GENERALIZATION OF H. MINC AND L. SATRIE'S INEQUALITY

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ABSTRACT. In the article, an inequality of H. Minc and L. Sathre (Proc. Edinburgh Math. Soc. 14 (1964/65), 41–46) is generalized: Let \( n \) and \( m \) be natural numbers, \( k \) a nonnegative integer, then we have

\[
\frac{n + k}{n + m + k} < \frac{\sqrt[n+m+k]{(n+k)!/k!}}{\sqrt[n+m+k]{(n+m+k)!/k!}} < 1.
\]

From this, some corollaries are deduced. At last, an open problem is proposed.

It is known that, for \( n \in \mathbb{N} \), the following inequalities were given in [3]:

\[
\frac{n}{n + 1} < \frac{\sqrt[n]{n!}}{\sqrt[n]{(n+1)!}} < 1.
\]  

(1)

In [1], the left inequality in (1) was refined by

\[
\frac{n}{n + 1} < \left( \frac{1}{n} \sum_{i=1}^{n} i^r \right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n]{(n+1)!}}
\]

(2)

for all positive real numbers \( r \). Both bounds are best possible.

In this article, using analytic method, we obtain

**Theorem.** Let \( n \) and \( m \) be natural numbers, \( k \) a nonnegative integer. Then we have

\[
\frac{n + k}{n + m + k} < \frac{\sqrt[n+m+k]{(n+k)!/k!}}{\sqrt[n+m+k]{(n+m+k)!/k!}} < 1.
\]  

(3)

**Proof.** The upper bound is obtained immediately from

\[
\frac{\sqrt[n+m+k]{(n+k)!/k!}}{\sqrt[n+m+k]{(n+m+k)!/k!}} = \left[ \left( \prod_{i=k+1}^{n+k} i \right) \left( \prod_{i=n+k+1}^{n+m+k} i \right) \right]^{1/(n+m)} < 1.
\]

The left inequality in (3) can be rearranged as

\[
\frac{n + k}{\sqrt{(n+k)!/k!}} < \frac{n + m + k}{\sqrt[n+m+k]{(n+m+k)!/k!}}.
\]

this is equivalent to

\[
\frac{n + k}{\sqrt{(n+k)!/k!}} < \frac{n + k + 1}{\sqrt[n](n+k+1)!/(n+1)!}
\]  

(4)

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When \( k = 0 \), inequality (4) follows from the left inequality in (1). When \( k \neq 0 \), the inequality (4) can be rewritten as

\[
\left( \frac{(n + k)!}{k!} \right)^{1/n} > \frac{(n + k)^{n+1}}{(n + k + 1)^n}.
\]  

(5)

In [4, p. 184], the following inequalities were given for \( n \in \mathbb{N} \)

\[
\sqrt{2\pi n} \left( \frac{n}{e} \right)^n < n! < \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \exp \frac{1}{12n}.
\]  

(6)

By substituting the inequalities in (6) into the left term of inequality (5), we see that it is sufficient to prove

\[
\left[ \sqrt{2\pi(n + k)} \left( \frac{n + k}{e} \right)^{n+k} \right]^{1/n} > \frac{(n + k)^{n+1}}{(n + k + 1)^n} \left[ \sqrt{2\pi k} \left( \frac{k}{e} \right)^k \exp \frac{1}{12k} \right]^{1/n}.
\]  

(7)

Simplifying (7) directly and standard arguments leads to

\[
n \ln \left( 1 + \frac{1}{n + k} \right) + \frac{2k + 1}{2n} \ln \left( 1 + \frac{n}{k} \right) - \frac{1}{12kn} - 1 > 0.
\]  

(8)

In [2, pp. 367–368], [4, pp. 273–274] and [8], we have for \( t > 0 \)

\[
\ln \left( 1 + \frac{1}{t} \right) > \frac{2}{2t + 1}.
\]

Thus, to get inequality (8), it suffices to show

\[
\frac{2n}{2(n + k) + 1} + \frac{2k + 1}{2n} \cdot \frac{2n}{2k + n} - \frac{1}{12kn} - 1 > 0.
\]

But this is equivalent to

\[
2(12k^2 - 1)n^2 + (12kn - 1)n + 4(6n - 1)k^2 + 2(3n - 1)k > 0.
\]

The proof is complete. ■

**Corollary 1.** For any given nonnegative integer \( k \), the sequences

\[
\sqrt[n]{(n + k)!/k!}, \quad \frac{n + k}{\sqrt[n]{(n + k)!/k!}}, \quad \frac{(n + k)!}{\sqrt[n]{(n + k)!/k!}}, \quad \frac{(n + k + 1)}{\sqrt[n]{(n + k)!/k!}}
\]

are strictly increasing with respect to \( n \in \mathbb{N} \).

**Corollary 2.** For any given \( n \in \mathbb{N} \), the sequences

\[
\sqrt[n]{(n + k)!/k!}, \quad \frac{(n + k)}{\sqrt[n]{(n + k)!/k!}}, \quad \frac{(n + k + 1)}{\sqrt[n]{(n + k)!/k!}}
\]

are strictly increasing with respect to the nonnegative integers \( k \).

**Remark.** Recently, the first author in [5] and [7], among other things, generalized the left inequality in (2) in new directions and got that, if \( n \) and \( m \) are natural numbers, \( k \) is a nonnegative integer, then

\[
\frac{n + k}{n + m + k} < \left( \frac{1}{n} \sum_{i=k+1}^{n+k} x^r / \frac{1}{n + m} \sum_{i=k+1}^{n+m+k} x^r \right)^{1/r},
\]  

(9)
where r is any given positive real number. The lower bound is best possible.

In [6], the first author further presented that, let n and m be natural numbers, suppose \( a = (a_1, a_2, \ldots) \) is a positive and increasing sequence satisfying

\[
a_{k+1}^2 a_k a_{k+2},
\]

\[
\frac{a_{k+1} - a_k}{a_{k+1}^2 - a_k a_{k+2}} \max \left\{ \frac{k + 1}{a_{k+1}}, \frac{k + 2}{a_{k+2}} \right\}
\]

for \( k \in \mathbb{N} \), then the inequality

\[
\frac{a_n}{a_{n+m}} \left( \frac{1}{n} \sum_{i=k+1}^{n+k} \frac{i}{n} \sum_{i=1}^{n+m+k} \frac{i}{r} \right)^{1/r} \leq \frac{n! (n + k)!/k!}{(n + m + k)!/k!}.
\]

holds for any given positive real number \( r \in \mathbb{R} \). The lower bound of (12) is best possible.

Using L’Hospital rule yields

\[
\lim_{r \to 0} \left( \frac{1}{n} \sum_{i=k+1}^{n+k} \frac{i}{r} \right) = \frac{n! (n + k)!/k!}{(n + m + k)!/k!}.
\]

thus, we propose the following

**Open Problem.** Let \( n \) and \( m \) be natural numbers, \( k \) a nonnegative integer. Then, for all real numbers \( r > 0 \), we have

\[
\left( \frac{1}{n} \sum_{i=k+1}^{n+k} \frac{i}{r} \right) < \frac{n! (n + k)!/k!}{(n + m + k)!/k!}.
\]

**The upper bound is best possible.**

**REFERENCES**


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