GENERALIZATIONS OF ALZER’S AND KUANG’S INEQUALITY

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Abstract. Let $f$ be a strictly increasing convex (or concave) function on $(0, 1]$, then, for $k$ being a nonnegative integer and $n$ a natural number, the sequence \( \frac{1}{n} \sum_{i=1}^{n+k} f \left( \frac{x_{i+k}}{n+1} \right) \) is decreasing in $n$ and $k$ and has a lower bound \( \int_0^1 f(t) \, dt \). From this, some new inequalities involving \( \sqrt{(n + k)/k!} \) are deduced. By the Hermite-Hadamard inequality, several inequalities are obtained.

1. Introduction

In [1], H. Alzer, using the mathematical induction and other techniques, proved that for $r > 0$ and $n \in \mathbb{N},$

\[
\frac{n}{n+1} \left( \frac{1}{n} \sum_{i=1}^{n} x^r \right) \left( \frac{1}{n+1} \sum_{i=1}^{n+1} x^r \right)^{1/r} < \frac{\sqrt{n!}}{n \sqrt{(n+1)!}}.
\]  

(1)

By the Cauchy’s mean-value theorem and the mathematical induction, the author in [7] presented that, if $n$ and $m$ are natural numbers, $k$ is a nonnegative integer, $r > 0$, then

\[
\frac{n + k}{n + m + k} < \left( \frac{1}{n} \sum_{i=k+1}^{n+k} x^r \right) \left( \frac{1}{n + m} \sum_{i=k+1}^{n+m+k} x^r \right)^{1/r}.
\]  

(2)

The lower bound is best possible.

From the Stirling’s formula, for all nonnegative integers $k$ and natural numbers $n$ and $m$, the author in [8] obtained

\[
\left( \prod_{i=k+1}^{n+k} i \right)^{1/n} / \left( \prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)} \sqrt{\frac{n + k}{n + m + k}}.
\]  

(3)

Let $f$ be a strictly increasing convex (or concave) function in $(0, 1]$, J.-C. Kuang in [2] verified that

\[
\frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) > \frac{1}{n+1} \sum_{k=1}^{n+1} f \left( \frac{k}{n+1} \right) > \int_0^1 f(x) \, dx.
\]  

(4)

1991 Mathematics Subject Classification. Primary 33D15.

Key words and phrases. Alzer’s inequality, Kuang’s inequality, convex function, Hermite-Hadamard inequality.

The author was supported in part by NSF of Henan Province, The People’s Republic of China.

This paper was typeset using \texttt{AmS-LaTeX}.
The study of Alzer’s and Minc-Sathre’s inequality has many literature, for examples, [1]–[9].

In this article, motivated by [2, 7], i.e. the inequalities in (2), (3) and (4), considering the convexity of a function, we get

**Theorem 1.** Let $f$ be a strictly increasing convex (or concave) function in $(0, 1]$, then the sequence $\frac{1}{n} \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right)$ is decreasing in $n$ and $k$ and has a lower bound $\int_{0}^{1} f(t) \, dt$, that is,

\[
\frac{1}{n} \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right) > \frac{1}{n + 1} \sum_{i=k+1}^{n+k+1} f\left(\frac{i}{n + k + 1}\right) > \int_{0}^{1} f(t) \, dt,
\]

where $k$ is a nonnegative integer, $n$ a natural number.

If let $f(x) = x^r$, $r > 0$, or let $k = 0$ in (5), then the inequalities in (1), (2) and (4) could be deduced. Therefore, inequality (5) generalizes Alzer’s and Kuang’s inequality in [1, 2] and inequality (2) above.

**Corollary 1.** For a nonnegative integer $k$ and a natural number $n > 1$, we have

\[
\frac{n + k}{n + k + 1} < \left[\frac{(2n + 2k)!}{(n + 2k)!}\right]^{1/n} \left[\frac{(2n + 2k + 2)!}{(n + 2k + 1)!}\right]^{1/(n+1)}
\]

\[
< \left[\frac{(n + k)!}{k!}\right]^{1/n} \left[\frac{(n + k + 1)!}{k!}\right]^{1/(n+1)} < \left[\frac{k!(k + 2)!}{(k + 3)!}\right]^{1/n(n+1)}.
\]

For a larger $n$, the upper bound in the third inequality of (6) is not better than (3) for $m = 1$.

From the Hermite-Hadamard inequality in [3] and [4, pp. 10–12], we get the following

**Theorem 2.** Let $f$ be a nonlinear convex function in $(0, 1]$, then

\[
\frac{1}{n + k} \sum_{i=k+1}^{n+k} \left[ f\left(\frac{i}{n+k}\right) - f\left(\frac{2i - 1}{2(n+k)}\right) \right] > \frac{1}{n + k} \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right) - \int_{k/(n+k)}^{1} f(t) \, dt
\]

\[
> \frac{1}{2(n+k)} \left[ f(1) - f\left(\frac{k}{n+k}\right) \right].
\]

Further, if $f$ satisfies the Lipschitz condition

\[
|f(x) - f(y)|M|x - y|^\alpha, \quad 0 < \alpha 1,
\]

then

\[
\frac{n}{n + k} \left[ \frac{M}{2(n+k)} \right]^\alpha > \frac{1}{n + k} \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right) - \int_{k/(n+k)}^{1} f(t) \, dt.
\]

If let $k = 0$ in theorem 2, the related result in [2] follows.
2. Proofs of Theorems

Proof of Theorem 1. Let us first assume that \( f \) be a strictly increasing convex function. Taking \( x_1 = \frac{i-1}{n+k} \), \( x_2 = \frac{i}{n+k} \), \( \alpha = \frac{i-k-1}{n} \) and using the convexity and monotonicity of \( f \) yields

\[
\frac{i-k-1}{n} f\left(\frac{i-1}{n+k}\right) + \left(1 - \frac{i-k-1}{n}\right) f\left(\frac{i}{n+k}\right) > f\left(\frac{n n - i + k + 1}{n(n+k)}\right) > f\left(\frac{i}{n+k+1}\right)
\]

for \( i = k + 1, k + 2, \ldots, n + k \). Summing up leads to

\[
\sum_{i=k+1}^{n+k} \frac{i-k-1}{n} f\left(\frac{i-1}{n+k}\right) + \left(1 - \frac{i-k-1}{n}\right) f\left(\frac{i}{n+k}\right) > \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k+1}\right),
\]

\[
\sum_{i=k+1}^{n+k} (i-k) f\left(\frac{i-1}{n+k}\right) + (n+k-i+1) f\left(\frac{i}{n+k}\right) > n \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k+1}\right),
\]

\[
\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k+1}\right) + n f(1) < (n+1) \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right),
\]

\[
\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k+1}\right) < (n+1) \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right).
\]

The inequality (5) is proved.

By similar procedure, if \( f \) is a strictly increasing concave function in \((0, 1]\), then for \( k < in + k \), we have

\[
\frac{i-k}{n+1} f\left(\frac{i+1}{n+k+1}\right) + \frac{n+k-i+1}{n+1} f\left(\frac{i}{n+k+1}\right)
\]

\[
f\left(\frac{i-k}{n+1} \frac{i+1}{n+k+1} + \frac{n+k-i+1}{n+1} \frac{i}{n+k+1}\right)
\]

\[
= f\left(\frac{ni + 2i - k}{(n+1)(n+k+1)}\right) < f\left(\frac{i}{n+k}\right),
\]

\[
\sum_{i=k+1}^{n+k} \frac{i-k}{n+1} f\left(\frac{i+1}{n+k+1}\right) + \frac{n+k-i+1}{n+1} f\left(\frac{i}{n+k+1}\right)
\]

\[
= \frac{n}{n+1} \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k+1}\right) + \frac{n}{n+1} f(1) < \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right),
\]

\[
\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k+1}\right) < (n+1) \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right).
\]

The proof is complete. \( \blacksquare \)

Proof of Corollary 1. Substituting \( f \) by \( \ln(1+x) \) or by \( \ln(x/(1+x)) \) in (5) and simplifying yields the first or the second inequality in (6), respectively.
Since
\[
\frac{[(n + k)!/k!]^{n+1}}{(n+k+1)!/k!} = \sum_{j=3}^{n} \left\{ \frac{([j+k]!/k!)^{n+1}}{([j+k+1]!/k!)^{n+1}} - \frac{([j+k-1]!/k!)^{n+1}}{([j+k]!/k!)^{n+1}} \right\} + \frac{[(k+2)!/k!]^{n+1}}{[(k+3)!/k!]^{n+1}} < \frac{k!(k+2)!}{(k+3)!},
\]
the third inequality in (6) is obtained. ■

Proof of Theorem 2. Using the Hermite-Hadamard inequality in [3] and [4, pp. 10–12], we have
\[
\sum_{i=k+1}^{n+k} f\left( \frac{2i-1}{2(n+k)} \right) < (n+k) \sum_{i=k+1}^{n+k} \int_{(i-1)/(n+k)}^{i/(n+k)} f(x) \, dx
\]
\[
< \frac{1}{2} \sum_{i=k+1}^{n+k} \left[ f\left( \frac{i}{n+k} \right) + f\left( \frac{i-1}{n+k} \right) \right]
\]
\[
= \sum_{i=k+1}^{n+k} f\left( \frac{i}{n+k} \right) - \frac{1}{2} \left[ f(1) - f\left( \frac{k}{n+k} \right) \right],
\]
that is
\[
\frac{1}{n+k} \sum_{i=k+1}^{n+k} f\left( \frac{2i-1}{2(n+k)} \right) < \int_{1/(n+k)}^{n/(n+k)} f(x) \, dx
\]
\[
< \frac{1}{n+k} \sum_{i=k+1}^{n+k} f\left( \frac{i}{n+k} \right) - \frac{1}{2(n+k)} \left[ f(1) - f\left( \frac{k}{n+k} \right) \right].
\]
The inequality (7) is proved.
Combining (8) with (7) yields inequality (9).
The proof of theorem 2 is complete. ■

References

[8] Feng Qi, Inequalities and monotonicity of sequences involving √[(n + k)!/k!], submitted.