# GENERALIZATIONS OF ALZER'S AND KUANG'S INEQUALITY

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Abstract. Let $f$ be a strictly increasing convex (or concave) functions on $(0,1]$, then, for $k$ being a nonnegative integer and $n$ a natural number, the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f\bigl(\frac{i}{n+k}\bigr)$ is decreasing in $n$ and $k$ and has a lower bound $\int_0^1 f(t)\,dt$. From this, some new inequalities involving $\sqrt[n]{(n+k)!/k!}$ are deduced. By the Hermie-Hadamard inequality, several inequalities are obtained.

## 1. Introduction

In [1], H. Alzer, using the mathematical induction and other techniques, proved that for $r>0$ and $n\in\mathbb{N}$,

$$\frac{n}{n+1}\left(\frac{1}{n}\sum_{i=1}^{n} i^r\Big/\frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1}$$

By the Cauchy's mean-value theorem and the mathematical induction, the author in [7] presented that, if $n$ and $m$ are natural numbers, $k$ is a nonnegative integer, $r>0$, then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r\Big/\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}. \tag{2}$$

The lower bound is best possible.

From the Stirling's formula, for all nonnegative integers $k$ and natural numbers $n$ and $m$, the author in [8] obtained

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}\Big/\left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \sqrt{\frac{n+k}{n+m+k}}. \tag{3}$$

Let $f$ be a strictly increasing convex (or concave) function in $(0,1]$, J.-C. Kuang in [2] verified that

$$\frac{1}{n}\sum_{k=1}^{n} f\Bigl(\frac{k}{n}\Bigr) > \frac{1}{n+1}\sum_{k=1}^{n+1} f\Bigl(\frac{k}{n+1}\Bigr) > \int_0^1 f(x)\,dx. \tag{4}$$

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The study of Alzer's and Minc-Sathre's inequality has many literature, for examples, [1]–[9].

In this article, motivated by [2, 7], i.e. the inequalities in (2), (3) and (4), considering the convexity of a function, we get

**Theorem 1.** *Let $f$ be a strictly increasing convex (or concave) function in $(0,1]$, then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f\bigl(\frac{i}{n+k}\bigr)$ is decreasing in $n$ and $k$ and has a lower bound $\int_0^1 f(t)\,dt$, that is,*

$$\frac{1}{n}\sum_{i=k+1}^{n+k} f\Bigl(\frac{i}{n+k}\Bigr) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} f\Bigl(\frac{i}{n+k+1}\Bigr) > \int_0^1 f(t)\,dt, \tag{5}$$

*where $k$ is a nonnegative integer, $n$ a natural number.*

If let $f(x) = x^r$, $r > 0$, or let $k = 0$ in (5), then the inequalities in (1), (2) and (4) could be deduced. Therefore, inequality (5) generalizes Alzer's and Kuang's inequality in [1, 2] and inequality (2) above.

**Corollary 1.** *For a nonnegative integer $k$ and a natural number $n > 1$, we have*

$$\begin{aligned}\frac{n+k}{n+k+1} &< \left[\frac{(2n+2k)!}{(n+2k)!}\right]^{1/n} \Bigg/ \left[\frac{(2n+2k+2)!}{(n+2k+1)!}\right]^{1/(n+1)} \\ &< \left[\frac{(n+k)!}{k!}\right]^{1/n} \Bigg/ \left[\frac{(n+k+1)!}{k!}\right]^{1/(n+1)} < \left[\frac{k!(k+2)!}{(k+3)^2}\right]^{1/n(n+1)}.\end{aligned} \tag{6}$$

For a larger $n$, the upper bound in the third inequality of (6) is not better than (3) for $m = 1$.

From the Hermite-Hadamard inequality in [3] and [4, pp. 10–12], we get the following

**Theorem 2.** *Let $f$ be a nonlinear convex function in $(0,1]$, then*

$$\begin{aligned}&\frac{1}{n+k}\sum_{i=k+1}^{n+k}\left[f\Bigl(\frac{i}{n+k}\Bigr) - f\biggl(\frac{2i-1}{2(n+k)}\biggr)\right] \\ >&\frac{1}{n+k}\sum_{i=k+1}^{n+k} f\Bigl(\frac{i}{n+k}\Bigr) - \int_{k/(n+k)}^1 f(t)\,dt \\ >&\frac{1}{2(n+k)}\left[f(1) - f\Bigl(\frac{k}{n+k}\Bigr)\right].\end{aligned} \tag{7}$$

*Further, if $f$ satisfies the Lipschitz condition*

$$|f(x) - f(y)| M |x-y|^{\alpha}, \quad 0 < \alpha 1, \tag{8}$$

*then*

$$\frac{n}{n+k}\cdot\frac{M}{[2(n+k)]^{\alpha}} > \frac{1}{n+k}\sum_{i=k+1}^{n+k} f\Bigl(\frac{i}{n+k}\Bigr) - \int_{k/(n+k)}^1 f(t)\,dt. \tag{9}$$

If let $k = 0$ in theorem 2, the related result in [2] follows.

## 2. Proofs of Theorems

*Proof of Theorem 1.* Let us first assume that $f$ be a strictly increasing convex function. Taking $x_1=\frac{i-1}{n+k}$, $x_2=\frac{i}{n+k}$, $\alpha=\frac{i-k-1}{n}$ and using the convexity and monotonicity of $f$ yields

$$\begin{aligned}
&\frac{i-k-1}{n}f\Big(\frac{i-1}{n+k}\Big)+\Big(1-\frac{i-k-1}{n}\Big)f\Big(\frac{i}{n+k}\Big)\\
&f\Big(\frac{i-k-1}{n}\cdot\frac{i-1}{n+k}+\frac{n-i+k+1}{n}\cdot\frac{i}{n+k}\Big)\\
=&f\Big(\frac{ni-i+k+1}{n(n+k)}\Big)>f\Big(\frac{i}{n+k+1}\Big)
\end{aligned}$$

for $i=k+1,k+2,\dots,n+k$. Summing up leads to

$$\sum_{i=k+1}^{n+k}\Big[\frac{i-k-1}{n}f\Big(\frac{i-1}{n+k}\Big)+\Big(1-\frac{i-k-1}{n}\Big)f\Big(\frac{i}{n+k}\Big)\Big]>\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k+1}\Big),$$

$$\sum_{i=k+1}^{n+k}\Big[(i-k-1)f\Big(\frac{i-1}{n+k}\Big)+(n+k-i+1)f\Big(\frac{i}{n+k}\Big)\Big]>n\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k+1}\Big),$$

$$n\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k+1}\Big)+nf(1)<(n+1)\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k}\Big),$$

$$n\sum_{i=k+1}^{n+k+1}f\Big(\frac{i}{n+k+1}\Big)<(n+1)\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k}\Big).$$

The inequality (5) is proved.

By similar procedure, if $f$ is a strictly increasing concave function in $(0,1]$, then for $k<in+k$, we have

$$\begin{aligned}
&\frac{i-k}{n+1}f\Big(\frac{i+1}{n+k+1}\Big)+\frac{n+k-i+1}{n+1}f\Big(\frac{i}{n+k+1}\Big)\\
&f\Big(\frac{i-k}{n+1}\cdot\frac{i+1}{n+k+1}+\frac{n+k-i+1}{n+1}\cdot\frac{i}{n+k+1}\Big)\\
=&f\Big(\frac{ni+2i-k}{(n+1)(n+k+1)}\Big)<f\Big(\frac{i}{n+k}\Big),\\
&\sum_{i=k+1}^{n+k}\Big[\frac{i-k}{n+1}f\Big(\frac{i+1}{n+k+1}\Big)+\frac{n+k-i+1}{n+1}f\Big(\frac{i}{n+k+1}\Big)\Big]\\
=&\frac{n}{n+1}\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k+1}\Big)+\frac{n}{n+1}f(1)<\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k}\Big),\\
&n\sum_{i=k+1}^{n+k+1}f\Big(\frac{i}{n+k+1}\Big)<(n+1)\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k}\Big).
\end{aligned}$$

The proof is complete. ∎

*Proof of Corallary 1.* Substituting $f$ by $\ln(1+x)$ or by $\ln(x/(1+x))$ in (5) and simplifying yields the first or the second inequality in (6), respectively.

Since

$$\frac{[(n+k)!/k!]^{n+1}}{[(n+k+1)!/k!]^n}=\sum_{j=3}^{n}\left\{\frac{[(j+k)!/k!]^{j+1}}{[(j+k+1)!/k!]^j}-\frac{[(j+k-1)!/k!]^j}{[(j+k)!/k!]^{j-1}}\right\}+\frac{[(k+2)!/k!]^3}{[(k+3)!/k!]^2}<\frac{k!(k+2)!}{(k+3)^2},$$

the third inequality in (6) is obtained. ∎

*Proof of Theorem 2.* Using the Hermite-Hadamard inequality in [3] and [4, pp. 10–12], we have

$$\begin{aligned}\sum_{i=k+1}^{n+k} f\left(\frac{2i-1}{2(n+k)}\right) &< (n+k)\sum_{i=k+1}^{n+k}\int_{(i-1)/(n+k)}^{i/(n+k)} f(x)\,\mathrm{d}x\\ &< \frac{1}{2}\sum_{i=k+1}^{n+k}\left[f\Big(\frac{i}{n+k}\Big)+f\Big(\frac{i-1}{n+k}\Big)\right]\\ &= \sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k}\Big)-\frac{1}{2}\left[f(1)-f\Big(\frac{k}{n+k}\Big)\right],\end{aligned}$$

that is

$$\frac{1}{n+k}\sum_{i=k+1}^{n+k} f\left(\frac{2i-1}{2(n+k)}\right) < \int_{k/(n+k)}^{1} f(x)\,\mathrm{d}x$$

$$<\frac{1}{n+k}\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k}\Big)-\frac{1}{2(n+k)}\left[f(1)-f\Big(\frac{k}{n+k}\Big)\right].$$

The inequality (7) is proved.

Combining (8) with (7) yields inequality (9).

The proof of theorem 2 is complete. ∎

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