3}$ *in* (2.14) *gives a Simpson-type rule that is worse than* (2.15), *remembering that here we are only using the assumption of a bounded first derivative rather than the more restrictive (though more accurate) result of a bounded fourth derivative.*

3. Application in Numerical Integration

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for any partition $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of $[a,b]$ and any intermediate point vector $\xi = (\xi_0, \xi_1, ..., \xi_{n-1})$ such that $\xi_i \in [x_i, x_{i+1}]$ for $i = 0, 1, ..., n-1$, we have:*

$$\int_a^b f(x)\,dx = A_c(f, I_n, \xi) + R_c(f, I_n, \xi), \tag{3.1}$$

where

$$\begin{aligned} & A_c(f, I_n, \xi) \\ = {} & (1-\gamma)\sum_{i=0}^{n-1} h_i f(\xi_i) + \gamma\left[\sum_{i=0}^{n-1}(\xi_i - x_i) f(x_i) + \sum_{i=0}^{n-1}(x_{i+1} - \xi_i) f(x_{i+1})\right] \\ = {} & (1-\gamma)\sum_{i=0}^{n-1} h_i f(\xi_i) + \gamma\left[\sum_{i=0}^{n-1}\xi_i\left(f(x_{i+1}) - f(x_i)\right) + bf(b) - af(a)\right] \end{aligned}$$

and the remainder

$$\begin{aligned}\left|R_c(f,I_n,\xi)\right| &\leq 2\left\|f'\right\|_\infty\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\sum_{i=0}^{n-1}\left[\left(\frac{h_i}{2}\right)^2+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\\ &\leq \frac{\left\|f'\right\|_\infty}{2}\sum_{i=0}^{n-1}h_i^2=\frac{\left\|f'\right\|_\infty}{2}n\nu^2(h),\end{aligned}$$

where $\nu(h)=\max_{i=0,\dots,n-1}h_i$.

Proof. Applying inequality (2.8) on the interval $[x_i,x_{i+1}]$ for $i=0,1,2,...,n-1$ we have

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}}f(x)\,dx\right.\\ &\left.-\left\{(1-\gamma)f(\xi_i)h_i+\gamma\left[(\xi_i-x_i)f(x_i)+(x_{i+1}-\xi_i)f(x_{i+1})\right]\right\}\right|\\ \leq\ &2\left\|f'\right\|_\infty\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\left[\left(\frac{h_i}{2}\right)^2+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\\ \leq\ &\frac{\left\|f'\right\|_\infty}{2}h_i^2,\end{aligned}$$

since the coarsest bound is obtained at $\gamma=0$ or 1 and $\xi_i\in[x_{li},x_{i+1}]$.
Summing over i for $i=0$ to $n-1$ we may deduce (3.1) and its subsequent elucidation. ∎

Corollary 3. *Let the assumptions of Theorem 3 hold. Then we have*

$$\int_a^b f(x)\,dx=A_c(f,I_n)+R_c(f,I_n)$$

where

$$A_c(f,I_n)=(1-\gamma)\sum_{i=0}^{n-1}h_if\left(\frac{x_i+x_{i+1}}{2}\right)+\frac{\gamma}{2}\sum_{i=0}^{n-1}h_i\left(f(x_i)+f(x_{i+1})\right),$$

which is a linear combination of the mid-point and trapezoidal rules and the remainder $R(f,I_n)$ *satisfies the relation*

$$\begin{aligned}\left|R_c(f,I_n)\right| &\leq \frac{\left\|f'\right\|_\infty}{2}\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\sum_{i=0}^{n-1}h_i^2\\ &= \frac{\left\|f'\right\|_\infty}{2}\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]n\nu^2(h),\end{aligned}$$

where $\nu(h)=\max_{i=0,\dots,n-1}h_i$.

Proof. Similar to Theorem 3 with $\xi_i=\frac{x_i+x_{i+1}}{2}$. ∎

4. A Generalized Ostrowski-Grüss Inequality Using Cauchy-Schwartz

The following result is known as the Grüss inequality which was proved by Grüss in 1935 (see for example [15, p. 296]).

Theorem 4. *Let f,g be two integrable functions defined on $[a,b]$, satisfying the conditions*

$$c \leq f(t) \leq C \text{ and } d \leq g(t) \leq D$$

for all $t \in [a,b]$. Then

$$|\mathfrak{T}(f,g)| \leq \frac{1}{4}(C-c)(D-d), \tag{4.1}$$

where

$$\mathfrak{T}(f,g) = \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt, \tag{4.2}$$

and the constant $\frac{1}{4}$ is the best possible.

The proof of this theorem is an extension of that for Theorem 9 and discussion will be delayed until then. See also Remark 14.

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{I}$ and let $a,b \in \mathring{I}$ with $a < b$. Further, let $f' \in L_1[a,b]$ and $d \leq f'(x) \leq D$, $\forall x \in [a,b]$. We have, then, the following inequality*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a)\right.\right.\right. \\ &\left.\left.\left. + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right) S\right| \\ \leq\ & \frac{(D-d)}{4}(b-a)\left\{\frac{b-a}{2} + \frac{1}{2}\left[\left|x - \frac{a+b}{2} + \left(\gamma - \frac{1}{2}\right)(b-a)\right|\right.\right. \\ &\left.\left. + \left|x - \frac{a+b}{2} - \left(\gamma - \frac{1}{2}\right)(b-a)\right|\right]\right\} \end{aligned} \tag{4.3}$$

where $S = \frac{f(b)-f(a)}{b-a}$ and $\gamma \in [0,1]$.

Proof. From the identity (2.3) with $\alpha(x)$ and $\beta(x)$ as defined in (2.11) we have

$$\begin{aligned} &\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} \\ &= -\int_a^b K(x,t)\,f'(t)\,dt \end{aligned} \tag{4.4}$$

where

$$K(x,t) = \begin{cases} t - [\gamma x + (1-\gamma) a], & t \in [a,x] \\ t - [\gamma x + (1-\gamma) b], & t \in (x,b] \end{cases}. \tag{4.5}$$

Now it is clear that for all $x \in [a,b]$ and $t \in [a,b]$ we have that

$$\phi(x) \leq K(x,t) \leq \Phi(x)$$

where

$$\phi(x) = -\max\{\gamma(x-a), (1-\gamma)(b-x)\}$$

and

$$\Phi(x) = \max\{(1-\gamma)(x-a), \gamma(b-x)\}.$$

Using the result that $\max\{X,Y\}=\frac{X+Y}{2}+\frac{1}{2}|Y-X|$ we have that

$$\Phi(x)=\frac{1}{2}[\gamma b-(1-\gamma)a+(1-2\gamma)x]+\frac{1}{2}|\gamma b+(1-\gamma)a-x|$$

and

$$-\phi(x)=\frac{1}{2}[(2\gamma-1)x+(1-\gamma)b-\gamma a]+\frac{1}{2}|\gamma a+(1-\gamma)b-x|.$$

Thus,

$$\begin{aligned}\Phi(x)-\phi(x) &= \frac{b-a}{2}+\frac{1}{2}\left\{\left|x-\frac{a+b}{2}+\left(\gamma-\frac{1}{2}\right)(b-a)\right|\right.\\ &\left.+\left|x-\frac{a+b}{2}-\left(\gamma-\frac{1}{2}\right)(b-a)\right|\right\}.\end{aligned} \tag{4.6}$$

Now,

$$\begin{aligned}&\int_a^b K(x,t)\,dt \\ &= \int_a^x [t-[\gamma x+(1-\gamma)a]]\,dt+\int_x^b [t-[\gamma x+(1-\gamma)b]]\,dt \\ &= \int_{-\gamma(x-a)}^{(1-\gamma)(x-a)} u du+\int_{-(1-\gamma)(b-x)}^{\gamma(b-x)} v dv \\ &= \frac{1}{2}\left[(1-\gamma)^2-\gamma^2\right]\left[(x-a)^2-(b-x)^2\right] \\ &= (b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right).\end{aligned} \tag{4.7}$$

Applying Theorem 4 of Grüss to the mappings $K(x,\cdot)$ and $f'(t)$, and using $S=\frac{1}{b-a}\int_a^b f'(t)\,dt$, we obtain from (4.6) and (4.7),

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b K(x,t)f'(t)\,dt-(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right| \\ &\le \frac{(D-d)}{4}\left\{\frac{b-a}{2}+\frac{1}{2}\left[\left|x-\frac{a+b}{2}+\left(\gamma-\frac{1}{2}\right)(b-a)\right|\right.\right. \\ &\left.\left.+\left|x-\frac{a+b}{2}-\left(\gamma-\frac{1}{2}\right)(b-a)\right|\right]\right\}.\end{aligned}$$

Then, using the identity (4.4) we obtain the result (4.3) as stated in the theorem. Hence, the theorem is completely proved. ∎

Remark 7. *It should be noted that the shifted quadrature rule that is obtained through the Grüss inequality still involves function evaluations at the end points and an interior point x. Thus, a simple grouping of terms would produce the left hand side of* (4.3) *in an alternate form*

$$\begin{aligned}&\int_a^b f(t)\,dt-\left\{(1-\gamma)(b-a)f(x)+\left[\gamma(b-x)+\left(x-\frac{a+b}{2}\right)\right]f(a)\right. \\ &\left.+\left[\gamma(x-a)-\left(x-\frac{a+b}{2}\right)\right]f(b)\right\}.\end{aligned}$$

Therefore, it is argued, the above quadrature rule is no more difficult to implement than the rule as given in Theorem 2 for example.

Corollary 4. *Let the conditions be as in Theorem 5. Then the following inequality holds for any* $x \in [a, b]$,

$$\left| \int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f(x) + (x-a) f(a) + (b-x) f(b)\right] \right| \leq \frac{(D-d)(b-a)}{4}\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]. \tag{4.8}$$

Proof. Setting $\gamma = \frac{1}{2}$ in (4.3) readily produces the result (4.8). ∎

Corollary 5. *Let the conditions be as in Theorem 5. Then the following inequality holds for any* $\gamma \in [0, 1]$,

$$\left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}\left[f(a) + f(b)\right]\right\} \right| \leq \frac{(D-d)(b-a)^2}{4}\left[\frac{1}{2} + \left(\gamma - \frac{1}{2}\right)^2\right]. \tag{4.9}$$

Proof. Choosing x to be at the mid-point of $[a, b]$ in (4.3) gives the result (4.9). ∎

Remark 8. *Placing* $\gamma = 0$ *in* (4.3) *produces an adjusted Ostrowski type rule, namely:*

$$\left| \int_a^b f(t)\,dt - \left[f(x) - \left(x - \frac{a+b}{2}\right) S\right] \right| \leq \frac{(D-d)}{8} \cdot (b-a) + \left[|x-b| + |x-a|\right].$$

This bound is sharpest at $x = \frac{a+b}{2}$, *thus producing the mid-point type rule*

$$\left| \int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{(D-d)(b-a)^2}{4}.$$

Remark 9. *Placing* $\gamma = 1$ *in* (4.3) *gives an adjusted generalized trapezoidal rule, namely:*

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right] - \left(x - \frac{a+b}{2}\right) S \right| \leq \frac{(D-d)}{8}\left[b - a + \left[|x-b| + |x-a|\right]\right].$$

This bound is sharpest at $x = \frac{a+b}{2}$ *giving the trapezoidal type rule*

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(a) + f(b)\right] \right| \leq \frac{(D-d)(b-a)^2}{4}.$$

Remark 10. *The sharpest bound on* (4.8) *and* (4.9) *are at* $x = \frac{a+b}{2}$ *and* $\gamma = \frac{1}{2}$ *respectively. The same result can be obtained directly from* (4.3), *giving as the best quadrature rule of this type*

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \right| \leq \frac{(b-a)^2}{8}(D-d) \tag{4.10}$$

which is an averaged mid-point and trapezoidal rule.

Again, as noted in Section 2 when it was assumed that $f' \in L_\infty(a,b)$, $\gamma = \frac{1}{3}$ (in (4.9)) produces a Simpson type rule which is worse than the optimal rule given by (4.10). Here, we are only assuming that $d < f'(x) < D$ rather than the more restrictive, though more accurate, assumptions in the development of a traditional Simpson's rule of a bounded fourth derivative.

The following two results by Ostrowski (1970) will be needed for the proof of the theorem that follows. An improvement by Lupas (1973) is also presented. These will be presented as theorems which are generalizations of the Grüss inequality. The notation of Pečarić, Proschan and Tong [16] will be used.

Theorem 6. *Let* f *be a bounded measurable function on* $I = (a,b)$ *such that* $c_1 \leq f(t) \leq c_2$ *for* $t \in I$ *and assume* $g'(t)$ *exists and is bounded on* I*. Then,*

$$|\mathfrak{T}(f,g)| \leq \frac{b-a}{8}(c_2 - c_1)\sup_{t\in I}|g'(t)|$$

and $\frac{1}{8}$ *is the best constant possible.*

Theorem 7. *Let* g *be locally absolutely continuous on* I *with* $g' \in L_2(I)$, *and let* f *be bounded and measurable on* $I = (a,b)$ *such that* $c_1 \leq f(t) \leq c_2$ *for* $t \in I$*. Then*

(*Ostrowski*, (1970))
$$|\mathfrak{T}(f,g)| \leq \frac{b-a}{4\sqrt{2}}(c_2 - c_1)\|g'\|_2,$$

and, the improved result,

(Lupas, (1973))
$$|\mathfrak{T}(f,g)| \leq \frac{b-a}{2\pi}(c_2 - c_1)\|g'\|_2,$$

where

$$\|g'\|_2 = \left(\frac{1}{b-a}\int_a^b |g'(t)|^2\,dt \right)^{\frac{1}{2}}.$$

Theorem 8. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $\mathring{I}$ *and let* $a,b \in \mathring{I}$ *with* $a < b$*. Further, let* $f' \in L_1[a,b]$ *and* $d \leq f'(x) \leq D$, $\forall x \in [a,b]$*. Then, the following inequality holds.*

$$\left| \int_a^b f(t)\,dt - (b-a)\left\{ (1-\gamma)f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b) \right] \right\} + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right)S \right| \leq \frac{D-d}{8}(b-a) \tag{4.11}$$

where $S = \frac{f(b)-f(a)}{b-a}$ *and* $\gamma \in [0,1]$.

Proof. Let $K(x,t)$ be as given by (4.5) and consider the interval $[a,x]$. Let $d_1 \leq f'(t) \leq D_1$ for $t \in [a,x]$. Then, from Theorem 4,

$$\left|\mathfrak{T}_{[a,x]}(f',K)\right| \leq \frac{x-a}{8}(D_1-d_1),$$

since

$$\sup_{t\in[a,x]} |K'(x,t)| = 1, \text{ as } K' \equiv 1.$$

Let $d_2 \leq f'(t) \leq D_2$ for $t \in (x,b]$. Then, in a similar fashion

$$\left|\mathfrak{T}_{(x,b]}(f',K)\right| \leq \frac{b-x}{8}(D_2-d_2).$$

Now, using the triangle inequality readily produces

$$\begin{aligned}\left|\mathfrak{T}_{[a,b]}(f',K)\right| &\leq \frac{(x-a)}{8}(D_1-d_1)+\frac{(b-x)}{8}(D_2-d_2)\\ &\leq \frac{b-a}{8}(D-d).\end{aligned}$$

Thus, from (4.7) and (4.5),

$$\begin{aligned}&\left|\mathfrak{T}_{[a,b]}(f',K)\right|\\ =& \left|\frac{1}{b-a}\int_a^b K(x,t)f'(t)\,dt-(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right|\\ \leq& \frac{b-a}{8}(D-d).\end{aligned}$$

Using identity (4.4) readily produces (4.11), and the theorem is proved. ∎

Remark 11. *On each of the intervals* $[a,x]$ *and* $(x,b]$

$$\sup_{t\in I}|k'(t)| = 1 = \|k'\|_2,$$

where $k(t) \equiv K(x,t)$. *Thus, using Theorem 6 is superior to using either of the two results of Theorem 7.*

Remark 12. *The bound obtained by* (4.11) *is uniform. The bound given by* (4.3) *attains its sharpest bound when* $x = \frac{a+b}{2}$ *and* $\gamma = \frac{1}{2}$, *producing* $\frac{(D-d)}{8}(b-a)^2$. *Thus, the current bound is better for* $b-a > 1$ *and for all* x.

Remark 13. *If Theorem 6 is used and* $T(K,f')$ *is considered, then a result similar to* (4.3) *would be obtained with* $\frac{D-d}{4}$ *being replaced by* $\frac{\|f''\|_\infty}{8}$. *This will not be investigated further since the second derivative is involved, thus placing it outside the scope of the present paper.*

The following result shall be termed as a **premature** *Grüss inequality* in that the proof of the Grüss inequality is not taken to its final conclusion but is stopped prematurely.

Theorem 9. *Let* f,g *be integrable functions defined on* $[a,b]$, *and let* $d \leq g(t) \leq D$. *Then*

$$|\mathfrak{T}(f,g)| \leq \frac{D-d}{2}\left[\mathfrak{T}(f,f)\right]^{\frac{1}{2}}, \tag{4.12}$$

where

$$\mathfrak{T}(f,f)=\mathfrak{M}\left[f^2\right]-\left[\mathfrak{M}(f)\right]^2 \tag{4.13}$$

with

$$\mathfrak{M}(f)=\frac{1}{b-a}\int_a^b f(t)\,dt \tag{4.14}$$

and $\frac{1}{2}$ *is the best possible constant.*

Proof. The proof follows that of the Grüss inequality as given in [16, p. 296]. The identity

$$\mathfrak{T}(f,g)=\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(\tau))(g(t)-g(\tau))\,dt d\tau \tag{4.15}$$

may easily be shown to be valid.
Now, applying the Cauchy-Schwartz-Buniakowsky integral inequality for double integrals, we have, on denoting the right hand side of (4.15) by $\mathfrak{T}_2(f,g)$,

$$\mathfrak{T}_2^2(f,g)\le\mathfrak{T}_2(f,f)\cdot\mathfrak{T}_2(g,g).$$

Therefore, from (4.15)

$$\mathfrak{T}^2(f,g)\le\mathfrak{T}(f,f)\cdot\mathfrak{T}(g,g). \tag{4.16}$$

Now,

$$\begin{aligned}&\mathfrak{T}(g,g) \\ =\ &(D-\mathfrak{M}(g))(\mathfrak{M}(g)-d)-\frac{1}{b-a}\int_a^b (D-g(t))(g(t)-d)\,dt \\ \le\ &(D-\mathfrak{M}(g))(\mathfrak{M}(g)-d)\end{aligned} \tag{4.17}$$

since $d\le g(t)\le D$.
In addition, using the elementary inequality for any real numbers p and q,

$$pq\le\left(\frac{p+q}{2}\right)^2,$$

we have, from (4.17),

$$\mathfrak{T}(g,g)\le\left(\frac{D-d}{2}\right)^2. \tag{4.18}$$

Combining (4.18) and (4.16), the results $(4.12-4.14)$ are obtained and the theorem is proved. To prove the sharpness of (4.12) simply take $f(t)=g(t)=sgn\left(t-\frac{a+b}{2}\right)$. ∎

Remark 14. *To prove* (4.1), *the bound* $\left(\frac{C-c}{2}\right)^2$ *for* $\mathfrak{T}(f,f)$ *would be obtained in a similar fashion to that of* $\mathfrak{T}(g,g)$, *and hence the term* ***premature****.*

Theorem 10. *Let the conditions be as in Theorem 5. The following sharper inequality holds:*

$$\begin{aligned}(4.19)\quad & \left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f(x)+\gamma\left[\left(\frac{x-a}{b-a}\right)f(a)\right.\right.\right.\\ & \left.\left.\left.+\left(\frac{b-x}{b-a}\right)f(b)\right]\right\}-(b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right|\\ \leq\ & \frac{(D-d)}{2\sqrt{3}}(b-a)\left\{\left(\frac{b-a}{2}\right)^2\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]\right.\\ & \left.+3\left(x-\frac{a+b}{2}\right)^2\left[\frac{1}{4}-\left(\gamma-\frac{1}{2}\right)^2\right]\right\}^{\frac{1}{2}}\end{aligned}$$

where $S=\frac{f(b)-f(a)}{b-a}$, *the secant slope.*

Proof. The proof of the current theorem follows along similar lines to that of Theorem 5 with the exception that a *premature* Grüss theorem (Theorem 6) is used. From the identity (4.4) the function $K(x,t)$ is known and it is as given by (4.5). Thus, applying the *premature* Grüss theorem (Theorem 6) to the mappings $K(x,\cdot)$ and $f'(\cdot)$ we obtain

$$\begin{aligned}(4.20)\quad & \left|\frac{1}{b-a}\int_a^b K(x,t)f'(t)\,dt-\frac{1}{b-a}\int_a^b K(x,t)\,dt\cdot\frac{1}{b-a}\int_a^b f'(t)\,dt\right|\\ \leq\ & \frac{D-d}{2}\cdot\left[\frac{1}{b-a}\int_a^b K^2(x,t)\,dt-\left(\frac{1}{b-a}\int_a^b K(x,t)\,dt\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Now, from (4.5),

$$\begin{aligned}& \frac{1}{b-a}\int_a^b K^2(x,t)\,dt\\ =\ & \frac{1}{b-a}\left\{\int_a^x[t-(\gamma x+(1-\gamma)a)]^2\,dt+\int_x^b[t-(\gamma x+(1-\gamma)b)]^2\,dt\right\}\\ =\ & \frac{1}{b-a}\left\{\int_{-\gamma(x-a)}^{(1-\gamma)(x-a)}u^2du+\int_{-(1-\gamma)(b-x)}^{\gamma(b-x)}v^2dv\right\}\\ =\ & \frac{1}{3(b-a)}\left[\gamma^3+(1-\gamma)^3\right]\left[(x-a)^3+(b-x)^3\right].\end{aligned}$$

The well-known identity

$$(4.21)\quad X^3+Y^3=(X+Y)\left[\left(\frac{X+Y}{2}\right)^2+3\left(\frac{X-Y}{2}\right)^2\right]$$

may be utilized to give

$$\begin{aligned}(4.22)\quad & \frac{1}{b-a}\int_a^b K^2(x,t)\,dt\\ =\ & \frac{1}{3}\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right].\end{aligned}$$

Thus, using the fact that

$$\frac{1}{b-a}\int_a^b f'(t)\,dt = \frac{f(b)-f(a)}{b-a} = S,$$

the secant slope together with (4.7) and (4.22) gives, from (4.20) :

$$\left|\frac{1}{b-a}\int_a^b K(x,t)\,f'(t)\,dt - (1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right| \tag{4.23}$$
$$\leq \frac{(D-d)}{2}\left\{\frac{1}{3}\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]\right.$$
$$\left.-4\left(\gamma-\frac{1}{2}\right)^2\left(x-\frac{a+b}{2}\right)^2\right\}^{\frac{1}{2}}$$
$$= \frac{(D-d)}{2\sqrt{3}}\left\{\left(\frac{b-a}{2}\right)^2\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]\right.$$
$$\left.+3\left(x-\frac{a+b}{2}\right)^2\left[\frac{1}{4}-\left(\gamma-\frac{1}{2}\right)^2\right]\right\}^{\frac{1}{2}}.$$

The term in the braces is, of course, positive since $b>a, \gamma\in[0,1], x\in[a,b]$ and

$$\frac{1}{4}-\left(\gamma-\frac{1}{2}\right)^2 = \gamma(1-\gamma).$$

Utilizing the identity (4.4) in (4.23) produces the result (4.19). Thus, the theorem is proved. ∎

Corollary 6. *Let the conditions be as in Theorem 5. Then the following inequality holds for all* $x\in[a,b]$,

$$\left|\int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a)f(x)+(x-a)f(a)+(b-x)f(b)\right]\right| \tag{4.24}$$
$$\leq \frac{(D-d)}{4\sqrt{3}}(b-a)\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}.$$

Proof. The result (4.24) is readily obtained from (4.19) by substituting $\gamma=\frac{1}{2}$. ∎

Corollary 7. *Let the conditions be as in Theorem 5. Then the following inequality holds for all* $\gamma\in[0,1]$

$$\left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f\left(\frac{a+b}{2}\right)+\frac{\gamma}{2}\left[f(a)+f(b)\right]\right\}\right| \tag{4.25}$$
$$\leq \frac{(D-d)}{4\sqrt{3}}(b-a)^2\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]^{\frac{1}{2}}.$$

Proof. Taking $x=\frac{a+b}{2}$ in (4.19) together with a minor rearrangement gives (4.25). ∎

Remark 15. *Result* (4.19) *is sharper than result* (4.3) *since the premature Grüss theorem is sharper than the Grüss theorem utilized to obtain* (4.3).

Remark 16. *Substituting $\gamma = 0$ into (4.19) gives an adjusted Ostrowski type rule, namely*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[f(x) - \left(x - \frac{a+b}{2} \right) S \right] \right| \le \frac{(D-d)}{4\sqrt{3}} (b-a)^2 .$$

This is a uniform bound which does not depend on the value of x. Thus, a mid-point rule would have the same bound as evaluating the function at any $x \in [a,b]$ together with an adjustment factor. Evaluation of the above result at $x = a$ or $x = b$ produces the standard trapezoidal rule.

Remark 17. *Taking $\gamma = 1$ in (4.19) gives*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[\left(\frac{x-a}{b-a} \right) f(a) + \left(\frac{b-x}{b-a} \right) f(b) + \left(x - \frac{a+b}{2} \right) S \right] \right|$$
$$\le \frac{(D-d)}{4\sqrt{3}} (b-a)^2 .$$

That is, using the fact that $S = \frac{f(b)-f(a)}{b-a}$, the trapezoidal rule,

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(a) + f(b) \right] \right| \le \frac{(D-d)}{4\sqrt{3}} (b-a)^2 ,$$

is recovered.

Remark 18. *The sharpest bounds for (4.24) and (4.25) are at $x = \frac{a+b}{2}$ and $\gamma = \frac{1}{2}$ respectively. This result can be obtained directly from (4.19) by taking x and γ at the mid-point, giving the best quadrature rule of this type as*

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f\left(\frac{a+b}{2} \right) + \frac{f(a)+f(b)}{2} \right] \right| \le \frac{(D-d)}{8\sqrt{3}} (b-a)^2 . \tag{4.26}$$

If $\gamma = \frac{1}{3}$ is taken in (4.25), then a Simpson type rule is obtained, giving

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{3}\left[2f\left(\frac{a+b}{2} \right) + \frac{f(a)+f(b)}{2} \right] \right| \le \frac{(D-d)}{8\sqrt{3}} (b-a)^2 \cdot \frac{2}{\sqrt{3}}.$$

This bound is worse than the optimal rule (4.26) by a relative amount of $\left(\frac{2}{\sqrt{3}} - 1 \right)$ which is approximately 15.5%. Computationally, the quadrature rule (4.26) is just as easy to apply as Simpson's rule since the only difference is the weights.

Remark 19. *The bound in (4.26) is $\frac{1}{\sqrt{3}}$ times better than that in (4.10). That is, the bound in (4.10) is worse than that in (4.26) by a relative amount of $\left(1 - \frac{1}{\sqrt{3}} \right)$.*

The *optimal* quadrature rule of this section will now be applied from (4.26) and it will be denoted by A_o.

Theorem 11. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{I}$ (the interior of I) and let $a, b \in \mathring{I}$ with $b > a$. Let $f' \in L_1[a,b]$ and $d \le f'(x) \le D, \forall x \in [a,b]$.*

Further, let I_n be any partition of $[a,b]$ such that $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. Then we have

$$\int_a^b f(x)\,dx = A_o(f, I_n) + R_o(f, I_n)$$

where

$$A_o(f, I_n) = \frac{1}{2}\sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[f(x_i) + f(x_{i+1})\right],$$

and

$$\begin{aligned}|R_o(f, I_n)| &\le \frac{(D-d)}{8\sqrt{3}}\sum_{i=0}^{n-1} h_i^2\\ &\le \frac{(D-d)}{8\sqrt{3}} n\nu^2(h)\end{aligned}$$

with $\nu(h) = \max_{i=0,...,n-1} h_i$.

Proof. Applying inequality (4.26) on the interval $[x_i, x_{i+1}]$ for $i = 0, 1, ..., n-1$ we have

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h_i}{4}\left[2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_i) + f(x_{i+1})\right]\right| \le \frac{(D-d)}{8\sqrt{3}} h_i^2$$

where $h_i = x_{i+1} - x_i$.
Summing over i for $i = 0$ to $n-1$ gives $A_o(f, I_n)$ and $|R_o(f, I_n)|$. ∎

Corollary 8. *Let the conditions of Theorem 11 hold. In addition, let I_n be the equidistant partition of $[a,b]$, $I_n : x_i = a + \left(\frac{b-a}{n}\right) i, i = 0, 1, ..., n$ then*

$$\left|\int_a^b f(x)\,dx - A_o(f, I_n)\right| \le \frac{(D-d)}{8\sqrt{3}}\frac{(b-a)^2}{n}.$$

Proof. From Theorem 11 with $h_i = \frac{b-a}{n}$ for all i such that

$$|R_o(f, I_n)| \le \frac{(D-d)}{8\sqrt{3}}\sum_{i=0}^{n-1}\left(\frac{b-a}{n}\right)^2 = \frac{(D-d)}{8\sqrt{3}}\cdot\frac{(b-a)^2}{n}$$

and hence the result is proved. ∎

Remark 20. *If we wish to qpproximate the integral $\int_a^b f(x)\,dx$ using the quadrature rule $A_o(f, I_n)$ of Corollary 8 with an accuracy of $\varepsilon > 0$, then we need $n_\varepsilon \in \mathbb{N}$ points for the equispaced partition I_n where*

$$n_\varepsilon \ge \left[\frac{(D-d)(b-a)^2}{8\sqrt{3}\,\varepsilon}\right] + 1$$

where $[x]$ denotes the integer part of x.

It should further be noted that the application of Corollary 8, in practice, is costly as it stands. The following corollary is more appropriate as it is more efficient.

Corollary 9. *Let the conditions of Theorem 11 hold and let I_{2m} be the equidistant partition of $[a,b]$, $I_{2m}: x_i = a + ih,\ i = 0,1,...,2m$ with $h = \frac{b-a}{2m}$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{h}{4}\left[f(x_0) + f(x_{2m})\right] - \frac{h}{2}\sum_{i=1}^{2m-1} f(x_i)\right| \le \frac{D-d}{16\sqrt{3}}\frac{(b-a)^2}{m}.$$

Proof. From Theorem 11

$$A_o(f, I_{2m}) = \frac{h}{2}\sum_{i=0}^{m-1} f(x_{2i+1}) + \frac{h}{4}\sum_{i=0}^{m-1}\left[f(x_{2i}) + f\left(x_{2(i+1)}\right)\right]$$

since

$$\frac{x_{2(i+1)} + x_{2i}}{2} = a + h(2i+1) = x_{2i+1}.$$

Now,

$$\begin{aligned}\sum_{i=0}^{m-1}\left[f(x_{2i}) + f\left(x_{2(i+1)}\right)\right] &= f(x_0) + f(x_{2m}) + \sum_{i=1}^{m-1} f(x_{2i}) + \sum_{i=0}^{m-2} f\left(x_{2(i+1)}\right) \\ &= f(x_0) + f(x_{2m}) + 2\sum_{i=1}^{m-1} f(x_{2i}).\end{aligned}$$

Thus

$$A_o(f, I_{2m}) = \frac{h}{4}\left[f(x_0) + f(x_{2m})\right] + \frac{h}{2}\sum_{i=1}^{2m-1} f(x_i),$$

where $h = \frac{b-a}{2m}$.
Further, from Theorem 11 with $h_i = \frac{b-a}{2m}$ for $i = 0,1,...,2m-1$,

$$|R_o(f, I_{2m})| \le \frac{(D-d)}{8\sqrt{3}}\sum_{i=0}^{2m-1}\left(\frac{b-a}{2m}\right)^2 = \frac{D-d}{16\sqrt{3}}\frac{(b-a)^2}{m}.$$

The corollary is thus proved. ∎

5. A Generalized Ostrowski-Grüss Inequality Via a New Identity

Traditionally, the Grüss inequality was effectively obtained by seeking a bound on $\mathfrak{T}^2(f,g)$ via a double integral identity and the Cauchy-Schwartz-Buniakowsky integral inequality to reduce the problem down to obtaining bounds for $\mathfrak{T}(f,f)$.

Recently, Dragomir and McAndrew [11] have obtained bounds on $\mathfrak{T}(f,g)$ as defined in (4.2), where f and g are integrable, by using the identity

$$\mathfrak{T}(f,g) = \frac{1}{b-a}\int_a^b \left[f(t) - \mathfrak{M}(f)\right]\left[g(t) - \mathfrak{M}(g)\right]dt. \tag{5.1}$$

Hence

$$|\mathfrak{T}(f,g)| \le \frac{1}{b-a}\int_a^b \left|\left(f(t) - \mathfrak{M}(f)\right)\left(g(t) - \mathfrak{M}(g)\right)\right| dt. \tag{5.2}$$

In particular, they apply the inequality when one of the functions is known and so effectively (although not explicitly stated) use

$$|\mathfrak{T}(f,g)| \le \sup_{t\in[a,b]}|g(t) - \mathfrak{M}(g)| \cdot \frac{1}{b-a}\int_a^b |f(t) - \mathfrak{M}(f)|\,dt, \tag{5.3}$$

where f is known.

Theorem 12. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{I}$ and let $a, b \in \mathring{I}$ with $a < b$. Furthermore, let $f' \in L_1[a,b]$ and $d \le f'(x) \le D$, $\forall x \in [a,b]$. Then the following inequality holds:*

$$\begin{aligned}(5.4)\quad & \left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\}\right. \\ & \left. + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right) S\right| \\ \le & \left[\frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|\right] I(\gamma, x) \\ \le & (D-d)\, I(\gamma, x),\end{aligned}$$

where

$$I(\gamma, x) = \int_a^b \left|K(x,t) - (1-2\gamma)\left(x - \frac{a+b}{2}\right)\right| dt, \tag{5.5}$$

$$K(x,t) = \begin{cases} t - (\gamma x + (1-\gamma) a), & t \le x \\ t - (\gamma x + (1-\gamma) b), & t > x \end{cases}, \tag{5.6}$$

$$\gamma \in [0,1],$$

and

$$S = \frac{f(b) - f(a)}{b-a}.$$

Proof. Applying (5.3) on the mappings $K(x,\cdot)$ and $f'(\cdot)$ gives
(5.7)

$$(b-a)\left|\mathfrak{T}(K, f')\right| \le \sup_{t\in[a,b]} \left|f'(t) - S\right| \cdot \int_a^b \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,u)\,du\right| dt.$$

Now,

$$\max\{D - S, S - d\} = \frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|$$

and, from (4.7),

$$\frac{1}{b-a}\int_a^b K(x,u)\,du = (1-2\gamma)\left(x - \frac{a+b}{2}\right),$$

and so

$$\int_a^b \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,u)\,du\right| dt = I(\gamma, x).$$

Hence,

$$\left|\mathfrak{T}(K, f')\right| \le \left[\frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|\right] I(\gamma, x). \tag{5.8}$$

Furthermore, using identity (4.4), (5.8) and the fact that $S = \frac{1}{b-a}\int_a^b f'(t)\,dt$, (5.4) results and the first part of the theorem is proved. Taking $S = d$ or D provides the upper bound given by the second inequality. ∎

We now wish to determine a closed form expression for $I(\gamma, x)$ as given by (5.5) where $K(x,t)$ is from (5.6).

Now, $I(\gamma, x)$ may be written as

$$I(\gamma, x) = \int_a^x |t - \phi(x)|\, dt + \int_x^b |t - \psi(x)|\, dt \tag{5.9}$$

where

$$\phi(x) = (1-\gamma)x + \gamma b - \frac{b-a}{2},\ \ \psi(x) = (1-\gamma)x + \gamma a + \frac{b-a}{2}. \tag{5.10}$$

In (5.9), let $(b-a)u = t - \phi(x)$ and $(b-a)v = t - \psi(x)$, such that

$$I(\gamma, x) = (b-a)^2 \left\{ \int_{\frac{a-\phi(x)}{b-a}}^{\frac{x-\phi(x)}{b-a}} |u|\, du + \int_{\frac{x-\psi(x)}{b-a}}^{\frac{b-\psi(x)}{b-a}} |v|\, dv \right\}. \tag{5.11}$$

To simplify the problem it is worthwhile to parameterize the partitioning of the interval $[a,b]$. To that end let

$$x = \delta b + (1-\delta)a,\ \ \delta \in [0,1]$$

so that

$$\delta = \frac{x-a}{b-a},\ 1-\delta = \frac{b-x}{b-a} \text{ and } x - \frac{a+b}{2} = (b-a)\left(\delta - \frac{1}{2}\right). \tag{5.12}$$

Now, from (5.10)

$$\begin{aligned} \frac{a-\phi(x)}{b-a} &= \left(\frac{\frac{a+b}{2} - x}{b-a}\right) - \gamma\left(\frac{b-x}{b-a}\right) \\ &= \frac{1}{2} - \delta - \gamma(1-\delta) \\ &= (1-\gamma)\left[1 - \frac{1}{2(1-\gamma)} - \delta\right], \end{aligned} \tag{5.13}$$

$$\begin{aligned} \frac{x-\phi(x)}{b-a} &= \frac{1}{2} - \left(\frac{b-x}{b-a}\right)\gamma \\ &= \frac{1}{2} - (1-\delta)\gamma \\ &= \gamma\left[\delta - \left(1 - \frac{1}{2\gamma}\right)\right], \end{aligned} \tag{5.14}$$

$$\begin{aligned} \frac{x-\psi(x)}{b-a} &= \gamma\left(\frac{x-a}{b-a}\right) - \frac{1}{2} \\ &= \gamma\delta - \frac{1}{2} \\ &= \gamma\left[\delta - \frac{1}{2\gamma}\right] \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} \frac{b-\psi(x)}{b-a} &= \frac{\left(\frac{a+b}{2}-x\right)}{b-a}+\gamma\left(\frac{x-a}{b-a}\right) \\ &= \frac{1}{2}-\delta+\gamma\delta \\ &= (1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]. \end{aligned} \tag{5.16}$$

Thus, (5.11) becomes, for $x=\delta b+(1-\delta)a$, on using $(5.13)-(5.16)$,

$$I(\gamma,x)=J(\gamma,\delta)=(b-a)^2\left[J_1(\gamma,\delta)+J_2(\gamma,\delta)\right], \tag{5.17}$$

where

$$J_1(\gamma,\delta)=\int_{(1-\gamma)\left[1-\frac{1}{2(1-\gamma)}-\delta\right]}^{\gamma\left[\delta-\left(1-\frac{1}{2\gamma}\right)\right]}|u|\,du \tag{5.18}$$

and

$$J_2(\gamma,\delta)=\int_{\gamma\left[\delta-\frac{1}{2\gamma}\right]}^{(1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]}|v|\,dv. \tag{5.19}$$

It should be noted that

$$J_2(\gamma,\delta)=J_1(1-\gamma,1-\delta)$$

and

$$J_1(\gamma,\delta)=J_2(1-\gamma,1-\delta), \tag{5.20}$$

so that only one of (5.18) or (5.19) need be evaluated explicitly and the other may be obtained in terms of it.

We shall consider $J_2(\gamma,\delta)$ in some detail. There are three possibilities to consider. The limits in (5.19) are either both negative, one negative and one positive, or are both positive. We note that the top limit is always greater than the bottom since $\frac{1}{2}-(1-\gamma)\delta>\gamma\delta-\frac{1}{2}$.

Thus, over the three different regions we have:

$$\begin{aligned} J_2(\gamma,\delta) &= \begin{cases} \int_{\gamma\left[\delta-\frac{1}{2\gamma}\right]}^{(1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]} -v\,dv, & \frac{1}{2(1-\gamma)}<\delta<\frac{1}{2\gamma} \\ \int_{\gamma\left[\delta-\frac{1}{2\gamma}\right]}^{0} -v\,dv+\int_0^{(1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]} v\,dv, & \delta<\frac{1}{2\gamma},\ \delta<\frac{1}{2(1-\gamma)} \\ \int_{\gamma\left[\delta-\frac{1}{2\gamma}\right]}^{(1-\gamma)\left[\frac{1}{2(1-\gamma)}-\delta\right]} v\,dv, & \frac{1}{2\gamma}<\delta<\frac{1}{2(1-\gamma)} \end{cases} \\ &= \begin{cases} \frac{1}{2}\left[\left(\gamma\delta-\frac{1}{2}\right)^2-\left(\frac{1}{2}-(1-\gamma)\delta\right)^2\right], & \frac{1}{2(1-\gamma)}<\delta<\frac{1}{2\gamma} \\ \frac{1}{2}\left[\left(\gamma\delta-\frac{1}{2}\right)^2+\left(\frac{1}{2}-(1-\gamma)\delta\right)^2\right], & \delta<\frac{1}{2\gamma},\ \delta<\frac{1}{2(1-\gamma)} \\ \frac{1}{2}\left[\left(\frac{1}{2}-(1-\gamma)\delta\right)^2-\left(\gamma\delta-\frac{1}{2}\right)^2\right], & \frac{1}{2\gamma}<\delta<\frac{1}{2(1-\gamma)}. \end{cases} \end{aligned}$$

Now, using the result $\frac{X^2-Y^2}{2} = \frac{1}{2}(X-Y)(X+Y)$ and (2.5), the above expressions may be simplified to give

$$(5.21)\qquad J_2(\gamma,\delta) = \begin{cases} -(1-\delta)\left(\gamma-\frac{1}{2}\right)\delta, & \frac{1}{2(1-\gamma)} < \delta < \frac{1}{2\gamma} \\ \left(\frac{1-\delta}{2}\right)^2 + \left(\gamma-\frac{1}{2}\right)^2\delta^2, & \delta < \frac{1}{2\gamma},\ \delta < \frac{1}{2(1-\gamma)} \\ (1-\delta)\left(\gamma-\frac{1}{2}\right)\delta, & \frac{1}{2\gamma} < \delta < \frac{1}{2(1-\gamma)}. \end{cases}$$

Further, using (5.20), an expression for $J_1(\gamma,\delta)$ may be readily obtained from (5.21) to give

$$(5.22)\qquad J_1(\gamma,\delta) = \begin{cases} -\delta\left(\frac{1}{2}-\gamma\right)(1-\delta), & 1-\frac{1}{2(1-\gamma)} < \delta < 1-\frac{1}{2\gamma} \\ \left(\frac{\delta}{2}\right)^2 + \left(\frac{1}{2}-\gamma\right)^2(1-\delta)^2, & \delta > 1-\frac{1}{2\gamma},\ \delta > 1-\frac{1}{2(1-\gamma)} \\ \delta\left(\frac{1}{2}-\gamma\right)(1-\delta), & 1-\frac{1}{2\gamma} < \delta < 1-\frac{1}{2(1-\gamma)}. \end{cases}$$

For an explicit evaluation of $I(\gamma,x)$, (5.17) needs to be determined. This involves the addition of $J_1(\gamma,\delta)$ and $J_2(\gamma,\delta)$. This may best be accomplished by reference to a diagram. Figure 5.1 shows five regions on the $\gamma\delta-$plane defined by the curves $\delta = \frac{1}{2(1-\gamma)}$, $\delta = \frac{1}{2\gamma}$, $\delta = 1-\frac{1}{2\gamma}$, $\delta = 1-\frac{1}{2(1-\gamma)}$, where $\gamma = 0,\ \gamma = 1,\ \delta = 0,\ \delta = 1$ define the outside boundary. The regions are defined as follows

$$(5.23)\qquad \begin{cases} A: & \delta > \frac{1}{2(1-\gamma)}, & \delta < \frac{1}{2\gamma}, & \delta > 1-\frac{1}{2\gamma}, & \delta > 1-\frac{1}{2(1-\gamma)}; \\ B: & \delta > \frac{1}{2\gamma}, & \delta < \frac{1}{2(1-\gamma)}, & \delta > 1-\frac{1}{2\gamma}, & \delta > 1-\frac{1}{2(1-\gamma)}; \\ C: & \delta < \frac{1}{2(1-\gamma)}, & \delta < \frac{1}{2\gamma}, & \delta < 1-\frac{1}{2\gamma}, & \delta > 1-\frac{1}{2(1-\gamma)}; \\ D: & \delta < \frac{1}{2\gamma}, & \delta < \frac{1}{2(1-\gamma)}, & \delta < 1-\frac{1}{2(1-\gamma)}, & \delta > 1-\frac{1}{2\gamma}; \\ \text{and} \\ E: & \delta < \frac{1}{2(1-\gamma)}, & \delta < \frac{1}{2\gamma}, & \delta < 1-\frac{1}{2\gamma}, & \delta > 1-\frac{1}{2(1-\gamma)}. \end{cases}$$

It is important to note that the first two inequalities in each of the regions define the contributions from $J_2(\gamma,\delta)$ and the second two, that of $J_1(\gamma,\delta)$. Thus using $(5.21)-(5.23)$, (5.17) is given by

$$(5.24)\qquad \frac{J(\gamma,\delta)}{(b-a)^2} = \begin{cases} \left(\gamma-\frac{1}{2}\right)(1-\delta)\left[\left(\gamma-\frac{1}{2}\right)(1-\delta)-\delta\right] + \left(\frac{\delta}{2}\right)^2 & \text{on } A \\ \left(\gamma-\frac{1}{2}\right)(1-\delta)\left[\left(\gamma-\frac{1}{2}\right)(1-\delta)+\delta\right] + \left(\frac{\delta}{2}\right)^2 & \text{on } B \\ \left(\gamma-\frac{1}{2}\right)\delta\left[\left(\gamma-\frac{1}{2}\right)\delta+(1-\delta)\right] + \left(\frac{1-\delta}{2}\right)^2 & \text{on } C \\ \left(\gamma-\frac{1}{2}\right)\delta\left[\left(\gamma-\frac{1}{2}\right)\delta-(1-\delta)\right] + \left(\frac{1-\delta}{2}\right)^2 & \text{on } D \\ \left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\delta-\frac{1}{2}\right)^2\right] & \text{on } E \end{cases}.$$

FIGURE 5.1. Diagram showing regions of validity for $\frac{J(\gamma,\delta)}{(b-a)^2}$ as given by (5.23) and (5.24), as well as its contours.

Remark 21. *We may now proceed in one of two ways. One approach is to transform to an expression involving x, thus giving $I(\gamma, x)$, and so (5.4) may be used. The second approach is to work in terms of δ so that Theorem 12 would be converted to an expression involving δ. We will take a modification of the first approach. Once a particular value γ is determined which dictates the type of rule δ is transformed in terms of x using the relation $J(\gamma,\delta) = I(\gamma, x)$, where $\delta = \frac{x-a}{b-a}$.*

Remark 22. *Taking different values of γ will produce bounds for various inequalities.*
For $\gamma = 0$, then from Figure 5.1 it may be seen that we are on the left boundary of region A and D and obtain, from (5.24), a uniform bound independent of δ,

$$J(0,\delta) = \frac{(b-a)^2}{4}.$$

Thus, from (5.4), a perturbed Ostrowski inequality is obtained on noting from (5.17) that $I(\gamma, x) = J(\gamma,\delta)$,

$$\left|\int_a^b f(t)\,dt - (b-a)\left[f(x) - \left(x - \frac{a+b}{2}\right)S\right]\right| \leq \frac{(b-a)^2}{4}\left[\frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|\right]. \tag{5.25}$$

For $\gamma = 1$, it may be noticed from Figure 5.1 that we are now on the right boundary of B and D so that from (5.24), a uniform bound independent of δ is obtained viz.,

$$J(1,\delta) = \frac{(b-a)^2}{4}.$$

Thus, from (5.4), *a perturbed generalized trapezoidal inequality is obtained, namely*

$$\left|\int_a^b f(t)\,dt-(b-a)\left[\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-\left(x-\frac{a+b}{2}\right)S\right]\right| \leq \frac{(b-a)^2}{4}\left[\frac{D-d}{2}+\left|S-\frac{d+D}{2}\right|\right]. \tag{5.26}$$

Taking $x=\frac{a+b}{2}$ *reproduces the result of Dragomir and McAndrew* [11]. *Again, it may be noticed that the above result is a uniform bound for any* $x\in[a,b]$.

Corollary 10. *Let the conditions of* f *be as in Theorem12. Then the following inequality holds for any* $x\in[a,b]$:

$$\left|\int_a^b f(t)\,dt-\frac{1}{2}\left[(b-a)f(x)+(x-a)f(a)+(b-x)f(b)\right]\right| \leq \frac{1}{4}\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{D-d}{2}+\left|S-\frac{d+D}{2}\right|\right], \tag{5.27}$$

where $S=\frac{f(b)-f(a)}{b-a}$.

Proof. Letting $\gamma=\frac{1}{2}$ in (5.4) readily produces the result (5.27) from (5.24), on noting that $I\left(\frac{1}{2},x\right)=J\left(\frac{1}{2},\delta\right)=\frac{b-a}{4}\left[\frac{1}{4}+\left(\delta-\frac{1}{2}\right)^2\right]$ where $(b-a)\left(\delta-\frac{1}{2}\right)=x-\frac{a+b}{2}$. ∎

Corollary 11. *Let the conditions of* f *be as in Theorem 12. Then the following inequality holds for any* $\gamma\in[0,1]$:

$$\left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f\left(\frac{a+b}{2}\right)+\frac{\gamma}{2}\left[f(a)+f(b)\right]\right\}\right| \leq \frac{(b-a)^2}{4}\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\left[\frac{D-d}{2}+\left|S-\frac{d+D}{2}\right|\right]. \tag{5.28}$$

Proof. Letting $x=\frac{a+b}{2}$ in (5.4) produces the result (5.28) from (5.24) on noting $I\left(\gamma,\frac{a+b}{2}\right)=J\left(\gamma,\frac{1}{2}\right)=\frac{(b-a)^2}{4}\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]$ in region E. ∎

Remark 23. *Taking* $x=\frac{a+b}{2}$ *in* (5.27) *or* $\gamma=\frac{1}{2}$ *in* (5.28) *is equivalent to taking both these values in* (5.4). *This produces the sharpest bound in this class, giving*

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{2}\left\{f\left(\frac{a+b}{2}\right)+\frac{1}{2}\left[f(a)+f(b)\right]\right\}\right| \leq \frac{(b-a)^2}{16}\left[\frac{D-d}{2}+\left|S-\frac{d+D}{2}\right|\right]. \tag{5.29}$$

For the bound, it is equivalent to taking the point $\left(\frac{1}{2},\frac{1}{2}\right)$ *in region* E *from* (5.24) *and Figure 5.1, thus giving* $J\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{16}$. *For a Simpson type rule, taking the point* $\left(\frac{1}{3},\frac{1}{2}\right)$ *in region* E *from* (5.24), *and Figure 5.1 gives* $J\left(\frac{1}{3},\frac{1}{2}\right)=\frac{1}{16}+\frac{1}{144}$ *which is a coarser bound than* $J\left(\frac{1}{2},\frac{1}{2}\right)$ *at which the minimum occurs (the centre point in Figure 5.1).*

Remark 24. *It should be noted that the best bound possible with the* ***premature*** *Grüss is given by* (4.26). *This may be compared with the current bound* (5.29). *Now,* (5.29) *is computationally more expensive, but even the* ***worst*** *bound,* $\frac{(b-a)^2}{16}(D-d)$ *in* (5.29) *is* ***better*** *than that of* (4.26).

Remark 25. *A generalized Simpson type rule may be obtained by taking* $\gamma = \frac{1}{3}$ *for unprescribed* x. *Thus, from* (5.24),

$$\frac{J\left(\frac{1}{3},\delta\right)}{(b-a)^2} = \begin{cases} \frac{1-\delta}{6}\cdot\frac{1+5\delta}{6} + \left(\frac{\delta}{2}\right)^2 & \frac{3}{4} < \delta < 1 \\ \frac{5}{18}\left[\frac{1}{4} + \left(\delta - \frac{1}{2}\right)^2\right] & \frac{1}{4} < \delta < \frac{3}{4} \\ \frac{\delta}{6}\cdot\frac{6-5\delta}{6} + \left(\frac{1-\delta}{2}\right)^2 & 0 < \delta < \frac{1}{4} \end{cases}, \tag{5.30}$$

and so from (5.4):

$$\left|\int_a^b f(t)\,dt - \frac{(b-a)}{3}\left[2f(x) + \left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right] + \frac{b-a}{3}\left(x - \frac{a+b}{2}\right)S\right| \le \left[\frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|\right] I\left(\frac{1}{3},x\right), \tag{5.31}$$

where

$$I\left(\frac{1}{3},x\right) = (b-a)^2 J\left(\frac{1}{3},\delta\right)$$

with

$$\delta = \frac{x-a}{b-a}.$$

That is, from (5.30),

$$I\left(\frac{1}{3},x\right) = \begin{cases} \frac{(b-x)}{6}\frac{[(b-a)+5(x-a)]}{6} + \left(\frac{x-a}{2}\right)^2, & \frac{3}{4} < \frac{x-a}{b-a} \\ \frac{5}{18}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right], & \frac{1}{4} < \frac{x-a}{b-a} < \frac{3}{4} \\ \frac{x-a}{6}\cdot\frac{6(b-a)-5(x-a)}{6} + \left(\frac{b-x}{2}\right)^2, & 0 < \frac{x-a}{b-a} < \frac{1}{4} \end{cases}.$$

Remark 26. *It may have been noticed from Figure 5.1 or, for that matter, directly from* (5.24). *Replacing* $1-\delta$ *by* δ *in* A *and* B *would give the regions* D *and* C *respectively. Also, replacing* $1-\gamma$ *by* γ *in* B *and* C *would give the regions* A *and* D *respectively. Thus, it would have been possible to investigate the region* $\frac{1}{2} \le \gamma \le 1$ *and* $\frac{1}{2} \le \delta \le 1$ *since we may readily transform any point* (γ', δ') *in the* $\gamma\delta$*−plane to one in this region. Thus, only the regions* B *and* E^* *would need to be analyzed where for* $\frac{1}{2} \le \gamma, \delta \le 1$,

$$B : \delta > \frac{1}{2\gamma},$$

and

$$E^* : \delta < \frac{1}{2\gamma}.$$

This approach was not followed since we were more interested in evaluation along lines perpendicular to the axes.

Remark 27. *For practical implementation of the above in numerical integration it would be expensive to calculate the bounds as given. However, instead of* $\frac{D-d}{2} + \left|S - \frac{d+D}{2}\right|$ *being used, the coarser bound of* $D-d$ *may be more suitable.*

The optimal quadrature rule of this section will now be applied from (5.29) and it will be denoted by A_0.

Theorem 13. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $\mathring{I}$ *(the interior of* I*) and let* $a, b \in \mathring{I}$ *with* $b > a$*. Let* $f' \in L_1[a,b]$ *and* $d \leq f'(x) \leq D$, $\forall x \in [a,b]$*. In addition, let* I_n *be a partition of* $[a,b]$ *such that* $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$*. Then we have*

$$\int_a^b f(x)\,dx = A_0(f, I_n) + R_0(f, I_n),$$

where

$$A_0(f, I_n) = \frac{1}{2}\sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[f(x_i) + f(x_{i+1})\right]$$

and

$$\begin{aligned} |R_0(f, I_n)| &\leq \frac{D-d}{32}\sum_{i=0}^{n-1} h_i^2 + \frac{1}{16}\sum_{i=0}^{n-1} h_i \sigma_i \\ &\leq \frac{D-d}{16}\sum_{i=0}^{n-1} h_i^2 \leq \left(\frac{D-d}{16}\right) n\nu^2(h), \end{aligned}$$

where

$$\sigma_i = |f(x_{i+1}) - f(x_i) - h_i(d+D)|$$

and

$$\nu(h) = \max_{i=0,\dots,n-1} h_i.$$

Proof. Applying inequality (5.29) on the interval $[x_i, x_{i+1}]$ for $i = 0, ..., n-1$ we have

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h_i}{4}\left\{2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_i) + f(x_{i+1})\right\}\right| \\ \leq\ & \frac{h_i^2}{16}\left[\frac{D-d}{2} + \left|S_i - \frac{d+D}{2}\right|\right], \end{aligned}$$

where

$$S_i = \frac{f(x_{i+1}) - f(x_i)}{h_i},\ h_i = x_{i+1} - x_i.$$

Summing over i for $i = 0, 1, ..., n-1$ gives $A_0(f, I_n)$ and the first bound for $|R_0(f, I_n)|$.
Now, consider the right hand side of the inequality above. Then

$$\frac{h_i^2}{16}\left[\frac{D-d}{2} + \left|S_i - \frac{d+D}{2}\right|\right] \leq \frac{h_i^2}{16}(D-d),$$

since

$$\left|S_i - \frac{d+D}{2}\right| \leq \frac{D-d}{2}.$$

Summing over i produces the last two upper bounds for the error. ∎

Corollary 12. *Let the conditions of Theorem 13 hold. Also, let I_{2m} be the equidistant partition of $[a,b]$, $I_{2m} : x_i = a + ih,\ i = 0, 1, ..., 2m$ with $h = \frac{b-a}{2m}$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{h}{4}[f(x_0) + f(x_{2m})] - \frac{h}{2}\sum_{i=1}^{2m-1} f(x_i)\right| \leq \frac{D-d}{32} \cdot \frac{(b-a)^2}{m}.$$

Proof. From Theorem 13 with $h_i = \frac{b-a}{2m}$ for all i and using the expression for $A_0(f, I_{2m})$ as given in Corollary 9 produces the desired result. ∎

Remark 28. *If we wish to approximate the integral $\int_a^b f(t)\,dt$ using the above quadrature rule in Corollary 12, with an accuracy of $\varepsilon > 0$, then we need $2m_\varepsilon \in \mathbb{N}$ points for the equispaced partition I_{2m},*

$$m_\varepsilon \geq \left[\frac{D-d}{32}\frac{(b-a)^2}{\varepsilon}\right] + 1,$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

6. Inequalities for which the First Derivative Belongs to $L_1[a,b]$

In this section we discuss the situation in which $f' \in L_1[a,b]$ which is a linear space of all absolutely integrable functions on $[a,b]$. We use the usual norm notation $\|\cdot\|_1$, where, we recall, $\|g\|_1 := \int_a^b |g(s)|\,ds,\ g \in L_1[a,b]$.

Theorem 14. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and $a, b \in \mathring{I}$ are such that $b > a$. If $f' \in L_1[a,b]$, then the following inequality holds for all $x \in [a,b]$, $\alpha(x) \in [a,x]$ and $\beta(x) \in [x,b]$,*

$$\left|\int_a^b f(t)\,dt - [(\beta(x) - \alpha(x))f(x) + (\alpha(x) - a)f(a) + (b - \beta(x))f(b)]\right| \tag{6.1}$$
$$\leq \frac{\|f'\|_1}{2}\left\{\frac{b-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| + \left|\beta(x) - \frac{x+b}{2}\right| + \left|x - \frac{a+b}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| - \left|\beta(x) - \frac{b+x}{2}\right|\right|\right\}.$$

Proof. Let $K(x,t)$ be as defined in (2.2). An integration by parts produces the identity as given by (2.3). Thus, from (2.3),

$$\begin{aligned}(6.2)\quad & \left|\int_a^b f(t)\,dt - \left[(\beta(x)-\alpha(x))\,f(x) + (\alpha(x)-a)\,f(a) + (b-\beta(x))\,f(b)\right]\right| \\ = \quad & \left|\int_a^b K(x,t)\,f'(t)\,dt\right|.\end{aligned}$$

Now, using (2.2),

$$\begin{aligned}(6.3)\qquad & \left|\int_a^b K(x,t)\,f'(t)\,dt\right| \\ \leq\ & \int_a^x |t-\alpha(x)|\,|f'(t)|\,dt + \int_x^b |t-\beta(x)|\,|f'(t)|\,dt \\ =\ & \int_a^{\alpha(x)} (\alpha(x)-t)\,|f'(t)|\,dt + \int_{\alpha(x)}^x (t-\alpha(x))\,|f'(t)|\,dt \\ & + \int_x^{\beta(x)} (\beta(x)-t)\,|f'(t)|\,dt + \int_{\beta(x)}^b (t-\beta(x))\,|f'(t)|\,dt \\ \leq\ & (\alpha(x)-a)\int_a^{\alpha(x)} |f'(t)|\,dt + (x-\alpha(x))\int_{\alpha(x)}^x |f'(t)|\,dt \\ & + (\beta(x)-x)\int_x^{\beta(x)} |f'(t)|\,dt + (b-\beta(x))\int_{\beta(x)}^b |f'(t)|\,dt \\ \leq\ & M(x)\,\|f'\|_1\end{aligned}$$

where

$$M(x) = \max\{M_1(x), M_2(x)\}$$

with

$$M_1(x) = \max\{\alpha(x)-a, x-\alpha(x)\}$$

and

$$M_2(x) = \max\{\beta(x)-x, b-\beta(x)\}.$$

The well-known identity

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|$$

may be used to give

$$M_1(x) = \frac{x-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right|$$

and

$$M_2(x) = \frac{b-x}{2} + \left|\beta(x) - \frac{x+b}{2}\right|.$$

Thus, using the identity again gives

$$M(x) = \frac{M_1(x)+M_2(x)}{2} + \left|\frac{M_1(x)-M_2(x)}{2}\right| \tag{6.4}$$
$$= \frac{1}{2}\left\{\frac{b-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| + \left|\beta(x) - \frac{x+b}{2}\right| + \left|x - \frac{a+b}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| - \left|\beta(x) - \frac{b+x}{2}\right|\right|\right\}.$$

On substituting (6.4) into (6.3) and using (6.2), result (6.1) is produced and thus the theorem is proved. ∎

Corollary 13. *Let f satisfy the conditions of Theorem 14. Then $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{b+x}{2}$ give the best bound for any $x \in [a,b]$ and so*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(x) + \left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right| \le \frac{\|f'\|_1}{2}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]. \tag{6.5}$$

Proof. From (6.1) the minimal value each of the moduli can take is zero. Hence the result. ∎

Remark 29. *An even tighter bound may be obtained from* (8.5) *if x is taken to be at the mid-point giving*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{4}\|f'\|_1. \tag{6.6}$$

This result corresponds to the average of a mid-point and trapezoidal quadrature rule for which $f' \in L_1[a,b]$.

Theorem 15. *Let f satisfy the conditions as stated in Theorem 18. Then the following inequality holds for any $\gamma \in [0,1]$ and $x \in [a,b]$:*

$$\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma)f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right| \le \|f'\|_1\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]. \tag{6.7}$$

Proof. Let $\alpha(x)$ and $\beta(x)$ be as in (2.11). Then,

$$\beta(x) - \alpha(x) = (1-\gamma)(b-a),$$

$$\alpha(x) - a = \gamma(x-a),$$

$$b - \beta(x) = \gamma(b-x),$$

$$\alpha(x) - \frac{a+x}{2} = \left(\gamma - \frac{1}{2}\right)(x-a)$$

and

$$\beta(x) - \frac{x+b}{2} = -\left(\gamma - \frac{1}{2}\right)(b-x).$$

Now,

$$\begin{aligned} & \frac{b-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| + \left|\beta(x) - \frac{x+b}{2}\right| \\ = \ & (b-a)\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right] \end{aligned}$$

and

$$\begin{aligned} & \left|x - \frac{a+b}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| - \left|\beta(x) - \frac{x+b}{2}\right|\right| \\ = \ & \left|x - \frac{a+b}{2} + \left|\gamma - \frac{1}{2}\right|(x-a) - \left|\gamma - \frac{1}{2}\right|(b-x)\right| \\ = \ & \left|x - \frac{a+b}{2} + 2\left|\gamma - \frac{1}{2}\right|\left(x - \frac{a+b}{2}\right)\right| \\ = \ & 2\left|x - \frac{a+b}{2}\right|\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]. \end{aligned}$$

Substitution of the above results into (6.1) gives (6.7), thus proving the theorem. ∎

Remark 30. *If* $\gamma = \frac{1}{2}$ *in* (6.7) *then* $\alpha(x) = \frac{a+x}{2}$ *and* $\beta(x) = \frac{x+b}{2}$ *and so result* (6.5) *is rightly recovered. The best quadrature rule of this type is given by* (6.6) *which is obtained by taking the optimal* γ *and* x *values at their respective mid-points of* $\frac{1}{2}$ *and* $\frac{a+b}{2}$ *in* (6.7).

Remark 31. *Taking* $\gamma = 0$ *in* (6.7) *gives Ostrowski's inequality for* $f' \in L_1[a,b]$ *as obtained by Dragomir and Wang* [1]. *If* $\gamma = 1$ *in* (6.7), *then a generalized trapezoidal rule is obtained for which the best bound occurs when* $x = \frac{a+b}{2}$ *giving the classical trapezoidal type rule for functions* $f' \in L_1[a,b]$.

Corollary 14. *Let* f *satisfy the conditions as stated in Theorem 14. Then the following inequality holds for* $\gamma \in [0,1]$:

$$\begin{aligned} (6.8) \qquad & \left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}[f(a)+f(b)]\right\}\right| \\ \leq \ & \|f'\|_1 \frac{b-a}{2}\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]. \end{aligned}$$

Proof. Simply evaluating (8.7) at $x = \frac{a+b}{2}$ gives (6.8). ∎

Remark 32. *Taking* $\gamma = 0$ *and* 1 *into* (8.8) *gives the mid-point and trapezoidal type rules respectively.*

Remark 33. *Taking* $\gamma = \frac{1}{2}$ *in* (6.8) *gives the optimal quadrature rule shown in* (6.6). *Placing* $\gamma = \frac{1}{3}$ *gives a Simpson type rule with an error bound of* $\|f'\|_1 \cdot \frac{b-a}{3}$. *Thus, a Simpson type rule is relatively worse (by* $\frac{1}{3}$*) when compared with the optimal rule* (6.6). *In addition, the optimal rule is just as easy to implement as the Simpson rule. All that is different are the weights.*

The following results investigate the implementation of the above inequalities to numerical integration.

Theorem 16. *For any $a, b \in \mathbb{R}$ with $a < b$ let $f : (a, b) \to \mathbb{R}$ be a differentiable mapping. Let $f' \in L_1[a, b]$, then, for any partition $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of $[a, b]$ and any intermediate point vector $\xi = (\xi_0, \xi_1, ..., \xi_{n-1})$ such that $\xi_i \in [x_i, x_{i+1}]$ for $i = 0, 1, ..., n-1$, we have, for $\gamma \in [0, 1]$,*

$$\begin{aligned}
&\left| \int_a^b f(x)\,dx - A_c(f, I_n, \xi) \right| \\
\leq\ & \|f'\|_1 \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \max_{0 \leq i \leq n-1} \left\{ \frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right\} \\
\leq\ & \|f'\|_1 \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \nu(h)
\end{aligned} \tag{6.9}$$

where $h_i = x_{i+1} - x_i, \nu(h) = \max_{0 \leq i \leq n-1} h_i$ and $A_c(f, I_n, \xi)$ is given by

$$\begin{aligned}
A_c(f, I_n, \xi) &= (1-\gamma) \sum_{i=0}^{n-1} h_i f(\xi_i) \\
&\quad + \gamma \left[\sum_{i=0}^{n-1} (\xi_i - x_i) f(x_i) + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i) f(x_{i+1}) \right].
\end{aligned}$$

Proof. Applying inequality (6.7) on the interval $[x_i, x_{i+1}]$ for $i = 0, 1, ..., n-1$ we have:

$$\begin{aligned}
&\left| \int_{x_i}^{x_{i+1}} f(x)\,dx \right. \\
&\left. - \{(1-\gamma) f(\xi_i) h_i + \gamma [(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})]\} \right| \\
\leq\ & \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \max_{0 \leq i \leq n-1} \left[\frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \int_{x_i}^{x_{i+1}} |f'(x)|\,dx.
\end{aligned}$$

Summing the above inequality, we have (6.9). Furthermore, observe that

$$\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{h_i}{2}$$

for $i = 0, 1, ..., n-1$. Therefore,

$$\max_{0 \leq i \leq n-1} \left[\frac{h_i}{2} + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \leq \max_{0 \leq i \leq n-1} h_i = \nu(h)$$

and hence the theorem is proved. ∎

Remark 34. *The coefficient of the* γ *term in* $A_c(f, I_n, \xi)$ *may be simplified to give*

$$\begin{aligned} & \sum_{i=0}^{n-1}(\xi_i - x_i) f(x_i) + \sum_{i=0}^{n-1}(x_{i+1} - \xi_i) f(x_{i+1}) \\ = & \sum_{i=0}^{n-1}\xi_i [f(x_i) - f(x_{i+1})] + \sum_{i=0}^{n-1}[x_{i+1} f(x_{i+1}) - x_i f(x_i)] \\ = & \sum_{i=1}^{n-1}(\xi_i - \xi_{i-1}) f(x_i) + \xi_0 f(a) - \xi_{n-1} f(b) + bf(b) - af(a) \\ = & \sum_{i=1}^{n-1}(\xi_i - \xi_{i-1}) f(x_i) + (\xi_0 - a) f(a) + (b - \xi_{n-1}) f(b). \end{aligned}$$

This version has the advantage in that the number of function evaluations is minimized. Thus,

$$\begin{aligned} A_c(f, I_n, \xi) = & (1-\gamma)\sum_{i=0}^{n-1} h_i f(\xi_i) + \gamma\left\{\sum_{i=1}^{n-1}(\xi_i - \xi_{i-1}) f(x_i)\right. \\ & \left. + (\xi_0 - a) f(a) + (b - \xi_{n-1}) f(b)\right\} \end{aligned} \tag{6.10}$$

Corollary 15. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and the mapping* $f : (a, b) \to \mathbb{R}$ *be differentiable. Further, let* $f' \in L_1[a, b]$. *Then, for any partition* $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ *of* $[a, b]$ *we have for any* $0 \leq \gamma \leq 1$,

$$\int_a^b f(x)\,dx = A_1(f, I_n) + R_1(f, I_n) \tag{6.11}$$

where

$$A_1(f, I_n) = (1-\gamma)\sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{\gamma}{2}\sum_{i=0}^{n-1} h_i [f(x_i) + f(x_{i+1})],$$

and

$$|R_1(f, I_n)| \leq \frac{\|f'\|_1}{2}\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]\nu(h).$$

Proof. The proof is straightforward. We either start with Corollary 14 and follow the procedure of Theorem 16, or we can take the easier option of placing $\xi_i = \frac{x_i + x_{i+1}}{2}$ in Theorem 16 to immediately produce the result. ∎

Remark 35. *The quadrature rule given by* (6.11) *is a composite mid-point and trapezoidal rule with* γ *determining the relative weighting of the two. The optimal rule is obtained when the composition is a straightforward average which is obtained by taking* $\gamma = \frac{1}{2}$.

Corollary 16. *Let the conditions of Corollary 15 hold, taking in particular* $\gamma = \frac{1}{2}$ *and the partition to be equidistant so that* $I_{2m} : x_i = a + ih, i = 0, 1, ..., 2m$ *with* $h = \frac{b-a}{2m}$. *Then*

$$\int_a^b f(x)\,dx = A_o(f, I_{2m}) + R_o(f, I_{2m}) \tag{6.12}$$

where

$$A_o\left(f, I_{2m}\right) = \frac{h}{4}\left[f\left(a\right) + f\left(b\right) + 2\sum_{i=1}^{2m-1} f\left(x_i\right)\right]$$

and

$$\left|R_o\left(f, I_{2m}\right)\right| \leq \frac{\|f'\|_1}{8}\left(\frac{b-a}{m}\right).$$

Proof. From Corollary 15, let a subscript of o signify the optimal quadrature rule obtained when $\gamma = \frac{1}{2}$ and so

$$A_o\left(f, I_{2m}\right) = \frac{h}{2}\sum_{i=0}^{m-1} f\left(\frac{x_{2i} + x_{2(i+1)}}{2}\right) + \frac{h}{4}\sum_{i=0}^{m-1}\left[f\left(x_{2i}\right) + f\left(x_{2(i+1)}\right)\right],$$

where

$$\frac{x_{2i} + x_{2(i+1)}}{2} = a + h\left(2i+1\right) = x_{2i+1}.$$

Now

$$\begin{aligned}\sum_{i=0}^{m-1}\left[f\left(x_{2i}\right) + f\left(x_{2(i+1)}\right)\right] &= f\left(x_0\right) + f\left(x_{2m}\right) + \sum_{i=1}^{m-1} f\left(x_{2i}\right) + \sum_{i=0}^{m-2} f\left(x_{2(i+1)}\right) \\ &= f\left(x_0\right) + f\left(x_{2m}\right) + 2\sum_{i=1}^{m-1} f\left(x_{2i}\right).\end{aligned}$$

Thus,

$$\begin{aligned}A_o\left(f, I_{2m}\right) &= \frac{h}{2}\left[\sum_{i=0}^{m-1} f\left(x_{2i+1}\right) + \sum_{i=1}^{m-1} f\left(x_{2i}\right)\right] + \frac{h}{4}\left[f\left(x_0\right) + f\left(x_{2m}\right)\right] \\ &= \frac{h}{4}\left[f\left(x_0\right) + f\left(x_{2m}\right) + 2\sum_{i=1}^{2m-1} f\left(x_i\right)\right].\end{aligned}$$

Further, from Corollary 15 with $h_i = \frac{b-a}{2m}$ for $i = 0, 1, ..., 2m$ and $\gamma = \frac{1}{2}$ we obtain

$$\left|R_o\left(f, I_{2m}\right)\right| \leq \frac{\|f'\|_1}{8}\left(\frac{b-a}{m}\right),$$

and hence the corollary is proved. ∎

Remark 36. *If we wish to approximate the integral $\int_a^b f\left(t\right)dt$ using the quadrature rule $A_o\left(f, I_{2m}\right)$ and (6.12) with an accuracy of $\varepsilon > 0$, then we need $2m_\varepsilon \in \mathbb{N}$ points for the equispaced partition I_{2m} where*

$$m_\varepsilon \geq \left[\frac{\|f'\|_1}{8}\frac{\left(b-a\right)}{\varepsilon}\right] + 1,$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

7. Grüss-type Inequalities for Functions whose First Derivative Belongs to $L_1[a,b]$

The identity of Dragomir and McAndrew [11] as given by (5.1) will now be utilized to obtain further inequalities. Define an operator σ such that

$$\sigma(f) = f - \mathfrak{M}(f), \tag{7.1}$$

where $\mathfrak{M}(f) = \frac{1}{b-a}\int_a^b f(u)\,du$.

Then, (5.1) may be written as

$$\mathfrak{T}(f,g) = \mathfrak{T}(\sigma(f), \sigma(g)), \tag{7.2}$$

where

$$\mathfrak{T}(f,g) = \mathfrak{M}(fg) - \mathfrak{M}(f)\,\mathfrak{M}(g).$$

It may be noticed from (7.1) and (7.2) that $\mathfrak{M}(\sigma(f)) = \mathfrak{M}(\sigma(g)) = 0$, so that (7.2) may be written in the alternative form:

$$\mathfrak{M}(fg) - \mathfrak{M}(f)\,\mathfrak{M}(g) = \mathfrak{M}(\sigma(f)\,\sigma(g)). \tag{7.3}$$

Theorem 17. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and $a, b \in \mathring{I}$ are such that $b > a$. If $f' \in L_1[a,b]$, then the following inequality holds for all $x \in [a,b]$ and $\gamma \in [0,1]$:*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} \right. \\ &\left. \quad + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right) S \right| \\ &\leq \|\sigma(f')\|_1\, \theta(\gamma, x), \end{aligned} \tag{7.4}$$

where

$$\theta(\gamma,x) = \sup_{t\in[a,b]} \left| K(x,t) - (1-2\gamma)\left(x - \frac{a+b}{2}\right)\right|, \tag{7.5}$$

$K(x,t)$ is as given by (5.6) and $S = \mathfrak{M}(f')$ with $\mathfrak{M}(\cdot)$ and $\sigma(\cdot)$ as given by (7.1).

Proof. Applying (7.2) or (7.3) on the mappings $K(x,\cdot)$ and $f'(\cdot)$ gives

$$\begin{aligned} \mathfrak{T}(K, f') &= \mathfrak{T}(\sigma(K), \sigma(f')) \\ &= \mathfrak{M}(\sigma(K)\,\sigma(f')). \end{aligned} \tag{7.6}$$

Thus,

$$(b-a)\,|\mathfrak{T}(K,f')| \leq \|\sigma(f')\|_1 \sup_{t\in[a,b]} |\sigma(K)|. \tag{7.7}$$

Now,

$$\theta(\gamma,x) = \sup_{t\in[a,b]} |\sigma(K)| = \sup_{t\in[a,b]} |K(x,t) - \mathfrak{M}(K)|, \tag{7.8}$$

where, from (4.7),

$$\mathfrak{M}(K) = \frac{1}{b-a}\int_a^b K(x,u)\,du = (1-2\gamma)\left(x - \frac{a+b}{2}\right).$$

Further, using identity (4.4), (7.7) and (7.8), the inequality (7.4) is derived and the theorem is hence proved. ∎

We now wish to obtain an explicit expression for $\theta(\gamma,x)$ as given by (7.5). Using (5.6) in (7.5) gives

$$\theta(\gamma,x)=\sup_{t\in[a,b]}|k(x,t)|, \tag{7.9}$$

where

$$k(x,t)=\begin{cases} t-\phi(x), & t\,[a,x] \\ t-\psi(x), & t\in(x,b] \end{cases} \tag{7.10}$$

and $\phi(x),\ \psi(x)$ are as given by (5.10).

Therefore,

$$\theta(\gamma,x)=\max\{|a-\phi(x)|,|x-\phi(x)|,|x-\psi(x)|,|b-\psi(x)|\} \tag{7.11}$$

since the extremum points from (7.9) and (7.10) are obtained at the ends of the intervals as $k(x,t)$ is piecewise linear.

The representation (7.11) may be explicit enough, but it is possible to proceed further, as in section 5, by making the transformation

$$x=\delta b+(1-\delta)a,\ \delta\in[0,1]. \tag{7.12}$$

Using (5.12) – (5.16) gives

$$\begin{aligned}\theta(\gamma,x) &= \Theta(\gamma,\delta) \\ &= (b-a)\max\left\{\left|\frac{1}{2}-\delta-\gamma(1-\delta)\right|,\left|\frac{1}{2}-\gamma(1-\delta)\right|,\right. \\ &\qquad \left.\left|\gamma\delta-\frac{1}{2}\right|,\left|\frac{1}{2}-\delta+\gamma(\delta)\right|\right\}.\end{aligned} \tag{7.13}$$

Now, the expressions in (7.13) can be either positive or negative depending on the region $A,\ B,...,E$ as defined by (5.23) and depicted in Figure 5.1. The well known result

$$\max\{X,Y\}=\frac{X+Y}{2}+\frac{1}{2}|X-Y|$$

may be applied twice to give

$$\begin{aligned}&\max\{X,Y,Z,W\} \\ =\ &\frac{1}{2}\left[\frac{X+Y+Z+W}{2}+\left|\frac{X-Y}{2}\right|+\left|\frac{Z-W}{2}\right|\right] \\ &+\frac{1}{2}\left|\frac{(X+Y)-(Z+W)}{2}+\left|\frac{X-Y}{2}\right|-\left|\frac{Z-W}{2}\right|\right|.\end{aligned} \tag{7.14}$$

Taking heed of Remark 26 then, since we are now dealing with the maximum in (7.13), that is, a point, then it is possible to investigate the regions B and E_B for $\frac{1}{2}\leq\gamma,\ \delta\leq 1$ where $B:\delta>\frac{1}{2\gamma}$ and $E_B:\delta<\frac{1}{2\gamma}$.

In region B, from (7.13),

$$\begin{aligned}&\Theta_B(\gamma,\delta) \\ =\ &(b-a)\max\left\{(1-\gamma)\delta+\left(\gamma-\frac{1}{2}\right),\gamma\delta-\left(\gamma-\frac{1}{2}\right),\gamma\delta-\frac{1}{2},\frac{1}{2}-(1-\gamma)\delta\right\}\end{aligned} \tag{7.15}$$

and associating these elements in order with those of (7.14) gives

$$\begin{aligned}X+Y &= \delta,\ X-Y=(2\gamma-1)(1-\delta) \\ Z+W &= (2\gamma-1)\delta,\ Z-W=-(1-\delta).\end{aligned}$$

Thus, after some simplification,

$$\frac{\Theta_B}{b-a}(\gamma,\delta)=\frac{\gamma}{2}+(1-\gamma)\left(\delta-\frac{1}{2}\right). \tag{7.16}$$

Similarly, in region E_B

$$\begin{aligned}&\Theta_{E_B}(\gamma,\delta)\\ =\;&(b-a)\max\left\{(1-\gamma)\delta+\gamma-\frac{1}{2},\gamma\delta-\left(\gamma-\frac{1}{2}\right),\frac{1}{2}-\gamma\delta,\frac{1}{2}-(1-\gamma)\delta\right\}\end{aligned}$$

and again associating these elements in order with those of (7.14) gives

$$\begin{aligned}X+Y&=\delta,\ X-Y=(2\gamma-1)(1-\delta)\\ Z+W&=1-\delta,\ Z-W=(1-2\gamma)\delta.\end{aligned}$$

Therefore, after some simplification, we obtain

$$\frac{\Theta_{E_B}(\gamma,\delta)}{b-a}=\frac{\gamma-\delta}{2}+(1-\gamma)\left|\delta-\frac{1}{2}\right|. \tag{7.17}$$

Now

$$\Theta_A(\gamma,\delta)=\Theta_B(1-\gamma,\delta),\ \Theta_C(\gamma,\delta)=\Theta_B(\gamma,1-\delta) \tag{7.18}$$

and

$$\Theta_D(\gamma,\delta)=\Theta_B(1-\gamma,1-\delta).$$

Let $E=E_1\cup E_2$ where E_1 represents the region of E for which $\gamma\le\frac{1}{2}$ and E_2 represents the region of E for which $\gamma>\frac{1}{2}$. That is, $E_1=E_A\cup E_D$ and $E_2=E_B\cup E_C$ where E_k is the remainder of the square region containing region $k=A,B,C,D$.

Hence, using (7.16) – (7.18) in (7.13) gives

$$\frac{\Theta(\gamma,\delta)}{b-a}=\begin{cases}\frac{1-\gamma}{2}+\gamma\left(\delta-\frac{1}{2}\right) & \text{on } A,\\ \frac{\gamma}{2}+(1-\gamma)\left(\delta-\frac{1}{2}\right) & \text{on } B,\\ \frac{\gamma}{2}+(1-\gamma)\left(\frac{1}{2}-\delta\right) & \text{on } C,\\ \frac{1-\gamma}{2}+\gamma\left(\frac{1}{2}-\delta\right) & \text{on } D,\\ \frac{1-\gamma}{2}+\gamma\left|\delta-\frac{1}{2}\right| & \text{on } E_1,\\ \frac{\gamma}{2}+(1-\gamma)\left|\delta-\frac{1}{2}\right| & \text{on } E_2.\end{cases} \tag{7.19}$$

Remark 37. *It may be noticed that* (7.19) *may be simplified to give*

$$\frac{\Theta(\gamma,\delta)}{b-a}=\begin{cases}\frac{1-\gamma}{2}+\gamma\left|\delta-\frac{1}{2}\right|, & \gamma\le\frac{1}{2}\\ \frac{\gamma}{2}+(1-\gamma)\left|\delta-\frac{1}{2}\right|, & \gamma\ge\frac{1}{2}\end{cases} \tag{7.20}$$

and so, using the fact that $(b-a)\left(\gamma-\frac{1}{2}\right)=x-\frac{a+b}{2}$ *in* (7.20) *gives,*

$$\theta(\gamma,x)=\begin{cases}\frac{b-a}{2}\cdot(1-\gamma)+\gamma\left|x-\frac{a+b}{2}\right|, & \gamma\le\frac{1}{2}\\ \frac{b-a}{2}\cdot\gamma+(1-\gamma)\left|x-\frac{a+b}{2}\right|, & \gamma\ge\frac{1}{2}.\end{cases} \tag{7.21}$$

Thus the bound in Theorem 17, namely (7.5), *is explicitly given by* (7.21).

Remark 38. *Taking different values if* γ *will produce bounds for various inequalities.*
For $\gamma = 0$ *in* (7.4) *and* (7.21), *a perturbed Ostrowski type inequality is obtained with a uniform bound. Namely,*

$$\left|\int_a^b f(t)\,dt - (b-a)\left[f(x) - \left(x - \frac{a+b}{2}\right)S\right]\right| \le \frac{b-a}{2}\left\|\sigma(f')\right\|_1.$$

For $\gamma = 1$ *in* (7.4) *and* (7.21) *a generalized perturbed trapezoidal rule is obtained with the same uniform bound viz.*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - (b-a)\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b) - \left(x - \frac{a+b}{2}\right)S\right]\right| \\ \le\ & \frac{b-a}{2}\left\|\sigma(f')\right\|_1.\end{aligned}$$

Corollary 17. *Let the conditions on* f *be as in Theorem 17. Then the following inequality holds for any* $x \in [a,b]$

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a)f(x) + (x-a)f(a) + (b-x)f(b)\right]\right| \\ \le\ & \frac{1}{2}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\left\|\sigma(f')\right\|_1.\end{aligned} \tag{7.22}$$

Proof. Letting $\gamma = \frac{1}{2}$ in (7.4) and (7.21) readily produces the result. ∎

Corollary 18. *Let the conditions on* f *be as in Theorem 17. Then the following inequality holds for any* $\gamma \in [0,1]$

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma)f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}\left[f(a)+f(b)\right]\right\}\right| \\ \le\ & \left\|\sigma(f')\right\|_1 \begin{cases} \frac{b-a}{2}(1-\gamma), & \gamma \le \frac{1}{2} \\ \frac{b-a}{2}\gamma, & \gamma \ge \frac{1}{2}. \end{cases}\end{aligned} \tag{7.23}$$

Proof. Taking $x = \frac{a+b}{2}$ in (7.4) and (7.21) gives the result as stated. ∎

Remark 39. *Taking* $x = \frac{a+b}{2}$ *in* (7.22) *or* $\gamma = \frac{1}{2}$ *in* (7.23) *is equivalent to taking both these in* (7.4) *and* (7.21). *This produces the sharpest bound in this case, giving*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left\{f\left(\frac{a+b}{2}\right) + \frac{1}{2}\left[f(a)+f(b)\right]\right\}\right| \\ \le\ & \frac{b-a}{4}\left\|\sigma(f')\right\|_1.\end{aligned}$$

A Simpson type rule is obtained from (7.23) *if* $\gamma = \frac{1}{3}$ *is taken, giving a bound consisting of* $\frac{b-a}{3}$ *rather than the* $\frac{b-a}{4}$ *obtained above.*

A perturbed generalized Simpson type rule may be demonstrated directly from (7.4) *and its bound from* (7.21) *by taking* $\gamma = \frac{1}{3}$ *to give*

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - \frac{b-a}{3} \left\{ 2f(a) + \left(\frac{x-a}{b-a}\right) f(a) \right.\right. \\ & \left.\left. + \left(\frac{b-x}{b-a}\right) f(b) - \left(x - \frac{a+b}{2}\right) S \right\} \right| \\ \leq \; & \frac{1}{3} \left[b - a + \left| x - \frac{a+b}{2} \right| \right] \|\sigma(f')\|_1 . \end{aligned}$$

Remark 40. *The numerical implementation of the inequalities obtained in the current section will not be followed up since they follow those of Section 6. It may be noticed that Corollaries 17 and 18 are similar to Corollaries 13 and 14 with* $\|\sigma(f')\|_1$ *replacing* $\|f'\|_1$ *. In a similar fashion, the implementation of the average of the mid-point and trapezoidal rules as developed in Corollary 16 may similarly be developed here with* $\|\sigma(f')\|_1$ *replacing* $\|f'\|_1$*. Each of these norms may be better for differing functions* f.

8. Inequalities for which the First Derivative Belongs to $L_p[a,b]$

Theorem 18. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *and* $f' \in L_p(a,b)$ *where* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the following inequality holds for all* $x \in [a,b]$, $\alpha(x) \in [a,x]$ *and* $\beta(x) \in [x,b]$,

$$\begin{aligned} (8.1) \quad & \left| \int_a^b f(t)\,dt - [(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b)] \right| \\ \leq \; & \Big[(\alpha(x) - a)^{q+1} + (x - \alpha(x))^{q+1} \\ & + (\beta(x) - x)^{q+1} + (b - \beta(x))^{q+1} \Big]^{\frac{1}{q}} (q+1)^{-\frac{1}{q}} \|f'\|_p \\ \leq \; & \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f'\|_p \\ \leq \; & (b-a) \left(\frac{b-a}{q+1} \right)^{\frac{1}{q}} \|f'\|_p \end{aligned}$$

where $\|f'\|_p := \left(\int_a^b |f'(t)|^p \,dt \right)^{\frac{1}{p}}$.

Proof. Let $K(x,t)$ be as defined by (2.2). Then an integration by parts of $\int_a^b K(x,t) f'(t)\,dt$ produces the identity as given by (2.3). Thus, from (2.3):

$$\begin{aligned} (8.2) \quad & \left| \int_a^b f(t)\,dt - [(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b)] \right| \\ = \; & \left| \int_a^b K(x,t) f'(t)\,dt \right| . \end{aligned}$$

Now, by Hölder's inequality we have:

$$\left|\int_a^b K(x,t) f'(t)\,dt\right| \le \left(\int_a^b |K(x,t)|^q\,dt\right)^{\frac{1}{q}} \|f'\|_p . \tag{8.3}$$

Now, from (2.2),

$$\begin{aligned} &\int_a^b |K(x,t)|^q\,dt \\ = &\int_a^{\alpha(x)} |t-\alpha(x)|^q\,dt + \int_{\alpha(x)}^x |t-\alpha(x)|^q\,dt \\ &+\int_x^{\beta(x)} |t-\beta(x)|^q\,dt + \int_{\beta(x)}^b |t-\beta(x)|^q\,dt \\ = &\int_a^{\alpha(x)} (\alpha(x)-t)^q\,dt + \int_{\alpha(x)}^x (t-\alpha(x))^q\,dt \\ &+\int_x^{\beta(x)} (\beta(x)-t)^q\,dt + \int_{\beta(x)}^b (t-\beta(x))^q\,dt. \end{aligned}$$

Therefore,

$$\begin{aligned} &(q+1)\int_a^b |K(x,t)|^q\,dt \\ = &(\alpha(x)-a)^{q+1} + (x-\alpha(x))^{q+1} + (\beta(x)-x)^{q+1} + (b-\beta(x))^{q+1} . \end{aligned} \tag{8.4}$$

Thus, using (8.2), (8.3) and (8.4) gives the first inequality in (8.1).
Now, using the inequality

$$(z-x)^n + (y-z)^n \le (y-x)^n \tag{8.5}$$

with $z \in [x,y]$ and $n > 1$, in (8.4) twice, taking $z = \alpha(x)$ and then $z = \beta(x)$, we have

$$(q+1)\int_a^b |K(x,t)|^q\,dt \le (x-a)^{q+1} + (b-x)^{q+1} \tag{8.6}$$

$$\le (b-a)^{q+1} \tag{8.7}$$

upon using (8.5) once more.
Hence, by utilizing (8.2), (8.3) with (8.6) and (8.7) we obtain the second bound and the third inequality in (8.1). ∎

Corollary 19. *Let the conditions on f of Theorem 18 hold. Then $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{b+x}{2}$ give the best bound for any $x \in [a,b]$ and so*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(x) + \left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right| \\ \le &\frac{1}{2}\left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \|f'\|_p . \end{aligned} \tag{8.8}$$

Proof. The inequality (8.5) produces an upper bound obtained with $z = x$ or y. For $z \in [x, y]$ and $n > 1$

$$(z-x)^n + (y-z)^n \geq 2\left(\frac{y-x}{2}\right)^n \tag{8.9}$$

where the lower bound is realized when $z = \frac{x+y}{2}$. Thus a tighter bound than the first inequality in (8.1) is obtained when, from (8.4) and using (8.9), $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{x+b}{2}$. Hence, (8.8) is obtained and the corollary is proved. ∎

Remark 41. *The best inequality we may obtain from* (8.8) *results from utilizing* (8.9) *again, giving, with* $x = \frac{a+b}{2}$,

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left\{f\left(\frac{a+b}{2}\right) + \frac{1}{2}\left[f(a)+f(b)\right]\right\}\right| \leq \frac{(b-a)}{4}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|f'\|_p. \tag{8.10}$$

Motivated by Theorem 18 and Corollary 19 we now take $\alpha(x)$ and $\beta(x)$ to be convex combinations of the end points so that they are as defined in (2.11). The following theorem then holds.

Theorem 19. *Let f satisfy the conditions as stated in Theorem 18. Then the following inequality holds for any $\gamma \in [0,1]$ and $x \in [a,b]$:*

$$\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma)f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right| \leq \left[\gamma^{q+1} + (1-\gamma)^{q+1}\right]^{\frac{1}{q}}\left[(x-a)^{q+1} + (b-x)^{q+1}\right]^{\frac{1}{q}}(q+1)^{-\frac{1}{q}}\|f'\|_p. \tag{8.11}$$

Proof. Using $\alpha(x)$ and $\beta(x)$ as defined in (2.11), then

$$\beta(x) - \alpha(x) = (1-\gamma)(b-a),$$

$$\alpha(x) - a = \gamma(x-a),$$

$$b - \beta(x) = \gamma(b-x),$$

$$x - \alpha(x) = (1-\gamma)(x-a)$$

and

$$\beta(x) - x = (1-\gamma)(b-x).$$

Substituting these results into the first inequality of Theorem 18 gives the stated result. ∎

Remark 42. *Taking $\gamma = 0$ or 1 in* (8.11) *produces the coarser upper bound as obtained in the second inequality of Theorem 18. In addition, taking $x = a$ or b in* (8.11) *gives the even coarser bound as given by the third inequality of Theorem 18. Here we are utilizing* (8.5) *where the upper bound is attained at $z = x$ or y, the end points.*

Corollary 20. *Let f satisfy the conditions of Theorems 18 and 19. Then, the following inequality holds for any $\gamma \in [0,1]$:*

$$\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}[f(a)+f(b)]\right\}\right| \tag{8.12}$$
$$\leq \left[\gamma^{q+1} + (1-\gamma)^{q+1}\right]^{\frac{1}{q}} \left(\frac{b-a}{2}\right)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|f'\|_p .$$

Proof. From identity (8.9) the minimum is obtained at the mid-point. Therefore, from (8.11),

$$\inf_{x\in[a,b]} \left[(x-a)^{q+1} + (b-x)^{q+1}\right] = 2\left(\frac{b-a}{2}\right)^{q+1},$$

when $x = \frac{a+b}{2}$. Hence the result (8.12). ∎

Remark 43. *Corollary 19 is recaptured if* (8.11) *is evaluated at $\gamma = \frac{1}{2}$, the mid-point.*

Remark 44. *Taking $\gamma = 0$ in* (8.11) *produces an Ostrowski type inequality for which $f' \in L_p[a,b]$ as obtained by Dragomir and Wang* [4]*. Furthermore, taking $x = \frac{a+b}{2}$ gives a mid-point rule.*

Remark 45. *Taking $\gamma = 1$ in* (8.11) *produces a generalized trapezoidal rule for which the best bound occurs when $x = \frac{a+b}{2}$, giving the standard trapezoidal rule with a bound of $\frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}$. This bound is twice as sharp as that obtained by Dragomir and Wang* [4] *since they used an Ostrowski type rule and obtained results at $x = a, x = b$ and utilized the triangle inequality.*

Remark 46. *Taking $\gamma = \frac{1}{2}$ and $x = \frac{a+b}{2}$ in* (8.11) *gives the best inequality as given by* (8.10)*. Taking $\gamma = \frac{1}{3}$ in* (8.12)
produces a Simpson type rule with a bound on the error of

$$\left(\frac{1}{3}\right)^{1+\frac{1}{q}} \left[1+2^{q+1}\right]^{\frac{1}{q}} \left(\frac{b-a}{2}\right)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|f'\|_p .$$

Taking $\gamma = \frac{1}{2}$ in (8.12) *gives the optimal rule with a bound on the error of*

$$\frac{1}{2}\left(\frac{b-a}{2}\right)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \|f'\|_p .$$

Thus, there is a relative difference of

$$\left|\frac{2}{3}\left(\frac{1+2^{q+1}}{3}\right)^{\frac{1}{q}} - 1\right|$$

between a Simpson type rule and the optimal. When $q = 2$ for example, the relative difference is $\frac{2}{\sqrt{3}} - 1 \approx 0.1547$. The greatest the relative difference can be is $\frac{1}{3}$.

The following particular instance for Euclidean norms is of interest.

Corollary 21. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) and $f'\in L_2(a,b)$. Then the following inequality holds for all $x\in[a,b]$ and $\gamma\in[0,1]$:*

$$\left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f(x)+\gamma\left[\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right| \tag{8.13}$$
$$\leq \left(\frac{b-a}{3}\right)^{\frac{1}{2}}\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}\|f'\|_2\,.$$

Proof. Applying Theorem 19 for $p=q=2$ immediately gives the left hand side of (8.13) with a bound of

$$\left[\gamma^3+(1-\gamma)^3\right]^{\frac{1}{2}}\left[(x-a)^3+(b-x)^3\right]^{\frac{1}{2}}\frac{\|f'\|_2}{\sqrt{3}}. \tag{8.14}$$

Now, using identity (4.21) we get:

$$\left[\gamma^3+(1-\gamma)^3\right]^{\frac{1}{2}}=\left[\frac{1}{4}+3\left(\gamma-\frac{1}{2}\right)^2\right]^{\frac{1}{2}}$$

and

$$\left[(x-a)^3+(b-x)^3\right]^{\frac{1}{2}}=\sqrt{b-a}\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}},$$

which, upon substitution into (8.14) gives (8.13). ∎

Remark 47. *The numerical implementation of the inequalities in this section follows along similar lines as treated previously. The only difference is in the approximation of the bound and knowledge of $\|f'\|_p$, which need to be determined* ***a priori*** *in order that the coarseness of the partition may be calculated, given a particular error tolerance.*

9. Grüss-type Inequalities for Functions whose First Derivative Belongs to $L_p[a,b]$

From (7.2) and (7.3) we have

$$\mathfrak{T}(f,g)=\mathfrak{M}(\sigma(f)\,\sigma(g)), \tag{9.1}$$

where $\sigma(f)$ represents a shift of the function by its mean, $\mathfrak{M}$ as given in (7.1).

Thus, using Hölder's inequality from (9.1) gives

$$(b-a)\left|\mathfrak{T}(f,g)\right|\leq\|\sigma(f)\|_q\,\|\sigma(g)\|_p\,, \tag{9.2}$$

where

$$\|h\|_p:=\left(\int_a^b|h(t)|^p\,dt\right)^{\frac{1}{p}}$$

and we say $h\in L_p[a,b]$.

Theorem 20. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) and $f' \in L_p(a,b)$ where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. The following inequality then, holds for all $x \in [a,b]$,*

$$\begin{aligned}(9.3)\quad & \left| \int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma \left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} \right. \\ & \left. (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right) S \right| \\ \leq\ & \|\sigma(K(x,\cdot))\|_q \|\sigma(f')\|_p ,\end{aligned}$$

where $\sigma(\cdot)$ is as given in (7.1) *and $K(x,\cdot)$, S are as in* (5.6).

Proof. Identifying $K(x,\cdot)$ with $f(\cdot)$ and $f'(\cdot)$ with $g(\cdot)$ in (9.2) gives

$$(9.4)\qquad (b-a)\left|\mathfrak{T}(K(x,\cdot), f')\right| \leq \|\sigma(K(x,\cdot))\|_q \|\sigma(f')\|_p .$$

Further, using identities (4.5), (4.8) and the fact that $S = \frac{1}{b-a}\int_a^b f'(t)\,dt$ in (9.4) readily produces (9.3) and hence the theorem is proved. ∎

We now wish to obtain a closed form expression for $\|\sigma(K(x,\cdot))\|_q$.
Notice that

$$\sigma(K(x,t)) = K(x,t) - \frac{1}{b-a}\int_a^b K(x,u)\,du,$$

where $K(x,t)$ is as given by (5.6) and so using (4.8)

$$(9.5)\qquad K_s(x,t) = \sigma(K(x,t)) = \begin{cases} t - \phi(x), & t \in [a,x] \\ t - \psi(x), & t \in (x,b] \end{cases},$$

where ϕ and ψ are as presented in (5.10).
Thus,

$$(9.6)\qquad \int_a^b |K_s(x,t)|^q\,dt = \int_a^x |t-\phi(x)|^q\,dt + \int_x^b |t-\psi(x)|^q\,dt.$$

Using (9.5) in (9.6) gives

$$\|\sigma(K(x,\cdot))\|_q = \|K_s(x,\cdot)\|_q = \left(\int_a^b |K_s(x,t)|^q\,dt\right)^{\frac{1}{q}},$$

and upon making the respective substitutions $(b-a)u = t - \phi(x)$ and $(b-a)v = t - \psi(x)$ for the integrals on the right hand side,

$$(9.7)\qquad \int_a^b |K_s(x,t)|^q\,dt = (b-a)^{q+1}\left\{\int_{\frac{a-\phi(x)}{b-a}}^{\frac{x-\phi(x)}{b-a}} |u|^q\,du + \int_{\frac{x-\phi(x)}{b-a}}^{\frac{b-\phi(x)}{b-a}} |v|^q\,dv\right\}.$$

Following the procedure of Section 5, we may make the substitution

$$(9.8)\qquad x = \delta b + (1-\delta)a,$$

to give, from (9.7), and using (5.12) – (5.16),

$$(9.9)\qquad I^{(q)}(\gamma,x) = (b-a)^{q+1} J^{(q)}(\gamma,\delta) = (b-a)^{q+1}\left[J_1^{(q)}(\gamma,\delta) + J_2^{(q)}(\gamma,\delta)\right],$$

where

$$I^{(q)}(\gamma,x)=\int_a^b |K_s(x,t)|^q\,dt, \tag{9.10}$$

$$\begin{aligned} J_2^{(q)}(\gamma,\delta) &= \int_w^{w+1-\delta} |v|^q\,dv, \\ w &= \gamma\delta-\frac{1}{2} \end{aligned} \tag{9.11}$$

and

$$J_1^{(q)}(\gamma,\delta)=J_2^{(q)}(1-\gamma,1-\delta). \tag{9.12}$$

Now, from (9.10), the limits may be both negative, one negative and one positive, or both positive. Therefore, with $w=\gamma\delta-\frac{1}{2}$,

$$\begin{aligned} &(q+1)\,J_2^{(q)}(\gamma,\delta) \\ &= \begin{cases} (-w)^{q+1}-(\delta-1-w)^{q+1}, & \frac{1}{2(1-\gamma)}<\delta<\frac{1}{2\gamma}; \\ (-w)^{q+1}+(w-1-\delta)^{q+1}, & \delta<\frac{1}{2\gamma},\ \delta<\frac{1}{2(1-\gamma)}; \\ (w+1-\delta)^{q+1}-w^{q+1}, & \frac{1}{2\gamma}<\delta<\frac{1}{2(1-\gamma)}. \end{cases} \end{aligned} \tag{9.13}$$

Further, from (9.10) and (9.11)

$$\begin{aligned} &(q+1)\,J_1^{(q)}(\gamma,\delta) \\ &= \begin{cases} (-\tilde{w})^{q+1}-(-\delta-\tilde{w})^{q+1}, & 1-\frac{1}{2(1-\gamma)}<\delta<1-\frac{1}{2\gamma}; \\ (-\tilde{w})^{q+1}+(\tilde{w}+\delta)^{q+1}, & \delta>1-\frac{1}{2\gamma},\ \delta>1-\frac{1}{2(1-\gamma)}; \\ (\tilde{w}+\delta)^{q+1}-\tilde{w}^{q+1}, & 1-\frac{1}{2\gamma}<\delta<\frac{1}{2(1-\gamma)}, \end{cases} \end{aligned} \tag{9.14}$$

where $\tilde{w}=(1-\gamma)(1-\delta)-\frac{1}{2}$.

We are now in a position to combine (9.13) and (9.14) by using the result (9.9) on each of the regions $A,...,E$ as given by (5.23) and depicted in Figure 5.1. Thus,

$$\begin{aligned} &(q+1)\,J^{(q)}(\gamma,\delta) \\ &= \begin{cases} (-w)^{q+1}-(\delta-1-w)^{q+1}+(-\tilde{w})^{q+1}+(\tilde{w}+\delta)^{q+1} & \text{on } A; \\ (w-1-\delta)^{q+1}-w^{q+1}+(-\tilde{w})^{q+1}+(\tilde{w}+\delta)^{q+1} & \text{on } B; \\ (-w)^{q+1}+(w+1-\delta)^{q+1}+(-\tilde{w})^{q+1}-(-\delta-\tilde{w})^{q+1} & \text{on } C; \\ (-w)^{q+1}+(w-1-\delta)^{q+1}+(\tilde{w}+\delta)^{q+1}-(-\tilde{w})^{q+1} & \text{on } D; \\ (-w)^{q+1}+(w+1-\delta)^{q+1}+(-\tilde{w})^{q+1}+(\tilde{w}+\delta)^{q+1} & \text{on } E. \end{cases} \end{aligned} \tag{9.15}$$

Hence $\|\sigma(K(x,\cdot))\|_q$ is explicitly determined from (9.6), (9.9) and (9.14) on using (9.8) and the fact that $w=\gamma\delta-\frac{1}{2}$ and $\tilde{w}=(1-\gamma)(1-\delta)-\frac{1}{2}$.

Remark 48. *It is instructive to take different values of γ to obtain various inequalities that lead to a variety of quadrature rules.*
For $\gamma = 0$ then, from Figure 5.1 it may be seen that we are on the left boundary of regions A and D so that, from (9.15), a uniform bound independent of δ is obtained to give

$$(q+1) J^{(q)}(0, \delta) = \left(\frac{1}{2}\right)^q .$$

Using (9.9), (9.6) and (9.3) produces a perturbed Ostrowski inequality

$$\left| \int_a^b f(t)\,dt - (b-a)\left[f(x) - \left(x - \frac{a+b}{2}\right) S\right]\right| \leq \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \left\|\sigma(f')\right\|_p . \tag{9.16}$$

Evaluation at $x = \frac{a+b}{2}$ gives the mid-point rule.
In a similar fashion, for $\gamma = 1$ the right hand boundary of B and C results to produce a perturbed generalized trapezoidal inequality

$$\left| \int_a^b f(t)\,dt - (b-a)\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) - \left(x - \frac{a+b}{2}\right) S\right]\right| \leq \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \left\|\sigma(f')\right\|_p . \tag{9.17}$$

Taking $x = \frac{a+b}{2}$ produces the trapezoidal rule for which $\sigma(f') \in L_p[a,b]$.

Corollary 22. *Let the conditions on f be as in Theorem 20. Then the following inequality holds for any $x \in [a,b]$.*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f(x) + (x-a) f(a) + (b-x) f(b)\right]\right| \\ &\leq \frac{1}{2}\left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left\|\sigma(f')\right\|_p \\ &\leq \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \left\|\sigma(f')\right\|_p . \end{aligned} \tag{9.18}$$

Proof. Placing $\gamma = \frac{1}{2}$ in (9.15) gives, after some simplification,

$$(q+1) J^{(q)}\left(\frac{1}{2}, \delta\right) = \left(\frac{1}{2}\right)^q \left[\delta^{q+1} + (1-\delta)^{q+1}\right].$$

Hence, from (9.8) and (9.9),

$$I^{(q)}\left(\frac{1}{2}, x\right) = \left(\frac{1}{2}\right)^q \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right].$$

From (9.6) and (9.10), taking the q^{th} root of the above expression gives (9.18) from (9.3) on taking $\gamma = \frac{1}{2}$.
The second inequality is obtained on using (8.5). ∎

Corollary 23. *Let the conditions on f be as in Theorem 20. Then the following inequality holds for any $\gamma \in [0,1]$.*

$$\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}[f(a)+f(b)]\right\}\right| \tag{9.19}$$
$$\leq \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\left[\gamma^{q+1}+(1-\gamma)^{q+1}\right]^{\frac{1}{q}}\|\sigma(f')\|_p$$
$$\leq \frac{b-a}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|\sigma(f')\|_p.$$

Proof. Taking $\gamma = \frac{1}{2}$ in (9.15) places us in the region E and so

$$(q+1)\,J^{(q)}\left(\gamma,\frac{1}{2}\right) = \left(\frac{1}{2}\right)^q\left[\gamma^{q+1}+(1-\gamma)^{q+1}\right]^{\frac{1}{q}}$$

and, from (9.8) and (9.9),

$$I^{(q)}\left(\gamma,\frac{a+b}{2}\right) = \frac{(b-a)^{q+1}}{q+1}\left(\frac{1}{2}\right)^q\left[\gamma^{q+1}+(1-\gamma)^{q+1}\right].$$

From (9.6) and (9.10), taking the q^{th} root of the above expression produces (9.19) from (9.3) on taking $x = \frac{a+b}{2}$. The second inequality is easily obtained on using (8.5). ∎

Remark 49. *Taking $x = \frac{a+b}{2}$ in (9.18) or $\gamma = \frac{1}{2}$ in (9.19) is equivalent to taking both of these in (9.3) and using (9.6), (9.8) – (9.10) and*

$$(q+1)\,J^{(q)}\left(\frac{1}{2},\frac{1}{2}\right) = \left(\frac{1}{4}\right)^q$$

from (9.15). This produces the sharpest inequality (see (9.5)) in the class. Namely,

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left\{f\left(\frac{a+b}{2}\right) + \frac{1}{2}[f(a)+f(b)]\right\}\right| \tag{9.20}$$
$$\leq \frac{b-a}{4}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}\|\sigma(f')\|_p.$$

Result (9.20) may be compared with (8.10) and it may be seen that either one may be better, depending on the behaviour of f.
A Simpson type rule is obtained from (9.19) if $\gamma = \frac{1}{3}$, giving a bound consisting of $\frac{2}{3}\left[\frac{1+2^{q+1}}{3}\right]^{\frac{1}{q}}$ times the above bound for the average of a midpoint and trapezoidal rule. For the Euclidean norm, $q = 2$ and so Simpson's rule has a bound of $\frac{2}{\sqrt{3}}$ times that of the average of the midpoint and trapezoidal rule. A generalized Simpson type rule may be obtained by taking $\gamma = \frac{1}{3}$ in (9.3) and using (9.6), (9.8) – (9.10) and (9.15) in much the same way as Remark 39.

Remark 50. *Corollaries 22 and 23 may be implemented in a straight forward fashion as carried out in earlier sections. The bounds involve determining $\|\sigma(f')\|$ in advance to decide on the refinement of the grid that is required in order to achieve a particular accuracy.*

10. Three Point Inequalities for Mappings of Bounded Variation, Lipschitzian or Monotonic

The following result involving a Riemann-Stieltjes integral is well known. It will be proved here for completeness.

Lemma 1. *Let g, $v : [a,b] \to \mathbb{R}$ be such that g is continuous on $[a,b]$ and v is of bounded variation on $[a,b]$. Then $\int_a^b g(t)\,dv(t)$ exists and is such that*

$$\left|\int_a^b g(t)\,dv(t)\right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v), \tag{10.1}$$

where $\bigvee_a^b (v)$ is the total variation of v on $[a,b]$.

Proof. We only prove the inequality (10.1). Let $\Delta_n : a < x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of partitions of $[a,b]$ such that $\nu(\Delta_n) \to 0$ as $n \to \infty$ where $\nu(\Delta_n) := \max_{i\in\{0,1,\dots,n-1\}} h_i^{(n)}$ with $h_i^{(n)} = x_{i+1}^{(n)} - x_i^{(n)}$. Let $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$ for $i - 0, 1, ..., n-1$ then

$$\begin{aligned}\left|\int_a^b g(t)\,dv(t)\right| &= \left|\lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} g\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right]\right| \\ &\le \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|g\left(\xi_i^{(n)}\right)\right| \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right| \\ &\le \sup_{t\in[a,b]} |g(t)| \cdot \bigvee_a^b (v),\end{aligned}$$

where

$$\bigvee_a^b (v) = \sup_{\Delta_n} \sum_{i=0}^{n-1} \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right|, \tag{10.2}$$

and Δ_n is any partition of $[a,b]$. ∎

Theorem 21. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - [(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b)]\right| \\ &\le \frac{\bigvee_a^b (f)}{2}\left\{\frac{b-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right| + \left|\beta(x) - \frac{x+b}{2}\right| \right. \\ &\quad \left. + \left|\frac{b+a}{2} - x + \left|\beta(x) - \frac{x+b}{2}\right| - \left|\alpha(x) - \frac{a+x}{2}\right|\right|\right\},\end{aligned} \tag{10.3}$$

where $\alpha(x) \in [a,x]$ and $\beta(x) \in [x,b]$.

Proof. Let the Peano kernel be as defined in (2.2), then consider the Riemann-Stieltjes integral $\int_a^b K(x,t)\,df(t)$ giving

$$\begin{aligned}\int_a^b K(x,t)\,df(t) &= \int_a^x (t-\alpha(x))\,df(t) + \int_x^b (t-\beta(x))\,df(t)\\ &= (t-\alpha(x))\,f(t)]_{t=a}^{x} - \int_a^x f(t)\,dt\\ &\quad + (t-\beta(x))\,f(t)]_{t=x}^{b} - \int_x^b f(t)\,dt.\end{aligned}$$

Simplifying and grouping some of the terms together produces the identity

$$\begin{aligned}&\int_a^b K(x,t)\,df(t)\\ &= [\beta(x)-\alpha(x)]\,f(x) + [\alpha(x)-a]\,f(a) + [b-\beta(x)]\,f(b) - \int_a^b f(t)\,dt.\end{aligned} \tag{10.4}$$

Now, to obtain the bounds from our identity (10.4),

$$\begin{aligned}\left|\int_a^b K(x,t)\,df(t)\right| &= \left|\int_a^x (t-\alpha(x))\,df(t) + \int_x^b (t-\beta(x))\,df(t)\right|\\ &\le \left|\int_a^x (t-\alpha(x))\,df(t)\right| + \left|\int_x^b (t-\beta(x))\,df(t)\right|.\end{aligned}$$

Further, using the result of Lemma 1, namely (10.1) on each of the intervals $[a,x]$ and $[x,b]$ by associating $g(t)$ with $t-\alpha(x)$ and $t-\beta(x)$ respectively gives, on taking $dv(t) \equiv df(t)$,

$$\begin{aligned}&\left|\int_a^b K(x,t)\,df(t)\right|\\ &\le \sup_{t\in[a,x]}|t-\alpha(x)| \bigvee_a^x(f) + \sup_{t\in[x,b]}|t-\beta(x)| \bigvee_x^b(f)\\ &\le m(x)\bigvee_a^b(f).\end{aligned} \tag{10.5}$$

Let

$$m_1(x) = \sup_{t\in[a,x]}|t-\alpha(x)| = \max\{\alpha(x)-a,\, x-\alpha(x)\}$$

and so

$$m_1(x) = \frac{x-a}{2} + \left|\alpha(x) - \frac{a+x}{2}\right|. \tag{10.6}$$

Similarly,

$$m_2(x) = \sup_{t\in[x,b]}|t-\beta(x)| = \max\{\beta(x)-b,\, b-\beta(x)\}$$

and so

$$m_2(x) = \frac{b-x}{2} + \left|\beta(x) - \frac{x+b}{2}\right|. \tag{10.7}$$

Thus, from (10.5)

$$\left|\int_a^b K(x,t)\,df(t)\right| \le m_1(x)\bigvee_a^x(f)+m_2(x)\bigvee_x^b(f) \le m(x)\bigvee_a^b(f), \tag{10.8}$$

where

$$m(x)=\max\{m_1(x),m_2(x)\}.$$

Therefore,

$$m(x)=\frac{m_1(x)+m_2(x)}{2}+\left|\frac{m_1(x)-m_2(x)}{2}\right|. \tag{10.9}$$

Substitution of $m_1(x)$ and $m_2(x)$ from (10.6) and (10.7) into (10.8) and using (10.4) gives inequality (10.3), and the theorem is proved. ∎

It should be noted that it is now possible to take various $\alpha(x)$ and $\beta(x)$ to obtain the previous results. For example, taking $\alpha(x)=\beta(x)=x$ produces the results of Dragomir, Cerone and Pearce [10] involving the generalized trapezoidal rule. Further, evaluation at $x=\frac{a+b}{2}$ of this result gives the classical trapezoidal type rule as obtained by Dragomir [12]. Taking $\alpha(x)=a$ and $\beta(x)=b$ reproduces the Ostrowski rule for functions of bounded variation [6]. In particular, we shall take $\alpha(x)$ and $\beta(x)$ to be convex combinations of the end points to obtain the following theorem.

Theorem 22. *Let f satisfy the conditions of Theorem 21. Then the following inequality holds for any $\gamma\in[0,1]$ and $x\in[a,b]$:*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt\right.\\ &\left.-(b-a)\left\{(1-\gamma)f(x)+\gamma\left[\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right|\\ &\le\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f).\end{aligned} \tag{10.10}$$

Proof. Let $\alpha(x)$, $\beta(x)$ be as in (2.11). Then, from Theorem 21

$$\beta(x)-\alpha(x)=(1-\gamma)(b-a),$$

$$\alpha(x)-a=\gamma(x-a),$$

and

$$b-\beta(x)=\gamma(b-x).$$

Further, from (10.6), (10.7), and (10.9),

$$m_1(x)=\left(\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right)(x-a)$$

$$m_2(x)=\left(\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right)(b-x)$$

and

$$m(x)=\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right].$$

Substitution of $m(x)$ into (10.8), and using the identity (10.4) gives the result (10.10) and the theorem is thus proved. ∎

Remark 51. *We note that coarser uniform bounds may be obtained on using the fact that*

$$\max_{X\in[A,B]}\left|X-\frac{A+B}{2}\right|=\frac{B-A}{2}.$$

Remark 52. *A tighter bound is obtained when*

$$\min_{X\in[A,B]}\left|X-\frac{A+B}{2}\right|.$$

The minimum of 0 *is attained when* $X=\frac{A+B}{2}$.

Corollary 24. *Let f satisfy the conditions of Theorems 21 and 22. Then the following inequality holds for any* $\gamma\in[0,1]$:

$$\left|\int_a^b f(t)\,dt-(b-a)\left\{(1-\gamma)f\left(\frac{a+b}{2}\right)+\frac{\gamma}{2}[f(a)+f(b)]\right\}\right| \tag{10.11}$$
$$\leq \frac{b-a}{2}\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]\bigvee_a^b(f).$$

Proof. From (10.8) and Remark 52,

$$\min_{x\in[a,b]}\left|x-\frac{a+b}{2}\right|=0$$

when $x=\frac{a+b}{2}$. Hence the result (10.9) is obtained. ∎

Corollary 25. *Let f satisfy the conditions of Theorems 21 and 22. Then the following inequality holds for all* $x\in[a,b]$,

$$\left|\int_a^b f(t)\,dt-\frac{1}{2}\{(b-a)f(x)+(x-a)f(a)+(b-x)f(b)\}\right| \tag{10.12}$$
$$\leq \frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f).$$

Proof. From (10.10) and Remark 52,

$$\min_{\gamma\in[0,1]}\left|\gamma-\frac{1}{2}\right|=0,$$

when $\gamma=\frac{1}{2}$. Thus, placing $\gamma=\frac{1}{2}$ in (10.10) gives the result (10.12). ∎

Remark 53. *The sharpest bounds on* (10.11) *and* (10.12) *occur when* $\gamma=\frac{1}{2}$ *and* $x=\frac{a+b}{2}$ *as may be concluded from the result of Remark 52. The same result can be* *obtained directly from* (10.10), *giving the quadrature rule with the sharpest bound as*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{4}\bigvee_a^b (f)\,. \tag{10.13}$$

It should be noted, as previously on similar occasions, that taking $\gamma = \frac{1}{3}$ *in* (10.11) *produces a Simpson-type rule as obtained by Dragomir* [9] *which is worse than the optimal 3 point Lobatto rule as given by* (10.13).
Namely,

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{3}\left[2f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right| \le \frac{b-a}{3}\bigvee_a^b (f)\,, \tag{10.14}$$

which is worse than (10.13) *by an absolute amount of* $\frac{1}{12}$.
Computationally speaking, the Simpson type rule (10.14) *is just as efficient and easy to apply as the optimal rule* (10.13) *which is the average of a trapezoidal and mid- point rule.*

Remark 54. *Taking various values of* $\gamma \in [0,1]$ *and/or* $x \in [a,b]$ *will reproduce earlier results.*
Taking $\gamma = 0$ *in* (10.10) *will reproduce the results of Dragomir* [6]*, giving an Ostrowski integral inequality for mappings of bounded variation. In addition, taking* $x = \frac{a+b}{2}$ *would give a mid-point rule.*
If $\gamma = 1$ *is substituted into* (10.11), *then the results of Dragomir, Cerone and Pearce* [10] *are recovered, giving a generalized trapezoidal inequality for any* $x \in [a,b]$. *Furthermore, fixing* x *at its optimal value of* $\frac{a+b}{2}$ *would give the results of Dragomir* [6].
Putting $\gamma = \frac{1}{3}$ *in* (10.10), *we obtain*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - \frac{1}{3}\left[2(b-a)f(x)+(x-a)f(a)+(b-x)f(b)\right]\right| \\ \le\;& \frac{2}{3}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b (f)\,,\end{aligned}$$

which is a generalized Simpson type rule. Also, taking $x = \frac{a+b}{2}$ *gives the result* (10.14)*, which was also produced in Dragomir* [9].

Remark 55. *If* f *is absolutely continuous on* $[a,b]$ *and* $f' \in L_1[a,b]$, *then* f *is of bounded variation. By applying the theorems of this section, the theorems of Section 6 are hence recovered. Thus, replacing* $\bigvee_a^b (f)$ *by* $\|f'\|_1$ *in this section reproduces the results of Section 6 and vice versa, provided that the conditions on* f *are satisfied. Further, the perturbed three point quadrature rules obtained in Section 7 through Grüss-type inequalities may be obtained here, where, instead of* $\|\sigma(f')\|_1$ *in identity* (7.4), *we would have* $\bigvee_a^b (\sigma(f)) \equiv \bigvee_a^b (f)$. *Thus the following theorem would result.*

Theorem 23. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}(10.15)\quad & \left|\int_a^b f(t)\,dt\right. \\ & -(b-a)\left\{(1-\gamma)f(x)+\gamma\left[\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)\right]\right\} \\ & \left.+(b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right| \\ \leq\ & \theta(\gamma,x)\bigvee_a^b(f),\end{aligned}$$

where $\theta(\gamma,x)$ is as given by (7.21) and S is the secant slope.

Proof. Identifying $\sigma(K(x,\cdot))$ with $g(\cdot)$ and $\sigma(f(\cdot))$ with $v(\cdot)$ in (10.1) gives, upon noting that $\bigvee_a^b(\sigma(f))\equiv\bigvee_a^b(f)$, and $d\sigma(f)=df$ since the σ operator merely shifts a function by its mean,

$$(10.16)\qquad \left|\int_a^b \sigma(K(x,t))\,df(t)\right|\leq\sup_{t\in[a,b]}|\sigma(K(x,t))|\bigvee_a^b(f),$$

where

$$\sigma(K(x,t))=\begin{cases} t-\phi(x), & t\in[a,x], \\ t-\psi(x), & t\in(x,b]\end{cases}$$

and $\phi(x)$, $\psi(x)$ are as given in (5.10).

The Riemann-Stieltjes integral may be integrated by parts to produce an identity similar to (10.4) with $\alpha(x)$ and $\beta(x)$ replaced by $\phi(x)$ and $\psi(x)$ respectively, since we are now considering $\sigma(K(x,\cdot))$ rather than $K(x,\cdot)$. In other words,

$$\begin{aligned}(10.17)\quad & \int_a^b\sigma(K(x,t))\,df(t) \\ =\ & [\psi(x)-\phi(x)]f(x)+[\phi(x)-a]f(a)+[b-\psi(x)]f(b)-\int_a^b f(t)\,dt,\end{aligned}$$

which becomes, on using (5.10),

$$\begin{aligned}(10.18)\quad & \int_a^b\sigma(K(x,t))\,df(t) \\ =\ & (b-a)(1-\gamma)f(x)+\left[\gamma(b-x)+\left(x-\frac{a+b}{2}\right)\right]f(a) \\ & +\left[\gamma(x-a)-\left(x-\frac{a+b}{2}\right)\right]f(b)-\int_a^b f(t)\,dt.\end{aligned}$$

A straightforward reorganization of (10.17), on noting that $S=\frac{f(b)-f(a)}{b-a}$ and using (10.16) readily produces (10.15) where $\theta(\gamma,x)=\sup_{t\in[a,b]}|\sigma(K(x,t))|$, and hence the theorem is proved. ∎

Remark 56. *Identity* (10.17) *(or indeed* (10.4)*) demonstrates that a three point quadrature rule may be obtained for arbitrary functions $\phi(\cdot)$ and $\psi(\cdot)$ (or $\alpha(\cdot)$ and $\beta(\cdot)$).*

Definition 1. *The mapping $u:[a,b]\to\mathbb{R}$ is said to be $L-$Lipschitzian on $[a,b]$ if*

$$|u(x)-u(y)|\leq L|x-y| \text{ for all } x,y\in[a,b]. \tag{10.19}$$

The following lemma holds.

Lemma 2. *Let $g,v:[a,b]\to\mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is L-Lipschitzian on $[a,b]$. Then*

$$\left|\int_a^b g(t)\,dv(t)\right|\leq L\int_a^b |g(t)|\,dt. \tag{10.20}$$

Proof. Let $\Delta_n : a<x_0^{(n)}<x_1^{(n)}<...<x_{n-1}^{(n)}<x_n^{(n)}=b$ be a sequence of partitions of $[a,b]$ such that $\nu(\Delta_n)\to 0$ as $n\to\infty$, where $\nu(\Delta_n):=\max_{i\in\{0,1,\dots,n-1\}} h_i^{(n)}$ with $h_i^{(n)}=x_{i+1}^{(n)}-x_i^{(n)}$. Further, let $\xi_i^{(n)}\in\left[x_i^{(n)},x_{i+1}^{(n)}\right]$ such that

$$\begin{aligned}
\left|\int_a^b g(t)\,dv(t)\right| &= \left|\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1} g\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)\right]\right| \\
&\leq \lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)}\right)\right|\left|\frac{v\left(x_{i+1}^{(n)}\right)-v\left(x_i^{(n)}\right)}{x_{i+1}^{(n)}-x_i^{(n)}}\right|\left(x_{i+1}^{(n)}-x_i^{(n)}\right) \\
&\leq L\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)}-x_i^{(n)}\right) \\
&= L\int_a^b |g(t)|\,dt.
\end{aligned}$$

Hence the lemma is proved. ∎

Theorem 24. *Let $f:[a,b]\to\mathbb{R}$ be L-Lipschitzian on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt-\left[(\beta(x)-\alpha(x))f(x)+(\alpha(x)-a)f(a)+(b-\beta(x))f(b)\right]\right| \\
&\leq L\left\{\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]\right. \\
&\quad\left.+\left(\alpha(x)-\frac{a+x}{2}\right)^2+\left(\beta(x)-\frac{x+b}{2}\right)^2\right\},
\end{aligned} \tag{10.21}$$

where $\alpha(x)$, $\beta(x)$ are as given by (2.11).

Proof. The proof is straightforward from identity (10.2), giving, after taking the absolute value

$$\left|\int_a^b K(x,t)\,df(t)\right|\leq L\int_a^b |K(x,t)|\,dt,$$

since f is L-Lipschitzian, and thus Lemma 2 may be used. Now, $K(x,t)$ is as given by (2.2) and $\int_a^b |K(x,t)|\,dt=Q(x)$ given in (2.6). Using identity (2.5) simplifies the expression for $Q(x)$ in (2.6) to give result (10.21). Hence the theorem is proved. ∎

Remark 57. *If f is L-Lipschitzian on $[a,b]$, then the bound on the Riemann-Stieltjes integral*

$$\left|\int_a^b K(x,t)\,df(t)\right| \le L\int_a^b |K(x,t)|\,dt.$$

On the other hand, if f is differentiable on $[a,b]$ and $f' \in L_\infty[a,b]$, then the Riemann integral

$$\left|\int_a^b K(x,t)\,f'(t)\,dt\right| \le \|f'\|_\infty \int_a^b |K(x,t)|\,dt.$$

Thus, all the theorems and bounds obtained in Section 2 are applicable here if f is L-Lipschitzian. The $\|f'\|_\infty$ norm is simply replaced by L.

Theorem 25. *Let $f:[a,b]\to\mathbb{R}$ be $L-$Lipschitzian on $[a,b]$. Then*

$$\left|\int_a^b f(t)\,dt - [(\psi(x)-\phi(x))\,f(x) + (\phi(x)-a)\,f(a) + (b-\psi(x))\,f(b)]\right| \le L\,\|\sigma(K(x,\cdot))\|_1, \tag{10.22}$$

where

$$\sigma(K(x,t)) = \begin{cases} t-\phi(x), & t\in[a,x], \\ t-\psi(x), & t\in(x,b]. \end{cases}$$

Proof. Consider

$$\int_a^b \sigma(K(x,t))\,df(t),$$

giving identity (10.17) and using (10.20) readily produces result (10.22). Thus, the theorem is proved. ∎

Remark 58. *If $\phi(x)$ and $\psi(x)$ are taken as in (5.10), then*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma)\,f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right)f(a) \right.\right.\right. \\ &\left.\left.\left. + \left(\frac{b-x}{b-a}\right)f(b)\right]\right\} + (b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right)S\right| \\ \le\ & L\cdot I(\gamma,x), \end{aligned} \tag{10.23}$$

where

$$I(\gamma,x) = \|\sigma(K(x,\cdot))\|_1 = J\left(\gamma, \frac{x-a}{b-a}\right)$$

which is given in (5.24) and $S = \frac{f(b)-f(a)}{b-a}$.

Lemma 3. *Let $g,v \in [a,b]\to\mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$. Then*

$$\left|\int_a^b g(t)\,dv(t)\right| \le \int_a^b |g(t)|\,dv(t). \tag{10.24}$$

Proof. Let $\Delta_n : a < x_0^{(n)} < x_1^{(n)} < \dots < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of partitions of $[a,b]$ such that $\nu(\Delta_n) \to 0$ as $n \to \infty$ where $\nu(\Delta_n) := \max_{i \in \{0,1,\dots,n-1\}} h_i^{(n)}$ with $h_i^{(n)} = x_{i+1}^{(n)} - x_i^{(n)}$. Now, let $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$ so that

$$\begin{aligned}\left|\int_a^b g(t)\,dv(t)\right| &= \left|\lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} g\left(\xi_i^{(n)}\right)\left[v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right]\right| \\ &\le \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|g\left(\xi_i^{(n)}\right)\right| \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right|.\end{aligned}$$

Now, using the fact that v is monotonic nondecreasing, then

$$\left|\int_a^b g(t)\,dv(t)\right| \le \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|g\left(\xi_i^{(n)}\right)\right| \left(v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right).$$

Making use of the definition of the integral, the lemma is proved. ∎

Theorem 26. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then the following inequality holds:*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - \left[(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b)\right]\right| \quad (10.25)\\ &\le [2x - (\alpha(x) + \beta(x))] f(x) + (b - \beta(x)) f(b) - (\alpha(x) - a) f(a) \\ &\quad - \int_a^b sgn(K(x,t)) f(t)\,dt\end{aligned}$$

$$\begin{aligned}&\le [2x - (\alpha(x) + \beta(x))] f(x) + (b - \beta(x)) f(b) - (\alpha(x) - a) f(a) \quad (10.26)\\ &\quad + [2\alpha(x) - (a+x)] f(\alpha(x)) + [2\beta(x) - (x+b)] f(\beta(x)),\end{aligned}$$

where $K(x,t)$ is as given by (2.2) and $\alpha(x) \in [a,x]$, $\beta(x) \in [x,b]$.

Proof. Let the Peano kernel $K(x,t)$ be as given by (2.2). Then the identity (10.4) is obtained upon integration by parts of the Riemann-Stieltjes integral $\int_a^b K(x,t)\,df(t)$. Now, since f is monotonic nondecreasing, then, using Lemma 3 and identifying $f(\cdot)$ with $v(\cdot)$ and $K(x,\cdot)$ with $g(\cdot)$ in (10.24) gives:

$$\left|\int_a^b K(x,t)\,df(t)\right| \le \int_a^b |K(x,t)|\,df(t). \quad (10.27)$$

Now, from (2.2),

$$\begin{aligned}\left|\int_a^b K(x,t)\,df(t)\right| &\le \int_a^x |t - \alpha(x)|\,df(t) + \int_x^b |t - \beta(x)|\,df(t) \\ &= \int_a^{\alpha(x)} (\alpha(x) - t)\,df(t) + \int_{\alpha(x)}^x (t - \alpha(x))\,df(t) \\ &\quad + \int_x^{\beta(x)} (\beta(x) - t)\,df(t) + \int_{\beta(x)}^b (t - \beta(x))\,df(t).\end{aligned}$$

Integration by parts and some grouping of terms gives

$$\begin{aligned}(10.28)\quad & \left|\int_a^b K(x,t)\,df(t)\right| \\ \leq\ & [2x-(\alpha(x)+\beta(x))]\,f(x)+(b-\beta(x))\,f(b)-(\alpha(x)-a)\,f(a) \\ & +\left\{\int_a^{\alpha(x)} f(t)\,dt-\int_{\alpha(x)}^{x} f(t)\,dt+\int_x^{\beta(x)} f(t)\,dt-\int_{\beta(x)}^{b} f(t)\,dt\right\}.\end{aligned}$$

Using the fact that the terms in the braces are equal to $-\int_a^b sgn\,(K(x,t))\,f(t)\,dt$, where

$$sgn_{x\in[a,b]}u(x)=\begin{cases}1 \text{ if } & u>0 \\ -1 \text{ if } & u<0\end{cases},$$

then, from identity (10.4) and equation (10.27) we obtain (10.25) and thus, the first part of the theorem is proved.

Now for the second part. Since $f(\cdot)$ is monotonic nondecreasing,

$$\int_a^{\alpha(x)} f(t)\,dt \leq (\alpha(x)-a)\,f(\alpha(x)),$$

$$\int_{\alpha(x)}^{x} f(t)\,dt \geq (x-\alpha(x))\,f(\alpha(x)),$$

$$\int_x^{\beta(x)} f(t)\,dt \leq (\beta(x)-b)\,f(\beta(x)),$$

and

$$\int_{\beta(x)}^{b} f(t)\,dt \geq (b-\beta(x))\,f(\beta(x)).$$

Thus, from (10.28)

$$\begin{aligned}(10.29)\quad & \int_a^b |K(x,t)|\,df(t) \\ \leq\ & [2x-(\alpha(x)+\beta(x))]\,f(x)+(b-\beta(x))\,f(b)-(\alpha(x)-a)\,f(a) \\ & +[2\alpha(x)-(a+x)]\,f(\alpha(x))+[2\beta(x)-(x+b)]\,f(\beta(x)).\end{aligned}$$

Substituting (10.29) into (10.27) and utilizing (10.4), we obtain (10.26). Thus, the second part of the theorem is proved. ∎

Remark 59. *It is now possible to recapture previous results for monotonic nondecreasing mappings. If $\alpha(x)=\beta(x)=x$, then the result obtained by Dragomir, Cerone and Pearce* [10] *for the generalized trapezoidal rule is recovered. Moreover, taking $x=\frac{a+b}{2}$ gives the trapezoidal-type rule. Taking $\alpha(x)=a$ and $\beta(x)=b$ reproduces an Ostrowski inequality for monotonic nondecreasing mappings which was developed by Dragomir* [7]. *Taking $\alpha(x)=\frac{a+x}{2}$ and $\beta(x)=\frac{x+b}{2}$ gives from*

(10.26) :

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a)\,f(x) + (x-a)\,f(a) + (b-x)\,f(b)\right] \right| \\ \leq & \left(x - \frac{a+b}{2}\right) f(x) + (b-x)\,f(b) - (x-a)\,f(a), \end{aligned}$$

which is the Lobatto type rule obtained by Milovanović and Pečarić [14, p. 470]. *However, here it is for monotonic functions.*

As discussed earlier, it is much more enlightening to take $\alpha(x)$ and $\beta(x)$ to be a linear combination of the end points, and so the following theorem can be shown to hold.

Theorem 27. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then the following inequality exists*

$$\begin{aligned} & \left| \int_a^b f(t)\,dt \right. \\ & \left. - (b-a)\left\{(1-\gamma)\,f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} \right| \qquad (10.30) \\ \leq\; & 2(1-\gamma)\left(x - \frac{a+b}{2}\right) f(x) + \gamma\left[(b-x)\,f(b) - (x-a)\,f(a)\right] \\ & - \int_a^b sgn(K(x,t))\,f(t)\,dt \\ \leq\; & (x-a)\left\{(1-\gamma)\left[f(x) - f(\alpha(x))\right] + \gamma\left[f(\alpha(x)) - f(a)\right]\right\} \qquad (10.31) \\ & + (b-x)\left\{(1-\gamma)\left[f(\beta(x)) - f(x)\right] + \gamma\left[f(b) - f(\beta(x))\right]\right\} \\ \leq\; & \left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right]\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right](f(b) - f(a)), \qquad (10.32) \end{aligned}$$

where $K(x,t)$ is as given by (2.2) *and $\alpha(x), \beta(x)$ by* (2.11).

Proof. Let $\alpha(x), \beta(x)$ be as in (2.11). Then, from Theorem 26,

$$\begin{aligned} \beta(x) - \alpha(x) &= (1-\gamma)(b-a), \\ \alpha(x) - a &= \gamma(x-a) \text{ and} \\ b - \beta(x) &= \gamma(b-x). \end{aligned}$$

In addition,

$$2x - (\alpha(x) + \beta(x)) = 2(1-\gamma)\left(x - \frac{a+b}{2}\right),$$

and so using these results in (10.25), (10.30) is obtained and the first part is proved. Now for the second part. Note that

$$2\alpha(x) - (a+x) = (2\gamma - 1)(x-a)$$

and

$$2\beta(x) - (x+b) = (1-2\gamma)(b-x).$$

Substituting the above expressions into the right hand side of (10.26) gives

$$\begin{aligned} & 2(1-\gamma)\left(x-\frac{a+b}{2}\right) f(x)+\gamma[(b-x) f(b)-(x-a) f(a)] \\ & +(2\gamma-1)(x-a) f(\alpha(x))+(1-2\gamma)(b-x) f(\beta(x)), \end{aligned}$$

which, upon rearrangement, produces (10.31) .
Now, to prove (10.32) , the well known result for the maximum may be used, namely

$$\max\{X, Y\}=\frac{X+Y}{2}+\left|\frac{X-Y}{2}\right| .$$

Thus, from the right hand side of (10.31) , using

$$\max\{\gamma,(1-\gamma)\}=\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|,$$

gives

$$\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]\{(x-a)(f(x)-f(a))+(b-x)(f(b)-f(x))\} .$$

Furthermore, using

$$\max\{x-a, b-x\}=\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|$$

readily produces (10.32) where $f(b)>f(a)$, since f is monotonic nondecreasing. Hence the theorem is completely proved. ∎

Remark 60. *Taking $\alpha(x)$ and $\beta(x)$ to be a convex combination of the endpoints produces, it is argued, a more elegant bound* (10.32) *than would otherwise be the case. The right hand side of* (10.29) *may easily be shown to equal*

$$\begin{aligned} & (\alpha(x)-a)[f(\alpha(x))-f(a)]+(x-\alpha(x))[f(x)-f(\alpha(x))] \\ & +(\beta(x)-x)[f(\beta(x))-f(x)]+(b-\beta(x))[f(b)-f(\beta(x))] \\ \leq\ & M(x)[f(b)-f(a)] \end{aligned}$$

where

$$M(x)=\max\{\alpha(x)-a, x-\alpha(x), \beta(x)-x, b-\beta(x)\}$$

which is as given in (6.4) .

It is further argued that the product form bound in (10.30) – (10.32) when a convex combination of the end points for $\alpha(x)$ and $\beta(x)$ is taken, it is much more enlightening than the bound given by (10.25) and (10.26) .

Corollary 26. *Let f satisfy the conditions as stated in Theorem 27. Then the following inequalities hold for any $x \in[a, b]$:*

$$\begin{aligned} & \left|\int_a^b f(t)\, dt-\frac{1}{2}[(b-a) f(x)+(x-a) f(a)+(b-x) f(b)]\right| \\ \leq\ & \frac{(x-a)}{2}[f(x)-f(a)]+\frac{(b-x)}{2}[f(b)-f(x)] \qquad (10.33) \\ \leq\ & \frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](f(b)-f(a)) . \qquad (10.34) \end{aligned}$$

Proof. Placing $\gamma=\frac{1}{2}$ in (10.31) and (10.32) readily produces (10.33) and (10.34) . ∎

Corollary 27. *Let f satisfy the conditions of Theorem 27. Then the following inequalities hold for all $\gamma \in [0,1]$.*

$$\left| \int_a^b f(t)\,dt - (b-a)\left\{ (1-\gamma) f\left(\frac{a+b}{2}\right) + \frac{\gamma}{2}[f(a)+f(b)] \right\} \right|$$

$$\leq \frac{b-a}{2}\left\{ \gamma [f(b)-f(a)] + (1-2\gamma)\left[f\left(\beta\left(\frac{a+b}{2}\right)\right) - f\left(\alpha\left(\frac{a+b}{2}\right)\right)\right]\right\} \tag{10.35}$$

$$\leq \frac{(b-a)}{2}\left[\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right](f(b)-f(a)). \tag{10.36}$$

Proof. Taking $x = \frac{a+b}{2}$ into (10.31) readily produces (10.35) after some minor simplification. Placing $x = \frac{a+b}{2}$ into (10.32) gives (10.36). ∎

Remark 61. *The monotonicity properties of $f(\cdot)$ may be used to obtain bounds from* (10.31) *(or indeed,* (10.33) *and* (10.35)*).*
Now, since f is monotonic nondecreasing, then

$$f(a) \leq f(\alpha(x)) \leq f(x) \leq f(\beta(x)) \leq f(b),$$

for any $x \in [a,b]$ and $\alpha(x) \in [a,x]$, $\beta(x) \in [x,b]$. Hence, the right hand side of (10.31) *is bounded by*

$$(x-a)[f(x)-f(a)] + (b-x)[f(b)-f(x)],$$

for any $\gamma \in [0,1]$ (that is a uniform bound). This is further bounded by

$$\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right](f(b)-f(a)),$$

upon using the maximum identity, which is the coarsest bound possible from (10.32)*, and is obtained by only controlling the γ parameter.*

Remark 62. *Although* (10.30), (10.31) *and its particularizations* (10.33) *and* (10.35) *are of academic interest, their practical applicability in numerical quadrature is computationally restrictive. Hence, the bound* (10.32) *and its specializations* (10.34) *and* (10.36) *are emphasized.*

Remark 63. *In the foregoing work we have assumed that f is monotonic nondecreasing. If f is assumed to be simply monotonic, then the modulus sign is required for function differences. Thus, for example, in* (10.32), $|f(b)-f(a)|$ *would be required rather than simply* $f(b)-f(a)$. *Alternatively, if $f(\cdot)$ were monotonically nondecreasing, then $-f(\cdot)$ would be monotonically nonincreasing.*

Remark 64. *Taking various values of $\gamma \in [0,1]$ and/or $x \in [a,b]$ will produce some specific special cases.*
Placing $\gamma = 0$ in (10.32) *gives a generalized trapezoidal rule for monotonic mappings and the results of Dragomir, Cerone and Pearce* [10] *are recovered. If $x = \frac{a+b}{2}$, then the trapezoidal rule results.*

Remark 65. *The optimum result from inequality* (10.32) *is obtained when both γ and x are taken to be at the midpoints of their respective intervals. Thus, the best* *quadrature rule is:*

$$\left|\int_a^b f(t)\,dt - \frac{1}{2}\left[(b-a) f\left(\frac{a+b}{2}\right) + \frac{1}{2}[f(a)+f(b)]\right]\right| \leq \frac{b-a}{4}(f(b)-f(a)). \tag{10.37}$$

This result could equivalently be obtained by taking $\gamma = \frac{1}{2}$ *in* (10.36) *or* $x = \frac{a+b}{2}$ *in* (10.34).
Taking $\gamma = \frac{1}{3}$ *in* (10.32) *gives a generalized Simpson-type rule in which the interior point is unspecified. Namely,*

$$\left|\int_a^b f(t)\,dt - \frac{1}{3}[2(b-a) f(x) + (x-a) f(a) + (b-x) f(b)]\right| \leq \frac{2}{3}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right](f(b)-f(a)).$$

If x *is taken at the midpoint, then the Simpson-type rule is obtained viz.,*

$$\left|\int_a^b f(t)\,dt - \frac{(b-a)}{3}\left[2f\left(\frac{a+b}{2}\right) + \frac{1}{2}(f(a)+f(b))\right]\right| \leq \frac{b-a}{3}(f(b)-f(a)), \tag{10.38}$$

which is a worse bound than (10.37). *Computationally, there is no difference in implementing* (10.37) *or* (10.38), *and yet* (10.38) *is worse by an absolute amount of* $\frac{1}{12}$.

Theorem 28. *Let* $f : [a,b] \to \mathbb{R}$ *be a monotonic non-decreasing mapping on* $[a,b]$. *Then the following inequality holds.*

$$\left|\int_a^b f(t)\,dt - (b-a)\left\{(1-\gamma) f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b)\right]\right\} + (b-a)(1-2\gamma)\left(x - \frac{a+b}{2}\right) S\right|$$

$$\leq 2\gamma\left(x - \frac{a+b}{2}\right) f(x) + \left[\gamma(x-a) - \left(x - \frac{a+b}{2}\right)\right] f(b) - \left[\gamma(b-x) + \left(x - \frac{a+b}{2}\right)\right] f(a) - \int_a^b sgn(\sigma(K(x,t))) f(t)\,dt \tag{10.39}$$

$$\leq 2\gamma\left(x - \frac{a+b}{2}\right) f(x) + \left[\gamma(x-a) - \left(x - \frac{a+b}{2}\right)\right] f(b) - \left[\gamma(b-x) + \left(x - \frac{a+b}{2}\right)\right] f(a) + (2\gamma-1)(b-x) f(\phi(x)) + (1-2\gamma)(x-a) f(\psi(x)) \tag{10.40}$$

$$\leq \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\{\gamma(f(b)-f(a)) + (1-2\gamma)(f(\psi)-f(\phi))\}. \tag{10.41}$$

Proof. From (10.8), and identifying $f(\cdot)$ with $v(\cdot)$ and $\sigma(K(x,\cdot))$ with $g(\cdot)$ gives

$$\left|\int_a^b \sigma(K(x,t))\,df(t)\right| \le \int_a^b |\sigma(K(x,t))|\,df(t), \tag{10.42}$$

which is equivalent to (10.27) with $\sigma(K(x,t))$ replacing $K(x,t)$. The results of Theorem 26 are obtained, namely, by (10.12) and (10.26) with $\phi(\cdot)$, $\psi(\cdot)$ and $\sigma(K(x,\cdot))$ replacing $\alpha(\cdot)$, $\beta(\cdot)$ and $K(x,\cdot)$ respectively. Taking $\phi(x)$ and $\psi(x)$ as given by (5.10) then gives

$$\begin{aligned}
\psi(x)-\phi(x) &= (1-\gamma)(b-a),\\
\phi(x)-a &= \gamma(b-x)+\left(x-\frac{a+b}{2}\right),\\
b-\psi(x) &= \gamma(x-a)-\left(x-\frac{a+b}{2}\right),\\
2x-(\phi(x)+\psi(x)) &= 2\gamma\left(x-\frac{a+b}{2}\right).
\end{aligned}$$

Thus, from (10.25), with the appropriate changes to ϕ, ψ and $\sigma(K)$, we have

$$\begin{aligned}
&(1-\gamma)(b-a)f(x)+\left[\gamma(b-x)+\left(x-\frac{a+b}{2}\right)\right]f(a)\\
&+\left[\gamma(x-a)-\left(x-\frac{a+b}{2}\right)\right]f(b)\\
=\;&(1-\gamma)(b-a)f(x)+\gamma[(x-a)f(a)+(b-x)f(b)]\\
&-(b-a)(1-2\gamma)\left(x-\frac{a+b}{2}\right)S
\end{aligned}\tag{10.43}$$

and

$$\begin{aligned}
&[2x-(\phi(x)+\psi(x))]f(x)+(b-\psi(x))f(b)-(\phi(x)-a)f(a)\\
=\;&2\gamma\left(x-\frac{a+b}{2}\right)f(x)+\left[\gamma(x-a)-\left(x-\frac{a+b}{2}\right)\right]f(b)\\
&-\left[\gamma(b-x)+\left(x-\frac{a+b}{2}\right)\right]f(a).
\end{aligned}\tag{10.44}$$

Hence, combining (10.43) and (10.44) readily gives the first inequality.

Now, for the second inequality, we have from (5.10),

$$\begin{aligned}
2\phi(x)-(a+x) &= (2\gamma-1)(b-x)\\
\text{and } 2\psi(x)-(x+b) &= (1-2\gamma)(x-a).
\end{aligned}$$

Thus, from (10.26) and identifying α with ϕ and β with ψ readily gives the second inequality of the theorem.

The third inequality (10.41) is obtained by grouping the terms in (10.40) as the coefficients of $x-a$ and $b-x$, and then using the fact that

$$\max\{x-a,b-x\}=\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|.$$

Thus, the theorem is now completely proved. ∎

Remark 66. *From* (10.42), *we have that*

$$\begin{aligned}\left|\int_a^b \sigma(K(x,t))\,df(t)\right| &\le \int_a^b |\sigma(K(x,t))|\,df(t)\\ &\le \sup_{t\in[a,b]} |\sigma(K(x,t))| \int_a^b df(t)\\ &= \theta(\gamma,x)(f(b)-f(a)),\end{aligned}$$

where $\theta(\gamma,x)$ *is as given by* (7.21). *Thus, result* (10.15) *is obtained because, for monotonic nondecreasing functions,* $\bigvee_a^b(f) = f(b) - f(a)$.

Remark 67. *Applications in probability theory are worthy of a mention.*
For X *a random variable taking on values in the finite interval* $[a,b]$, *the cumulative distribution function* $F(x)$ *is defined by* $F(x) = \Pr(X \le x) = \int_a^x f(t)\,dt$. *Thus, from* (10.10), (10.22) *with* (2.11), *we obtain rules for evaluating the cumulative distribution function in terms of function evaluation of the density. Namely, for any* $y \in [a,x]$

$$\begin{aligned}&|F(x) - \{(1-\gamma)(x-a)f(y) + \gamma[(y-a)f(a) + (x-y)f(x)]\}|\\ \le &\begin{cases}\left[\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right]\left[\frac{x-a}{2}+\left|y-\frac{a+x}{2}\right|\right]\bigvee_a^x(f)\\ 2L\left[\frac{1}{4}+\left(\gamma-\frac{1}{2}\right)^2\right]\left[\left(\frac{x-a}{2}\right)^2+\left(y-\frac{a+x}{2}\right)^2\right].\end{cases}\end{aligned}$$

The above results could be used to approximate $\Pr(c \le X \le d)$ *where* $[c,d] \subseteq [a,b]$.
If $\gamma = 0$, *then the results of Barnett and Dragomir* [18] *are recaptured.*

If $f(t) \equiv F(t)$, then $\int_a^b F(t)\,dt = b - E[X]$ and so $F(t)$ is monotonic non-decreasing. In addition, $F(a) = 0$, $F(b) = 1$ give, from Theorem 28,

$$\begin{aligned}&|\beta(x) - E[X] - (1-\gamma)(b-a)F(x)|\\ \le\ & 2(1-\gamma)\left(x-\frac{a+b}{2}\right)F(x) + \gamma(b-x) - \int_a^b sgn(K(x,t))f(t)\,dt\\ \le\ & (x-a)[(1-\gamma)F(x) + (2\gamma-1)F(\alpha(x))]\\ &+(b-x)[(\gamma-1)F(x) + (1-2\gamma)F(\beta(x)) + \gamma]\\ \le\ & \left[\frac{1}{2}-\left|\gamma-\frac{1}{2}\right|\right]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right],\end{aligned}$$

where $\alpha(x)$ and $\beta(x)$ are as given in (2.11) and $K(x,t)$ by (2.2). Noting that $R(x) = \Pr\{X \ge x\} = 1 - F(x)$, then bounds for

$$|\alpha(x) - E[X] + (1-\gamma)(b-a)R(x)|$$

could be obtained from the above development. This is more suitable for work in reliability where $f(x)$ is a failure density. Taking various values of $\gamma \in [0,1]$ and $x \in [a,b]$ gives a variety of results, some of which $(\gamma = 0)$ have been obtained in Barnett and Dragomir [18].

11. Conclusion and Discusson

The current work has investigated three-point quadrature rules in which, at most, the first derivative is involved. The major thrust of the work aims at providing **a priori** error bounds so that a suitable partition may be determined that will provide an approximation which is within a particular specified tolerance. The work contains, as special cases, both open and closed Newton-Cotes formulae such as the mid-point, trapezoidal and Simpson rules. The results mainly involve Ostrowski-type rules which contain an arbitrary point $x \in [a,b]$. These rules may be utilised when data is only known at discrete points, which may be non- uniform, without first interpolating.

The approach taken has been through the use of appropriate Peano kernels, resulting in an identity. The identity is then exploited through the Theory of Inequalities to obtain bounds on the error, subject to a variety of norms. The results developed in the current work provide both Riemann and Riemann-Stieltjes quadrature rules.

Grüss-type results are obtained, giving perturbed quadrature rules. A **premature** Grüss approach has produced rules that have tighter bounds. A new identity introduced recently by Dragomir and McAndrew [11] is exploited in Sections 5, 7, 9 and within Section 10 to produce Ostrowski-Grüss type results that seem like a perturbation of the original three-point quadrature rule. A simple reorganisation of the rule to incorporate the perturbation produces a different three- point rule. In effect, the following identity holds,

$$\mathfrak{T}(f,g) = \mathfrak{T}(\sigma(f), g) = \mathfrak{T}(f, \sigma(g)) = \mathfrak{T}(\sigma(f), \sigma(g)), \tag{11.1}$$

where

$$\begin{aligned} \mathfrak{T}(f,g) &= \mathfrak{M}(fg) - \mathfrak{M}(f)\,\mathfrak{M}(g), \\ \sigma(f) &= f - \mathfrak{M}(f) \end{aligned}$$

and

$$\mathfrak{M}(f) = \frac{1}{b-a}\int_a^b f(u)\,du.$$

That is,

$$\mathfrak{M}(fg) - \mathfrak{M}(f)\,\mathfrak{M}(g) = \mathfrak{M}(\sigma(f)\,g) = \mathfrak{M}(f\sigma(g)) = \mathfrak{M}(\sigma(f)\,\sigma(g)), \tag{11.2}$$

since $\mathfrak{M}(\sigma(f)) = \mathfrak{M}(\sigma(g)) = 0$.

Relation (11.1) (or indeed (11.2)) does not seem to have been **fully** realised in the literature. Dragomir and McAndrew [11] used only the equality of the outside two terms in (11.1) (or equivalently (11.2)) to obtain results for a trapezoidal rule. As a matter of fact, there are only two rules that have been used in the current article in which $f(\cdot) \equiv K(x,\cdot)$ and $g(\cdot) \equiv f'(\cdot)$. For $K(x,t)$ as given by (2.2), identity (2.3) is obtained. A similar identity to (2.3) would be obtained if $\sigma(K(x,\cdot))$ were to be considered with $\alpha(x)$ and $\beta(x)$ being replaced by $\phi(x)$ and $\psi(x)$ respectively, where

$$\sigma(K(x,t)) = \begin{cases} t - \phi(x), & t \in [a,x] \\ t - \psi(x), & t \in (x,b]. \end{cases} \tag{11.3}$$

Hence, from (2.2),

$$\sigma(K(x,t)) = K(x,t) - \mathfrak{M}(K(x,\cdot)),$$

where

$$
\begin{aligned}
& \mathfrak{M}(K(x,\cdot)) \\
= & \frac{1}{b-a}\int_a^b K(x,u)\,du \\
= & \frac{1}{b-a}\left[\int_a^x (u-\alpha(x))\,du + \frac{1}{b-a}\int_x^b (u-\beta(x))\,du\right] \\
= & \left(\frac{a+x}{2}-\alpha(x)\right)\left(\frac{x-a}{b-a}\right) + \left(\frac{x+b}{2}-\beta(x)\right)\left(\frac{b-x}{b-a}\right) \\
= & \frac{a+b}{2} - \alpha(x)\left(\frac{x-a}{b-a}\right) - \beta(x)\left(\frac{b-x}{b-a}\right).
\end{aligned}
$$

Therefore,

$$
\phi(x) = \alpha(x) + \mathfrak{M}(K(x,\cdot)) = \frac{a+b}{2} + (\alpha(x)-\beta(x))\left(\frac{b-x}{b-a}\right) \tag{11.4}
$$

with

$$
\psi(x) = \beta(x) + \mathfrak{M}(K(x,\cdot)) = \frac{a+b}{2} + (\beta(x)-\alpha(x))\left(\frac{x-a}{b-a}\right). \tag{11.5}
$$

Thus, from (2.3) and (11.3) we have, on using (11.4) and (11.5) and identifying $\phi(\cdot)$ with $\alpha(\cdot)$ and $\psi(\cdot)$ with $\beta(\cdot)$,

$$
\begin{aligned}
& \int_a^b \sigma(K(x,t))\,f'(t)\,dt \\
= & (\psi(x)-\phi(x))\,f(x) + (\phi(x)-a)\,f(a) + (b-\psi(x))\,f(b) - \int_a^b f(t)\,dt \\
= & (\beta(x)-\alpha(x))\,f(x) + (\alpha(x)-a)\,f(a)\,(b-\beta(x))\,f(b) \\
& +\mathfrak{M}(K(x,\cdot))\,[f(b)-f(a)] - \int_a^b f(t)\,dt.
\end{aligned}
$$

Therefore, using $\mathfrak{T}(K(x,\cdot), f')$, giving rise to what seems to be a perturbed quadrature rule, is equivalent to considering a Peano kernel shifted by its mean. Namely,

$$
\mathfrak{T}(\sigma(K(x,\cdot)), f') = \mathfrak{M}(\sigma(K(x,\cdot)), f').
$$

Bounds on the above quadrature rule may be obtained using a variety of norms as shown in the article. Using the above identity, bounds involving the first derivative result. If in (11.1) or (11.2), $\sigma(f')$ is used rather than f', then bounds involving the norms of $\sigma(f')$ would result. Either choice may be superior depending on the particular function $f(\cdot)$. For the Riemann-Stieltjes integral $df \equiv d\sigma(f)$ and so the two cases are equivalent.

The current work has provided a means for estimating the partition required in order to be guaranteed a certain accuracy for Newton- Cotes quadrature rules. The efficiency, mainly in terms of the number of function evaluations to achieve a particular accuracy, is a very important practical consideration.

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