JENSEN-TYPE INEQUALITIES FOR INVEX FUNCTIONS

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Abstract. Jensen’s inequality for a real convex function $f$ on a convex domain is generalised in several ways to vector functions with a cone inequality, and to invex functions generalizing convex functions.

1. Introduction

Jensen’s inequality for a real convex function $f$ on a convex domain $C$ states that, whenever $x_1, x_2, \ldots \in C$ and $\alpha_1, \alpha_2, \ldots \geq 0$ with $\sum \alpha_i = 1$,

$$f \left( \sum \alpha_i x_i \right) \leq \sum \alpha_i f(x_i).$$

This paper presents several generalisations of this inequality, to vector functions $F$ with a cone inequality, and to invex functions generalizing convex functions. This introduces additional gradient terms into various inequalities.

2. Basic Concepts and Definitions

Definition 1. Let $X$ and $Z$ be normed spaces, $Q \subset Z$ a closed convex cone, and $E \subset X$ a convex open set. A function $F : E \to Z$ is $Q$-convex on $E$ if

$$(\forall x, y \in E) \ (\forall \alpha \in (0,1)) \ \alpha F(x) + (1 - \alpha) F(y) - F(\alpha x + (1 - \alpha) y) \in Q.$$ 

A (Fréchet or linear Gateaux) differentiable function $F : E \to Z$ is $Q$-invex (see [1], [2], [3]) if, for some scale function $\omega : E \times E \to X$,

$$(\forall x, y \in E) \ F(x) - F(y) - F'(y) \omega(x - y) \in Q.$$ 

Remark 1. $F'(y)$ denotes the derivative. Invex properties have been extensively used with optimization problems. It is well-known that $Q$-invex implies $Q$-convex if $(\forall x, y \in E) \ \omega(x - y, y) = x - y$. If $F$ is real-valued and $Q = \mathbb{R}_+$, then the usual convexity follows. From $Q$-convex there follows readily

$$(\forall x_i \in E) \left( \forall \alpha_i \geq 0, \ \sum \alpha_i = 1 \right) \ \sum \alpha_i F(x_i) - F \left( \sum \alpha_i x_i \right) \in Q.$$ 

In this paper, invex shall require the function to be differentiable. (Extensions to nondifferentiable functions will be considered elsewhere.)

Denote by $\geq_Q$ the ordering defined by $Q$ thus $b \geq_Q a \iff b - a \in Q$.

If $\omega(\cdot, y)$ is linear, then $(\forall z = x - y) \ F(y + z) - F(z) \geq_Q F'(y) C z$, where $C$ is a linear mapping (that may depend on $y$), hence $(\forall z) \ F'(y) z \geq_Q F'(y) C z$;

if $Q$ is pointed (thus if $Q \cap (-Q) = \{0\}$) then $F$ is convex, since

$$(\forall x) \ F(x) - F(y) \geq_Q F'(y) (x - y) + F'(y) (C - I) \ (x - y) = F'(y) (x - y).$$

Thus functions where $\omega(\cdot, y)$ is linear for each $y$ are equivalent to convex functions.

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Notation 1. Various results require a suitable notation, in order that the concept is not obscured by algebraic details. Let \( x = (x_1, x_2, \ldots, x_n) \), where each \( x_i \) is a vector in \( E \). Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) where each \( \alpha_i \geq 0 \) and \( \sum \alpha_i = 1 \). Denote by \( \alpha \cdot x \) the inner product \( \sum \alpha_i x_i \), and similarly for other inner products. Denote by \( S \) a sequence of indices \( (i_1, i_2, \ldots, i_r) \) taken from \( \{1, 2, \ldots, n\} \) with repetitions allowed; similarly denote by \( S' \) a sequence of indices \( \{i_1, i_2, \ldots, i_{r+1}\} \). Denote by \( x_S \) the vector of \( x_i \) when \( i \) runs through \( S \); and similarly define \( \alpha_S \). Similarly \( F'(x_S) \) denotes the sequence of \( F'(x_i) \) as \( i \) runs through \( S \). Denote by \( M_{x_S} \) the mean of the sequence \( x_S \), and by \( \Pi_{\alpha_S} \) the product of the items in \( \alpha_S \). If \( \Phi(S) \) denotes a function of \( S \), denote by \( A \Phi(S) \) the sum of the items in \( \Phi(S) \), when \( S \) runs over all sequences of length \( r \), and denote by \( A \Phi(S') \) the sum of the items in \( \Phi(S') \), where \( S' \) runs over the sequences of length \( r + 1 \). Thus \( A(\Pi_{\alpha_S}) \Phi(S) \) represents the sum of all terms
\[
\alpha_i \alpha_{i_2} \cdots \alpha_i \Phi(i_1, i_2, \ldots, i_r)
\]
as \( i_1, i_2, \ldots, i_r \) run through \( 1, 2, \ldots, n \).

Proposition 1. Let \( F : E \rightarrow Z \) be \( Q \)-invex; let \( x_1, x_2, \ldots \in E \) (allowing possible repetitions), and let \( \alpha_1, \alpha_2, \ldots \) be nonnegative with sum 1; let \( w = \sum \alpha_i x_i \). Then
\[
\sum \alpha_i F(x_i) - F(w) \geq_Q F'(w) \sum \omega(x_i - w, w).
\]
Proof. \( Q \)-invex requires, for each \( i \), that
\[
F(x_i) - F(w) \geq_Q F'(w) \omega(x_i - w, w).
\]
Multiplication by \( \alpha_i \) and summing gives the result. \( \blacksquare \)

Proposition 2. Let \( F : E \rightarrow Z \) be differentiable \( Q \)-invex. Let \( x_1, x_2, \ldots \in E \) (allowing possible repetitions), and let \( \alpha_1, \alpha_2, \ldots \) be nonnegative with sum 1. Then
\[
-\sum_j \alpha_j F'(x_j) \omega(\alpha \cdot (x - x_j), x_j)
\geq_Q \alpha \cdot F(x_S) - F(\alpha \cdot x)
\geq_Q F'(\alpha \cdot x) \sum j \alpha_j \omega(\alpha \cdot (x_j - x), \alpha \cdot x).
\]
Proof. Let \( w = \alpha \cdot x \). From invex,
\[
F'(x_j) \omega(x - x_j, x_j) \geq_Q F(x_j) - F(w) \geq_Q F'(w) \omega(x_i - w, w).
\]
Multiplication by \( \alpha_j \) and summing gives the result. \( \blacksquare \)

Corollary 1. Assume also that \( E \) is a linear subspace, and \( \omega(\cdot, w) \) is linear. Then
\[
\alpha \cdot F(x_S) - F(\alpha \cdot x) \geq_Q 0.
\]
Proof. \( \sum j \alpha_j \omega(\alpha \cdot (x - x_j), \alpha \cdot x) = \omega(\sum_j \alpha_j \alpha_i (x_i - x_j), \alpha \cdot x) = 0 \). \( \blacksquare \)

Corollary 2. Under hypothesis of Corollary 1,
\[
\sum j \alpha_j F'(x_j) \omega(x_j, x_j) - \sum i,j \alpha_i \alpha_j F'(x_j) \omega(x_j, x_j)
\geq_Q \sum j \alpha_j F(x_j) - F(\sum_i \alpha_i x_j) \geq_Q 0.
\]
Proof. Since $\omega(\cdot, w)$ is linear,

$$-\sum_j \alpha_j F'(x_j) \omega(\alpha \cdot (x - x_j), x_j)$$

$$= -\sum_j F'(x_j) \omega\left(\sum_i \alpha_j \alpha_i \cdot (x_i - x_j), x_j\right)$$

$$= \sum_j \alpha_j F'(x_j) \omega(x, x_j) - \sum_{i,j} \alpha_i \alpha_j F'(x_j) \omega(x_j, x_j).$$

Then the result follows from Corollary 1.

Corollary 3. If $F$ is $Q$-convex, then

$$\sum_j \alpha_j F'(x_j) \left(\sum_i \alpha_i x_i\right) - \sum_j \alpha_j F'(x_j) \left(\sum_i \alpha_i x_i\right)$$

$$\geq Q \sum_j \alpha_j F(x_j) - F\left(\sum_j \alpha_j x_j\right) \geq 0.$$ Proof. By substituting $\omega(\alpha \cdot (x - x_j), x_j) = \sum_j \alpha_j (x_i - x_j)$ into the left inequality of Proposition 2, then rearranging the terms.

3. Generalized Jensen Inequalities for Invex Functions

The following generalizations of Jensen’s inequalities, involving multiple summations, hold for $Q$-invex functions.

Proposition 3. Let $F : E \to Z$ be differentiable $Q$-invex; let $r \geq 1$. Then

$$-A(\Pi\alpha S) F'(M_{xS}) \omega((\alpha \cdot (x - M_{xS}), M_{xS}))$$

$$\geq Q A(\Pi\alpha S) F(M_{xS}) - F(\alpha \cdot x)$$

$$\geq Q F'(\alpha \cdot x) A(\Pi\alpha S) \omega((\alpha \cdot (M_{xS} - x), M_{xS})).$$

Proof. By the $Q$-invex property,

$$-F'(M_{xS}) \omega((\alpha \cdot (x - M_{xS}), M_{xS}))$$

$$\geq Q F(M_{xS}) - F(\alpha \cdot x)$$

$$\geq Q F'(\alpha \cdot x) \omega((M_{xS} - \alpha \cdot x), \alpha \cdot x).$$

The result follows by applying $J \equiv A^T \Pi\alpha S$ on the left, noting that cone inequalities are thus unchanged, and $J\xi = \xi$ for any argument $\xi$ independent of $S$.

Corollary 4. Assume the hypothesis of Proposition 3. If $E$ is a linear subspace and $\omega(\cdot, M_{xS})$ is linear, then

$$A(\Pi\alpha S) F(M_{xS}) - F(\alpha \cdot x) \geq 0.$$ Proof. From the linear assumption,

$$A(\Pi\alpha S) \omega((\alpha \cdot (M_{xS} - x), M_{xS})) = \omega(u, M_{xS}),$$

where, after simplification,

$$u = (A(\Pi\alpha S)) \alpha \cdot (M_{xS} - x) = \alpha \cdot (M_{xS} - x) = 0.$$
In the following Corollary, which generalizes Corollary 2, note that $\xi = x_i$, in the first summation, and $S$ has $r$ terms; in the second summation, $M_{x_S}$ is unchanged, but $S'$ has $r + 1$ terms, and $\eta = x_{i_r+1}$.

**Corollary 5.** Under the hypothesis of Corollary 4,
\[-A (\Pi \alpha S) F' (M_{x_S}) \omega (\xi, M_{x_S}) - A (\Pi \alpha S') F' (M_{x_{S'}}) \omega (\eta, M_{x_{S'}}) \geq Q A (\Pi \alpha S) F (M_{x_S}) - F (\alpha \cdot x).\]

**Proof.** Since $\omega (\cdot, M_{x_S})$ is linear,
\[-A (\Pi \alpha S) F' (M_{x_S}) \omega (\alpha \cdot (\xi - M_{x_S}), M_{x_S}) = A (\Pi \alpha S) F' (M_{x_S}) \omega (M_{x_S}, M_{x_S}) - A (\Pi \alpha S) F' (M_{x_S}) \alpha \cdot \omega (\xi, M_{x_S}) = A (\Pi \alpha S) F' (M_{x_S}) \omega (\xi, M_{x_S}) - A (\Pi \alpha S') F' (M_{x_{S'}}) \omega (\eta, M_{x_{S'}}).\]

The conclusion follows by applying Proposition 3. \(\blacksquare\)

**Corollary 6.** If $F$ is convex, then (denoting $\xi = x_i$)
\[-A (\Pi \alpha S) F' (M_{x_S}) \xi - A (\Pi \alpha S) F' (M_{x_S}) (M_{x_S}) \geq Q A (\Pi \alpha S) F (M_{x_S}) - F (\alpha \cdot x) \geq Q 0.\]

**Proof.** Substitute $\omega (\alpha \cdot (x - M_{x_S}), M_{x_S}) = \alpha (x - M_{x_S})$. The details are omitted. \(\blacksquare\)

### 4. Further Refinements

The notation of Corollary 5 is used to state Proposition 4, namely $\xi = x_i$, $\eta = x_{i_r+1}$. Note that $S$ has $r$ terms, $S'$ has $r + 1$ terms.

**Proposition 4.** Let $F : E \to Z$ be differentiable $Q$–invex. Then, with $\varepsilon = (r + 1)^{-1}$,
\[-A (\Pi \alpha S) F' (M_{x_S}) \omega (\varepsilon (\eta - M_{x_S}), M_{x_S}) \geq Q A (\Pi \alpha S) F' (M_{x_S}) - A (\Pi \alpha S') F (M_{x_{S'}}) \geq Q A (\Pi \alpha S') F' (M_{x_{S'}}) \omega (\varepsilon (M_{x_S} - \eta), M_{x_{S'}}).\]

**Proof.** \(\varepsilon\)From $Q$–invex,
\[F' (M_{x_S}) \omega (\varepsilon (\eta - M_{x_S}), M_{x_S}) \geq Q F (M_{x_S}) - F (M_{x_{S'}}) \geq Q F' (M_{x_{S'}}) \omega (\varepsilon (M_{x_S} - \eta), M_{x_{S'}}).\]

The result follows on applying $(\Pi \alpha S')$ and averaging with $A$. \(\blacksquare\)

**Corollary 7.** If $F$ is as in Proposition 4, and $\omega (\cdot, M_{x_{S'}})$ is linear, then, for each $r = 1, 2, \ldots$,
\[A (\Pi \alpha S) F (M_{x_S}) \geq Q A (\Pi \alpha S') F (M_{x_{S'}}).\]

**Proof.** This follows from the second inequality of Proposition 4, noting that, when $\omega (\cdot, M_{x_{S'}})$ is linear,
\[
\omega (\varepsilon (M_{x_S} - \eta), M_{x_{S'}}) = \left(\frac{\varepsilon}{r}\right) \sum_{i=1}^{n} \omega (x_i, M_{x_{S'}}) = \varepsilon \omega (x_{i_r+1}, M_{x_{S'}}),
\]
and application of $A(\Pi_\alpha S')$ gives an equal value for each summand, so the result is 0.

**Corollary 8.** Under the same hypotheses as Corollary 7,

$$A(\Pi_\alpha S') F'(M_x S) \xi - A(\Pi_\alpha S) F'(M_x S'(M_x S))$$

$$\geq Q \ A(\Pi_\alpha S) F(M_x S) - A(\Pi x S') F(M_x S')$$

$$\geq Q 0.$$

**Proof.** From the first inequality of Proposition 4, using the assumed linearity to expand $\omega(\cdot, M_x S)$ in its first argument.

For other recent results connected with Jensen's discrete inequality, see the papers [4]-[9], where further references are given.

**References**


