1. Introduction

In a recent paper [1], Matić, Pečarić and Ujević proved the following inequality, which has been called, in [2], the pre-Grüss inequality

\[
\left| \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt \right| \leq \frac{1}{4} (\phi - \gamma) \left[ \frac{1}{b-a} \int_a^b g^2(t) \, dt - \left( \frac{1}{b-a} \int_a^b g(t) \, dt \right)^2 \right]^{\frac{1}{2}},
\]

provided that \( \gamma \leq f(t) \leq \phi \) a.e. on \([a,b]\) and the integrals exist and are finite.

In [1], the authors used (1.1) to obtain some bounds for the remainder in certain Taylor like formulae whilst in [2], the authors applied (1.1) to estimation of the remainder in three point quadrature formulae.

Basically, (1.1) is a pre-Grüss inequality since, if we assume that \( \alpha \leq g(t) \leq \beta \) a.e. in \([a,b]\), then, by the well known fact that (see for example [8])

\[
\frac{1}{b-a} \int_a^b g^2(t) \, dt - \left( \frac{1}{b-a} \int_a^b g(t) \, dt \right)^2 \leq \frac{1}{4} (\beta - \alpha)^2,
\]

and, by (1.1) and (1.2), we can deduce the original Grüss inequality:

\[
\left| \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt \right| \leq \frac{1}{4} (\phi - \gamma) (\beta - \alpha).
\]
In [1], Matić, Pečarić and Ujević observed that if a factor is known, for example \( g(t) \), \( t \in [a, b] \), then instead of using (1.3) in estimating the difference
\[
\frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt,
\]
it is better to use (1.1), as they have shown in their paper that it improves some recent results by the second author [4].

In this paper, by using the same approach, we obtain some inequalities for the expectation \( E(X) \) and cumulative distribution function \( F(\cdot) \) of a random variable having the probability distribution function \( f : [a, b] \to \mathbb{R} \). It is assumed that we know the lower and the upper bound for \( f \), i.e., the real numbers \( \gamma, \phi \) such that \( 0 \leq \gamma \leq f(t) \leq \phi \leq 1 \) a.e. \( t \) on \( [a, b] \). Some related results are also established.

2. SOME INEQUALITIES FOR EXPECTATION AND DISPERSION

We start with the following result for expectation.

**Theorem 1.** Let \( X \) be a random variable having the probability density function \( f : [a, b] \to \mathbb{R} \). Assume that there exists the constants \( \gamma, \phi \) such that \( 0 \leq \gamma \leq f(t) \leq \phi \leq 1 \) a.e. \( t \) on \( [a, b] \). Then we have the inequality
\[
\left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2,
\]
where \( E(X) \) is the expectation of the random variable \( X \).

**Proof.** If we put \( g(t) = t \) in (1.1), we obtain
\[
\left| \frac{1}{b-a} \int_a^b tf(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b t \, dt \right|
\]
\[
\leq \frac{1}{2} (\phi - \gamma) \left[ \frac{1}{b-a} \int_a^b t^2 \, dt - \left( \frac{1}{b-a} \int_a^b t \, dt \right)^2 \right]^{1/2}
\]
and as
\[
\int_a^b tf(t) \, dt = E(X),
\]
\[
\int_a^b f(t) \, dt = 1, \quad \frac{1}{b-a} \int_a^b t \, dt = \frac{a+b}{2}
\]
and
\[
\frac{1}{b-a} \int_a^b t^2 \, dt - \left( \frac{1}{b-a} \int_a^b t \, dt \right)^2 = \frac{(b-a)^2}{12},
\]
then by (2.2) we deduce (2.1).

To point out a result for the \( p \)-moments of the random variable \( X \), \( p \in \mathbb{R} \setminus \{-1,0\} \), we need the following \( p \)-Logarithmic mean
\[
M_p (a, b) := \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p},
\]
where \( 0 < a < b \).
Theorem 2. Let $X$ and $f$ be as in Theorem 1 and $E_p(X)$ be the $p$-moment of $X$, i.e.,

$$E_p(X) := \int_a^b t^p f(t) \, dt,$$

which is assumed to be finite.

Then we have the inequality

$$|E_p(X) - M_p^p(a,b)| \leq \frac{1}{2} (\phi - \gamma) \left[ M_{2p}^{2p}(a,b) - M_{2p}^{2p}(a,b) \right]^{\frac{1}{2}}. \quad (2.3)$$

The proof is obvious by the inequality (1.1) in which we choose $g(t) = t^p$, $p \in \mathbb{R} \setminus \{-1,0\}$ and use the definition of $p$-logarithmic means.

If we consider the Logarithmic mean

$$M_{-1}(a,b) := L(a,b) = \frac{b - a}{\ln b - \ln a}, \quad 0 < a < b$$

and define the $(-1)$-moment of the random variable $X$ by

$$E_{-1}(X) := \int_a^b \frac{f(t)}{t} \, dt,$$

then we can also state the following theorem.

Theorem 3. Let $X$ and $f$ be as in Theorem 1. Then we have the inequality:

$$|E_{-1}(X) - M_{-1}^{-1}(a,b)| \leq \frac{1}{2} (\phi - \gamma) \left[ M_{-2}^{-2}(a,b) - M_{-2}^{-2}(a,b) \right]^{\frac{1}{2}}, \quad (2.4)$$

provided the $(-1)$-moment of $X$ is finite.

The proof is obvious by the inequality (1.1) and we omit the details.

The following theorem also holds.

Theorem 4. Let $X$ and $f$ be as above. If

$$\sigma_\mu(X) := \left[ \int_a^b (t - \mu)^2 f(t) \, dt \right]^{\frac{1}{2}}, \quad \mu \in [a,b],$$

then we have the inequality

$$\left| \sigma_\mu^2(X) - \left( \mu - \frac{a + b}{2} \right)^2 - \frac{(b - a)^2}{12} \right| \leq \frac{1}{2} (\phi - \gamma) (b - a)^2 \left[ \frac{1}{3} \left( \mu - \frac{a + b}{2} \right)^2 + \frac{1}{180} (b - a)^2 \right]^{\frac{1}{2}} \leq \frac{1}{3\sqrt{3}} (\phi - \gamma) (b - a)^3. \quad (2.5)$$

Proof. If we put $g(t) = (t - \mu)^2$ in (1.1) we get

$$\left| \frac{1}{b - a} \int_a^b g(t) f(t) (t - \mu)^2 \, dt - \frac{1}{b - a} \int_a^b g(t) f(t) \, dt \cdot \frac{1}{b - a} \int_a^b (t - \mu)^2 \, dt \right| \leq \frac{1}{2} (\phi - \gamma) \left[ \frac{1}{b - a} \int_a^b (t - \mu)^4 \, dt - \left( \frac{1}{b - a} \int_a^b (t - \mu)^2 \, dt \right)^2 \right]^{\frac{1}{2}}, \quad (2.6)$$
and as
\[
\frac{1}{b-a} \int_a^b (t - \mu)^2 \, dt = \frac{(b - \mu)^3 + (\mu - a)^3}{3 (b - a)} = \frac{(b - \mu)^2 - (b - \mu)(\mu - a) + (\mu - a)^2}{3} = \left( \mu - \frac{a + b}{2} \right)^2 + \frac{(b - a)^2}{12},
\]

\[
\frac{1}{b-a} \int_a^b (t - \mu)^4 \, dt = \frac{(b - \mu)^5 + (\mu - a)^5}{5 (b - a)} - \frac{\left[(b - \mu)^3 + (\mu - a)^3\right]^2}{3 (b - a)}
\]

\[
= \frac{1}{45} \left[ 9 (b - \mu)^4 - (b - \mu)^3 (\mu - a) + (b - \mu)^2 (\mu - a)^2 \\
- (b - \mu)(\mu - a)^3 + (\mu - a)^4 - 5 (b - \mu)^4 - 5 (b - \mu)^2 (\mu - a)^2 \\
- 5 (\mu - a)^4 + 10 (b - \mu)^3 (\mu - a) + 10 (b - \mu)(\mu - a)^3 - 10 (b - \mu)^2 (\mu - a)^2 \right]
\]

\[
= \frac{1}{45} \left[ 4 (b - \mu)^4 + 4 (\mu - a)^4 - 8 (b - \mu)^2 (\mu - a)^2 + 2 (b - \mu)^3 (\mu - a)^2 + (\mu - a) (b - \mu) \left[(b - \mu)^2 + (\mu - a)^2\right] \right]
\]

\[
= \frac{1}{45} \left[ 4 \left( (b - \mu)^2 - (\mu - a)^2 \right)^2 \\
+ 2 (b - \mu)^2 (\mu - a)^2 + (\mu - a) (b - \mu) \left[(b - \mu)^2 + (\mu - a)^2\right] \right]
\]

: \quad = A.

However,
\[
(b - \mu)^2 - (\mu - a)^2 = (b - a)(b + a - 2\mu) = 2(b - a)\left(\frac{b + a}{2} - \mu\right),
\]

\[
(b - \mu)(\mu - a) = \frac{1}{4} (b - a)^2 - \left(\mu - \frac{a + b}{2}\right)^2,
\]

\[
(b - \mu)^2 + (\mu - a)^2 = \frac{1}{2} (b - a)^2 + 2 \left(\mu - \frac{a + b}{2}\right)^2.
\]
Denote \( \delta := b - a \) and \( x = \mu - \frac{a+b}{2} \). Then we get
\[
\begin{align*}
45A &= 4 \left( 2 \delta x \right)^2 + 2 \left( \frac{1}{4} \delta^2 - x^2 \right)^2 + \frac{1}{4} \delta^2 - x^2 \left( \frac{1}{2} \delta + 2x \right) \\
&= 16 \delta^2 x^2 + \left( \frac{1}{4} \delta^2 - x^2 \right) \left[ 2 \left( \frac{1}{4} \delta^2 - x^2 \right) + \frac{1}{2} \delta^2 + 2x^2 \right] \\
&= \delta^2 \left( 15x^2 + \frac{1}{4} \delta^2 \right).
\end{align*}
\]

Then
\[
A = \frac{(b-a)^2}{45} \left[ 15 \left( \mu - \frac{a+b}{2} \right)^2 + \frac{1}{4} (b-a)^2 \right] \\
= (b-a)^2 \left[ \frac{1}{3} \left( \mu - \frac{a+b}{2} \right)^2 + \frac{1}{180} (b-a)^2 \right].
\]

Using the inequality (2.6), we deduce the desired inequality (2.5).

The best inequality we can obtain from (2.5) is that one for which \( \mu = \frac{a+b}{2} \) and therefore, we can state the following corollary.

**Corollary 1.** With the above assumptions and denoting \( \sigma_0(X) := \sigma_{\frac{a+b}{2}}(X) \), we have the inequality:
\[
(2.7) \quad \left| \sigma_0^2(X) - \frac{(b-a)^2}{12} \right| \leq \frac{1}{12 \sqrt{5}} (\phi - \gamma) (b-a)^3.
\]

The following theorem also holds.

**Theorem 5.** Let \( X \) and \( f \) be as above. If
\[
A_\mu(X) := \int_a^b |t - \mu| f(t) \, dt, \quad \mu \in [a,b],
\]
then we have the inequality
\[
(2.8) \quad \left| A_\mu(X) - \frac{1}{b-a} \left[ (b-a)^2 + \left( \mu - \frac{a+b}{2} \right)^2 \right] \right| \\
\leq \frac{1}{2} (\phi - \gamma) (b-a) \left[ \frac{(b-a)^2}{48} + \left( \frac{\mu - \frac{a+b}{2}}{b-a} \right)^2 \left[ \frac{1}{2} (b-a)^2 + \left( \frac{\mu - \frac{a+b}{2}}{2} \right)^2 \right] \right]^{\frac{1}{2}}.
\]

for all \( \mu \in [a,b] \).

**Proof.** If we put \( g(t) = |t - \mu| \) in (1.1), we have
\[
(2.9) \quad \left| \frac{1}{b-a} \int_a^b |t - \mu| f(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b |t - \mu| \, dt \right| \\
\leq \frac{1}{2} (\phi - \gamma) \left[ \frac{1}{b-a} \int_a^b |t - \mu|^2 \, dt - \left( \frac{1}{b-a} \int_a^b |t - \mu| \, dt \right)^2 \right]^{\frac{1}{2}}.
\]
and as
\[
\int_a^b f(t) \, dt = 1,
\]
\[
\frac{1}{b-a} \int_a^b |t - \mu| \, dt = \frac{1}{b-a} \left[ \int_a^\mu (\mu - t) \, dt + \int_\mu^b (t - \mu) \, dt \right]
\]
\[
= \frac{1}{b-a} \left[ \frac{(b-\mu)^2 + (\mu - a)^2}{2} \right]
\]
\[
= \frac{1}{b-a} \left[ \frac{(b-a)^2}{4} + \left( \mu - \frac{a+b}{2} \right)^2 \right].
\]
\[
\frac{1}{b-a} \int_a^b (t - \mu)^2 \, dt = \frac{(b-\mu)^3 + (\mu - a)^3}{3(b-a)}
\]
\[
= \frac{(b-a)^2}{12} \left( \mu - \frac{a+b}{2} \right)^2,
\]
\[
\frac{1}{b-a} \int_a^b |t - \mu|^2 \, dt - \left( \frac{1}{b-a} \int_a^b |t - \mu| \, dt \right)^2
\]
\[
= \frac{(b-a)^2}{12} + \left( \mu - \frac{a+b}{2} \right)^2 - \left[ \frac{(b-a)}{4} + \frac{1}{b-a} \left( \mu - \frac{a+b}{2} \right)^2 \right]^2
\]
\[
= \frac{(b-a)^2}{48} + \frac{1}{2} \left( \mu - \frac{a+b}{2} \right)^2 - \frac{1}{(b-a)^2} \left( \mu - \frac{a+b}{2} \right)^4
\]
\[
= \frac{(b-a)^2}{48} + \left( \frac{\mu - \frac{a+b}{2}}{b-a} \right)^2 \left[ \frac{1}{2} (b-a)^2 - \left( \mu - \frac{a+b}{2} \right)^2 \right].
\]

Finally, using (2.9) we deduce the desired inequality.

The best inequality we can get from (2.8) is embodied in the following corollary.

**Corollary 2.** The best inequality we can get from (2.8) is for \( \mu = \mu_0 := \frac{a+b}{2} \), obtaining

\[
|A_{\mu_0}(X) - \frac{b-a}{4}| \leq \frac{1}{8\sqrt{3}} (\phi - \gamma)(b-a)^2.
\]

**Proof.** Consider the mapping \( g(\mu) := \frac{(b-a)^2}{48} + \frac{1}{2} \left( \mu - \frac{a+b}{2} \right)^2 - \frac{1}{(b-a)^2} \left( \mu - \frac{a+b}{2} \right)^4. \)

We have
\[
\frac{dg(\mu)}{d\mu} = \left( \mu - \frac{a+b}{2} \right) - \frac{4}{(b-a)^2} \left( \mu - \frac{a+b}{2} \right)^3
\]
\[
= \left( \mu - \frac{a+b}{2} \right) \left[ 1 - \frac{4}{(b-a)^2} \left( \mu - \frac{a+b}{2} \right)^2 \right].
\]
We observe that \( \frac{dg(\mu)}{d\mu} = 0 \) if \( \mu = a \) or \( \mu = \frac{a+b}{2} \) or \( \mu = b \) and as
\[
\frac{dg(\mu)}{d\mu} < 0 \text{ for } \mu \in \left( a, \frac{a+b}{2} \right) \text{ and } \frac{dg(\mu)}{d\mu} > 0 \text{ for } \mu \in \left( \frac{a+b}{2}, \mu \right),
\]
we deduce that \( \mu = \frac{a+b}{2} \) is the point realizing the global minimum on \((a, b)\) and as \( g(\mu_0) = \frac{(b-a)^2}{48} \), the inequality (2.10) is indeed the best inequality we can get from (2.8).

Another inequality which can be useful for obtaining different inequalities for dispersion is the following weighted Grüss type result (see for example [8] or [6]).

**Lemma 1.** Let \( g, p : [a, b] \to \mathbb{R} \) be measurable functions and such that \( \alpha \leq g \leq \beta \) a.e., \( p \geq 0 \) a.e. on \([a, b]\) and \( \int_a^b p(x) \, dx > 0 \).

Then
\[
0 \leq \frac{\int_a^b p(x) g^2(x) \, dx}{\int_a^b p(x) \, dx} - \left( \frac{\int_a^b p(x) g(x) \, dx}{\int_a^b p(x) \, dx} \right)^2 \leq \frac{1}{4} (\beta - \alpha)^2,
\]
provided that all the integrals in (2.11) exist and are finite.

Using the above lemma we shall be able to prove the following result for dispersion.

**Theorem 6.** Let \( X \) be a random variable whose probability density function \( f \) is defined on the finite interval \([a, b]\) and \( \sigma(X) < \infty \). Then we have the inequality
\[
0 \leq \sigma^2_\mu(X) - (E(X) - \mu)^2 \leq \frac{1}{4} (b-a)^2
\]
for all \( \mu \in [a, b] \), or, equivalently,
\[
0 \leq \sigma(X) \leq \frac{1}{2} (b-a).
\]

**Proof.** Let us choose in (2.11), \( g(x) = x - \mu \), \( p(x) = f(x) \). Then, obviously, \( \sup_{x \in [a, b]} g(x) = b - \mu \), \( \inf_{x \in [a, b]} g(x) = a - \mu \), \( \int_a^b f(x) \, dx = 1 \), and then by (2.11), we get
\[
0 \leq \int_a^b (x - \mu)^2 f(x) \, dx - \left( \frac{\int_a^b (x - \mu) f(x) \, dx}{\int_a^b f(x) \, dx} \right)^2 \leq \frac{1}{4} (b-a)^2
\]
and the inequality (2.12) is proved.

The following inequality connecting \( \sigma_\mu(X) \) and \( A_\mu(X) \) also holds.

**Theorem 7.** Let \( X \) be as in Theorem 6 and assume that \( \sigma_\mu(X), A_\mu(X) < \infty \) for all \( \mu \in [a, b] \). Then we have the inequality
\[
0 \leq \sigma^2_\mu(X) - A^2_\mu(X) \leq \frac{1}{2} \left| \mu - \frac{a+b}{2} \right|
\]
for all \( \mu \in [a, b] \).
Proof. Choose in Lemma 1, \( p(x) = f(x) \), \( g(x) = |x - \mu| \), \( \mu \in [a, b] \). Then

\[
\beta = \sup_{x \in [a, b]} g(x) = \max \{ \mu - a, b - \mu \} = \frac{b - a + |\mu - a - b + \mu|}{2},
\]

\[
\alpha = \inf_{x \in [a, b]} g(x) = \min \{ \mu - a, b - \mu \} = \frac{b - a - |\mu - a - b + \mu|}{2},
\]

which gives us

\[
\beta - \alpha = 2 \left| \mu - \frac{a + b}{2} \right|.
\]

Applying (2.11), we deduce (2.14).

3. Some Inequalities for the Cumulative Distribution Function

The following theorem contains an inequality which connects the expectation \( E(X) \), the cumulative distribution function \( F(X) := \int_a^x f(t) \, dt \) and the bounds \( \gamma \) and \( \phi \) of the probability density function \( f : [a, b] \to \mathbb{R} \).

**Theorem 8.** Let \( X, f, E(X), F(\cdot) \) and \( \gamma, \phi \) be as above. Then we have the inequality:

\[
|E(X) + (b - a) F(x) - x - \frac{b - a}{2}| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b - a)^2,
\]

for all \( x \in [a, b] \).

**Proof.** We have the following equality established by Barnett and Dragomir in [3]

\[
(b - a) F(x) + E(X) - b = \int_a^b p(x, t) \, dF(t) = \int_a^b p(x, t) f(t) \, dt,
\]

where

\[
p(x, t) := \begin{cases} t - a & \text{if } a \leq t \leq x \\ t - b & \text{if } a \leq x < t \leq b \end{cases}.
\]

Now, if we apply the inequality (1.1) for \( g(t) = p(x, t) \), we get

\[
\left| \frac{1}{b - a} \int_a^b p(x, t) f(t) \, dt - \frac{1}{b - a} \int_a^b p(x, t) \, dt \cdot \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{1}{2} (\phi - \gamma) \left[ \frac{1}{b - a} \int_a^b p^2(x, t) \, dt - \left( \frac{1}{b - a} \int_a^b p(x, t) \, dt \right)^2 \right]^{\frac{1}{2}}.
\]

Now, we observe that

\[
\frac{1}{b - a} \int_a^b p(x, t) \, dt = x - \frac{a + b}{2},
\]

\[
\int_a^b f(t) \, dt = 1,
\]
and

\[
D = \frac{1}{b-a} \int_a^b p^2(x, t) \, dt - \left( \frac{1}{b-a} \int_a^b p(x, t) \, dt \right)^2
\]

\[
= \frac{1}{b-a} \left[ \frac{(b-x)^3 + (x-a)^3}{3} \right] - \left( x - \frac{a+b}{2} \right)^2
\]

\[
= \frac{(b-x)^2 - (b-x)(x-a) + (x-a)^2}{3} - \left( x - \frac{a+b}{2} \right)^2 .
\]

As a simple calculation shows that

\[
(b-x)^2 - (b-x)(x-a) + (x-a)^2
\]

\[
= 3 \left( x - \frac{a+b}{2} \right)^2 + \frac{1}{4} (b-a)^2 ,
\]

then we get

\[
D = \frac{1}{12} (b-a)^2.
\]

Using (3.3), we deduce (3.1). □

**Remark 1.** If in (3.1) we choose either \(x = a\) or \(x = b\), we get

\[
\left| \frac{E(X) - \frac{a+b}{2}}{b-a} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2,
\]

which is the inequality (2.1).

**Remark 2.** If in (3.1) we choose \(x = \frac{a+b}{2}\), then we get the inequality

\[
(3.4) \quad \left| \frac{E(X) + (b-a) \Pr \left( X \leq \frac{a+b}{2} \right) - b}{b-a} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2.
\]

The following theorem also holds.

**Theorem 9.** Let \(X, f, \gamma, \phi\) and \(F(\cdot)\) be as above. Then we have the inequality:

\[
(3.5) \quad \left| E(X) + \frac{b-a}{2} F(x) - \frac{b+x}{2} \right|
\]

\[
\leq \frac{1}{2\sqrt{3}} (\phi - \gamma) \left[ \frac{1}{4} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right]
\]

\[
\leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2 ,
\]

for all \(x \in [a,b]\).

**Proof.** We use the following identity proved by Barnett and Dragomir in [3]

\[
(3.6) \quad (b-a) F(x) + E(X) - b = \int_a^x (t-a) \, dF(t) + \int_x^b (t-b) \, dF(t)
\]

\[
= \int_a^x (t-a) f(t) \, dt + \int_x^b (t-b) f(t) \, dt
\]

for all \(x \in [a,b]\).
Applying the pre-Grüss inequality (1.1), we get for \( x \in [a, b] \)

\[
\left| \frac{1}{x-a} \int_a^x (t-a) f(t) \, dt - \frac{1}{x-a} \int_a^x (t-a) \, dt \cdot \frac{1}{x-a} \int_a^x f(t) \, dt \right| 
\leq \frac{1}{2} (\phi - \gamma) \left[ \frac{1}{x-a} \int_a^x (t-a)^2 \, dt - \left( \frac{1}{x-a} \int_a^x (t-a) \, dt \right)^2 \right]^\frac{1}{2} 
= \frac{1}{4\sqrt{3}} (\phi - \gamma) (x-a)
\]

and, similarly, we have

\[
\left| \frac{1}{b-x} \int_x^b (t-b) f(t) \, dt - \frac{1}{b-x} \int_x^b (t-b) \, dt \cdot \frac{1}{b-x} \int_x^b f(t) \, dt \right| 
\leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-x), \quad x \in (a, b).
\]

From (3.7) and (3.8) we can write

\[
\left| \int_a^x (t-a) f(t) \, dt - \frac{x-a}{2} F(x) \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (x-a)^2
\]

and

\[
\left| \int_x^b (t-b) f(t) \, dt + \frac{b-x}{2} (1-F(x)) \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-x)^2,
\]

for all \( x \in [a, b] \).

Summing (3.9) and (3.10) and using the triangle inequality, we deduce that

\[
\left| \int_a^x (t-a) f(t) \, dt + \int_x^b (t-b) f(t) \, dt - \frac{b-a}{2} F(x) + \frac{b-x}{2} \right| 
\leq \frac{1}{4\sqrt{3}} (\phi - \gamma) \left[ (x-a)^2 + (b-x)^2 \right] 
= \frac{1}{2\sqrt{3}} (\phi - \gamma) \left[ \frac{1}{4} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right].
\]

Using the identity (3.6), we deduce the desired inequality (3.5).

**Remark 3.** If we choose in (3.5), either \( x = a \) or \( x = b \), we get the inequality

\[
\left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2
\]

and thus recapture (2.1).

**Remark 4.** If we choose in (3.5), \( x = \frac{a+b}{2} \), then we get

\[
\left| E(X) + \left( \frac{b-a}{2} \right) \Pr \left( X \leq \frac{a+b}{2} \right) - \frac{a+3b}{4} \right| \leq \frac{1}{8\sqrt{3}} (\phi - \gamma) (b-a)^2,
\]

which is the best inequality that can be obtained.
REFERENCES


School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC 8001, Victoria, Australia

E-mail address: neil@matilda.vu.edu.au

E-mail address: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au