Weighted inequalities in triangle geometry

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Abstract

The paper contains two parts. In the first we point some applications of a weighted inequality and in the second part the equality conditions are obtained.

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In [1] it is proved the following:

Proposition 1 Let \(m, n, p\) be real numbers such that \(m + n > 0, n + p > 0, p + m > 0, mn + np + pm > 0\). Then in any triangle \(ABC\) the following inequality holds:

\[
ma^2 + nb^2 + pc^2 \geq 4\sqrt{mn + np + pm}S
\]

with standard notations.

Some applications are given in the cited paper:

(2) \(a^2 + b^2 + c^2 \geq 4\sqrt{3}S\) for \(m = n = p\)

(3) \(a^4 + b^4 + c^4 \geq 4\sqrt{a^2b^2 + b^2c^2 + c^2a^2}S\) for \(m = a^2, n = b^2, p = c^2\)

and therefore:

(3') \(a^4 + b^4 + c^4 \geq 16S^2\)

(4) \(9a^2 + 5b^2 - 3c^2 \geq 4\sqrt{3}S\) for \(m = 9, n = 5, c = -3\)

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27a^2 + 27b^2 - 13c^2 \geq 12\sqrt{3}S \quad \text{for } m = n = 27, p = -13

3a^2 - b^2 + 15c^2 \geq 12\sqrt{3}S \quad \text{for } m = 3, n = -1, p = 15.

Let us point some other applications of (1):

I) the problem O:553 from Gazeta Matematică, no. 5-6(1988), p. 260 (without author):

3a^2 + 3b^2 - c^2 \geq 4\sqrt{3}S \quad \text{for } m = n = 3, c = -1.

II) the problem E 3150 proposed by George A. Tsintsifas in American Mathematical Monthly, vol. 93(1986), p. 400:

\frac{m}{n+p}a^2 + \frac{n}{p+m}b^2 + \frac{p}{m+n}c^2 \geq 2\sqrt{3}S

where \( m, n, p \) are positive real numbers. From (1) we have:

\frac{m}{n+p}a^2 + \frac{n}{p+m}b^2 + \frac{p}{m+n}c^2 \geq 4\sqrt{\frac{mn}{(n+p)(p+m)} + \frac{np}{(p+m)(m+n)} + \frac{pm}{(n+p)(m+n)}} S.

Therefore it must be proved that:

\frac{mn}{(n+p)(p+m)} + \frac{np}{(p+m)(m+n)} + \frac{pm}{(m+n)(n+p)} \geq \frac{3}{4}

or, equivalent:

mn (m+n) + np (n+p) + pm (p+m) \geq \frac{3}{4} (m+n)(n+p)(p+m).

But the left-hand side of (9) is \( m^2n + mn^2 + n^2p + np^2 + p^2m + pm^2 \) and the right-hand side of (9) is \( \frac{3}{4} (2mnp + m^2n + mn^2 + p^2m + pm^2 + n^2p + np^2) \).

Then (9) is equivalent with \( m^2n + mn^2 + n^2p + np^2 + p^2m + pm^2 \geq 6mnp \) which is consequence of AM-GM inequality. For others three solutions of (8) see the cited journal, vol. 95(1988), p. 658-659.

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A natural question with respect to (1) is: when equality holds? The aim of this paper is to give the answer. More precisely, we will show:

**Proposition 2** In (1) there is equality if and only if:

\[
\frac{a}{\sqrt{n+p}} = \frac{b}{\sqrt{p+m}} = \frac{c}{\sqrt{m+n}}.
\]

**Proof** From generalized Pitagora’s theorem for \( c \) and expression \( S = \frac{1}{2}ab\sin C \) it results that in (1) is equality if and only if:

\[
ma^2 + nb^2 + p\left(a^2 + b^2 - 2ab\cos C\right) = 2\sqrt{mn + \ldots ab\sin C} \iff
\]

\[
(m + p)\frac{a}{b} + (n + p)\frac{b}{a} = 2\left(p\cos C + \sqrt{mn + \ldots \sin C}\right).
\]

From AM-GM inequality we have

\[
(m + n)\frac{a}{b} + (n + p)\frac{b}{a} \geq 2\sqrt{(m + n)(n + p)}
\]

and from Cauchy-Buniakowski-Schwartz inequality we get

\[
\sqrt{(m + n)(n + p)} \geq \left(p\cos C + \sqrt{mn + \ldots \sin C}\right).
\]

From last three relations it results that in (1) is equality if and only if

\[
(m + n)\frac{a}{b} = (n + p)\frac{b}{a}
\]

and \( \frac{\cos C}{p} = \frac{\sin C}{\sqrt{mn+\ldots}} \) which means:

\[
\text{(15}_1)\quad \frac{a}{\sqrt{n+p}} = \frac{b}{\sqrt{m+p}} \quad \text{denote} \quad k
\]

\[
\text{(15}_2)\quad \frac{\cos C}{p} = \frac{\sin C}{\sqrt{mn+\ldots}} = \frac{1}{\sqrt{(m+n)(n+p)}}.
\]

Replacing \( \cos C = \frac{p}{\sqrt{(m+n)(n+p)}} \) from (15_2) and \( b \) from (15_1) in generalized Pitagora’s theorem we have

\[
c^2 = a^2(k^2 + 1) - 2a^2k\frac{p}{\sqrt{(m+n)(n+p)}}.\]

But \( k = \)
\[
\sqrt{\frac{m+p}{n+p}} \quad \text{and then} \quad \left(\frac{e}{a}\right)^2 = 1 + \frac{m+n}{m+p} - 2\sqrt{\frac{m+n}{m+p} \left(\frac{n+p}{m+n}\right)} = \frac{m+n}{n+p}. \quad \text{Therefore} \quad \frac{a}{\sqrt{m+n}} = \frac{e}{\sqrt{m+n}} \quad \text{and this last relation with (152) gives the conclusion.} \quad \square
\]

Consequences:

\[
a^2 + b^2 + c^2 = 4\sqrt{3}S \iff a = b = c
\]

\[
\frac{a^4 + b^4 + c^4}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}} = 4S \iff \frac{a^2}{b^2 + c^2} = \frac{b^2}{c^2 + a^2} = \frac{c^2}{a^2 + b^2} \iff a = b = c
\]

\[
a^4 + b^4 + c^4 = 16S^2 \iff a = b = c
\]

\[
9a^2 + 5b^2 - 3c^2 = 4\sqrt{3}S \iff \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{6}} = \frac{c}{\sqrt{14}} \iff a = \frac{b}{\sqrt{3}} = \frac{c}{\sqrt{7}}
\]

\[
27a^2 + 27b^2 - 13c^2 = 12\sqrt{3}S \iff \frac{a}{\sqrt{14}} = \frac{b}{\sqrt{14}} = \frac{c}{\sqrt{54}} \iff \frac{a}{\sqrt{7}} = \frac{b}{\sqrt{7}} = \frac{c}{\sqrt{27}}
\]

\[
3a^2 - b^2 + 15c^2 = 12\sqrt{3}S \iff \frac{a}{\sqrt{14}} = \frac{b}{\sqrt{18}} = \frac{c}{\sqrt{2}} \iff \frac{a}{\sqrt{3}} = \frac{b}{\sqrt{3}} = c
\]

\[
3a^2 + 3b^2 - c^2 = 4\sqrt{3}S \iff \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{2}} = \frac{c}{\sqrt{6}} \iff a = b = \frac{c}{\sqrt{3}}.
\]

References
