BETTER BOUNDS IN SOME OSTROWSKI-GRÜSS TYPE INEQUALITIES

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Abstract. The main aim of this note is to point out some improvements of the recent results in [1].

1. INTRODUCTION

As in [1], let \( \{ P_n \}_{n \in \mathbb{N}} \) and \( \{ Q_n \}_{n \in \mathbb{N}} \) be two sequences of harmonic polynomials, that is, polynomials satisfying
\[
P_n'(t) = P_{n-1}(t), \ P_0(t) = 1, \ t \in \mathbb{R},
\]
(1.1)
\[
Q_n'(t) = Q_{n-1}(t), \ Q_0(t) = 1, \ t \in \mathbb{R}.
\]
(1.2)

In [1], the authors proved the following result.

Lemma 1. Let \( \{ P_n \}_{n \in \mathbb{N}} \) and \( \{ Q_n \}_{n \in \mathbb{N}} \) be two harmonic polynomials. Set
\[
S_n(t, x) := \begin{cases} P_n(t), & t \in [a, x] \\ Q_n(t), & t \in (x, b]. \end{cases}
\]
(1.3)

Then we have the equality
\[
\int_a^b f(t) \, dt = \sum_{k=1}^{n} (-1)^{k+1} \left[ Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ -P_k(a) f^{(k-1)}(a) \right] + (-1)^n \int_a^b S_n(t, x) f^{(n)}(t) \, dt,
\]
(1.4)

provided that \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n-1)} \) is absolutely continuous on \([a, b]\).

Using the following “pre-Grüss” inequality
\[
|T(f, g)| \leq \frac{1}{2} \sqrt{T(f, f)} (\Gamma - \gamma),
\]
(1.5)

where
\[
T(f, g) := \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{(b-a)^2} \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx
\]
is the Chebychev functional and \( f, g \) are such that the previous integrals exist and \( \gamma \leq g(x) \leq \Gamma \) for a.e. \( x \in [a, b] \), the authors of [1] proved basically the
Then for all \( x \) for the mappings \( f \) the Euclidean norm of \( f^{(n)} \) has been obtained in [1] as well.

**Theorem 1.** Assume that \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n)} \) is integrable and \( \gamma_n \leq f^{(n)}(t) \leq \Gamma_n \) for all \( t \in [a, b] \). Put

\[
U_n(x) := \frac{Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)}{b - a}.
\]

Then for all \( x \in [a, b] \), we have the inequality

\[
\left| \int_a^b f(t) \, dt - \sum_{k=1}^n (-1)^{k+1} \left[ Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a) \right] \right| \\
- (-1)^n U_n(x) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
\leq \frac{1}{2} K (\Gamma_n - \gamma_n) (b - a),
\]

where

\[
K := \left\{ \frac{1}{b - a} \int_a^x P_n^2(t) \, dt + \int_x^b Q_n^2(t) \, dt - [U_n(x)]^2 \right\}^{1/2}.
\]

A number of particular cases by choosing some appropriate harmonic polynomials have been obtained in [1] as well.

The main aim of this note is to point out a sharper bound in (1.6) in terms of the Euclidean norm of \( f^{(n)} \), which is valid also for a larger class of mappings, i.e., for the mappings \( f \) for which \( f^{(n)} \) is unbounded on \((a, b)\) but \( f^{(n)} \in L_2[a, b] \). Some particular cases as in [1], are also considered.

2. The Results

The following theorem holds.

**Theorem 2.** Assume that the mapping \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n-1)} \) is absolutely continuous on \([a, b]\) and \( f^{(n)} \in L_2[a, b] \) \( (n \geq 1) \). If we denote

\[
\left[ f^{(n-1)}; a, b \right] := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a},
\]

then we have the inequality

\[
\left| \int_a^b f(t) \, dt - \sum_{k=1}^n (-1)^{k+1} \left[ Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a) \right] \right| \\
- (-1)^n \left[ Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a) \right] \left[ f^{(n-1)}; a, b \right] \\
\leq K (b - a) \left( \frac{1}{b - a} \left\| f^{(n)} \right\|_2^2 - \left( \left[ f^{(n)}; a, b \right] \right)^2 \right)^{1/2} \\
\left( \leq \frac{1}{2} K (b - a) (\Gamma_n - \gamma_n) \text{ if } f^{(n)} \in L_\infty[a, b] \right)
\]
for all $x \in [a, b]$, where $K$ is defined in Theorem 1 (and $\gamma_n$, $\Gamma_n$ are as in the Introduction, i.e., $\gamma_n \leq f^{(n)}(t) \leq \Gamma_n$ for all $t \in [a, b]$).

Proof. Recall Korkine’s identity

$$T(h, g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (h(t) - h(s)) (g(t) - g(s)) \, dt ds,$$

where $T(\cdot, \cdot)$ is the Chebychev functional defined in the Introduction. Using (2.2) and the identity (1.4), we may write (see also [1])

$$\int_a^b f(t) \, dt - \sum_{k=1}^n (-1)^{k+1} \left[ Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a) \right] - (-1)^n \left[ Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a) \right] \left[ f^{(n-1)}; a, b \right] = (b-a) T\left(S_n(\cdot, x), f^{(n)} \right) = \frac{1}{2(b-a)} \int_a^b \int_a^b \left( S_n(t, x) - S_n(s, x) \right) \left( f^{(n)}(t) - f^{(n)}(s) \right) \, dt ds,$$

which is an identity that is interesting in itself as well.

Using the Cauchy-Buniakowski-Schwartz integral inequality for double integrals, we may write

$$\left| \int_a^b \int_a^b \left( S_n(t, x) - S_n(s, x) \right) \left( f^{(n)}(t) - f^{(n)}(s) \right) \, dt ds \right| \leq \left( \int_a^b \int_a^b \left( S_n(t, x) - S_n(s, x) \right)^2 \, dt ds \right)^{\frac{1}{2}} \times \left( \int_a^b \int_a^b \left( f^{(n)}(t) - f^{(n)}(s) \right)^2 \, dt ds \right)^{\frac{1}{2}} = \left[ 2(b-a)^2 T\left(S_n(\cdot, x), S_n(\cdot, x) \right) \right]^{\frac{1}{2}} \left[ 2(b-a)^2 T\left(f^{(n)}, f^{(n)} \right) \right]^{\frac{1}{2}} = 2(b-a)^2 K \left[ \frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left( \left[ f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}}.$$

Now, taking the modulus in (2.3) and using the estimate (2.4), we may deduce the first inequality in (2.1).

If we assume that $f^{(n)} \in L_\infty [a, b] (\subset L_2 [a, b]$ and the inclusion is strict), then applying the Grüss inequality

$$0 \leq \frac{1}{b-a} \int_a^b \left| f^{(n)}(t) \right|^2 \, dt - \left( \frac{1}{b-a} \int_a^b f^{(n)}(t) \, dt \right)^2 \leq \frac{1}{4} (\Gamma_n - \gamma_n)^2,$$

we deduce the last part in (2.1). $\blacksquare$

We are now able to improve the Corollaries 1-3 and Theorem 2 from [1] as follows.
Under the assumptions of Theorem 2, we have

Corollary 1. Under the assumptions of Theorem 2, we have

\begin{equation}
\int_a^b f(t) \, dt - \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left[ (b - B)^k \, f^{(k-1)}(a) \right] \\
+ \left[ (x - A)^k - (x - B)^k \right] f^{(k-1)}(x) - (a - A)^k \, f^{(k-1)}(a) \\
- \frac{(-1)^n}{(n+1)!} \left[ (b - B)^{n+1} - (x - B)^{n+1} \\
+ (x - A)^{n+1} - (a - A)^{n+1} \right] \left[ f^{(n-1)}; a, b \right] \\
\leq \left( b - a \right) K_1 \left[ \frac{1}{b - a} \left\| f^{(n)} \right\|_2^2 - \left( \left[ f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}},
\end{equation}

where \( K_1 \) is, as defined in [1]

\begin{equation}
K_1 := \frac{1}{n!} \left[ \frac{(x - A)^{2n+1} - (a - A)^{2n+1} + (b - B)^{2n+1} - (x - B)^{2n+1}}{(2n+1)(b - a)} \\
- \frac{\left( (b - B)^{n+1} - (x - B)^{n+1} + (x - a)^{n+1} - (a - A)^{n+1} \right)^2}{(n+1)(b - a)} \right]^{\frac{1}{2}}
\end{equation}

and \( x \in [a, b], A, B \in \mathbb{R} \).

The proof follows from Theorem 2 with the polynomial choices of \( P_n(t) = \frac{(t-a)^n}{n!} \) and \( Q_n(t) = \frac{(t-B)^n}{n!} \) (see also [1, Corollary 1]).

Corollary 2. Under the assumptions of Theorem 2, we have

\begin{equation}
\int_a^b f(t) \, dt - \sum_{k=1}^{n} \frac{(-1)^{k+1}(b - a)^k}{k! (p + q)^k} \left[ q^k \left( f^{(k-1)}(b) - (-1)^k \, f^{(k-1)}(a) \right) \right] \\
+ \left( \frac{p - q}{2} \right)^k \left[ 1 - (-1)^k \right] f^{(k-1)} \left( \frac{a + b}{2} \right) \\
- \frac{(-1)^n(b - a)^{n+1} \left( 1 + (-1)^n \right)}{(n+1)! (p + q)^{n+1}} \left[ 2^{n+1} + \left( \frac{p - q}{2} \right)^{n+1} \right] \left[ f^{(n-1)}; a, b \right] \\
\leq \left( b - a \right) K_2 \left[ \frac{1}{b - a} \left\| f^{(n)} \right\|_2^2 - \left( \left[ f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}},
\end{equation}

for \( p, q \in \mathbb{R} \) (\( p, q > 0 \)), where

\begin{equation}
K_2 := \frac{(b - a)^n}{n! (p + q)^n} \left[ \frac{2 \left( q^{2n+1} + \left( \frac{p-q}{2} \right)^{2n+1} \right)}{(p + q)(2n+1)} - 2 \left[ 1 + (-1)^n \right] \left( \frac{q^{n+1}}{(n+1)^2 (p + q)^2} \right)^{\frac{1}{2}} \right].
\end{equation}

The proof follows by Corollary 1 with \( A = \frac{2a+qb}{p+q}, x = \frac{a+b}{2} \) and \( B = \frac{2a+qb}{p+q} \) where \( p, q \in \mathbb{R} \) and \( p + q > 0 \) (see also [1, Corollary 2]).

For \( x = b \), Theorem 2 gives the following.
Theorem 3. With the assumptions in Theorem 2, we have:

\[
\left\| \int_a^b f(t) \, dt - \sum_{k=1}^{n} (-1)^{k+1} \left[ P_k(b) f^{(k-1)}(b) - P_k(a) f^{(k-1)}(a) \right] \right\|_2 \\
- (-1)^n [P_{n+1}(b) - P_{n+1}(a)] \left[ f^{(n-1)}; a, b \right] \\
\leq K_3 (b-a) \left[ \frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left( \left[ f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}},
\]

where \( K_3 \) is given by (see [1, Theorem 2])

\[
K_3 := \left[ \frac{1}{b-a} \int_a^x P_n^2(t) \, dt - \left( \frac{P_{n+1}(b) - P_{n+1}(a)}{b-a} \right)^2 \right]^{\frac{1}{2}}.
\]

The choice \( P_n(t) = \frac{1}{n!} \left(t - \frac{a+b}{2}\right)^n \) provides the following corollary.

Corollary 3. Under the assumptions of Theorem 2, we have:

\[
\left\| \int_a^b f(t) \, dt - \sum_{k=1}^{n} (-1)^{k+1} \left( b-a \right)^k \left[ f^{(k-1)}(b) - (-1)^k f^{(k-1)}(a) \right] \right\|_2 \\
- \frac{(-1)^n (1 + (-1)^n)}{2n+1 (n+1)!} \left( b-a \right)^{n+1} \left[ f^{(n-1)}; a, b \right] \\
\leq K_4 (b-a) \left[ \frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left( \left[ f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}},
\]

where \( K_4 \) is given by (see [1, Corollary 3])

\[
K_4 := \frac{(b-a)^n}{n! 2^n} \left[ \frac{1}{2n+1} - \frac{(1 + (-1)^n)^2}{(n+1)^2} \right]^{\frac{1}{2}}.
\]

Remark 1. All the other results from Sections 4 and 5 can be improved accordingly. For example, if we assume that the derivative \( f^{(n)} \in L_2[a, b] \) \((n \in \{1, 2, 3, 4\})\), then we have the Simpson’s inequality \((n \in \{1, 2, 3\})\)

\[
\left| \int_a^b f(t) \, dt - \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \\
\leq \tilde{c}_n (b-a)^n \sigma \left( f^{(n)}; a, b \right)
\]

where

\[
\tilde{c}_1 = \frac{1}{6}, \quad \tilde{c}_2 = \frac{1}{12\sqrt{30}}, \quad \tilde{c}_3 = \frac{1}{48\sqrt{105}}
\]

and

\[
\sigma \left( f^{(n)}; a, b \right) := \left[ \frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left( \left[ f^{(n)}; a, b \right] \right)^2 \right]^{\frac{1}{2}}, \quad n \in \{1, 2, 3, 4\}.
\]
For \( n = 4 \), we have the perturbed Simpson’s inequality:

\[
\left| \int_a^b f(t) \, dt - \frac{b - a}{6} \left[ f(a) + 4f\left(\frac{a + b}{2}\right) + f(b) \right] + \frac{(b - a)^5}{2880} \left[ f^{(3)}; a, b \right] \right| \leq \frac{1}{2880} \sqrt{\frac{11}{14}} (b - a)^4 \sigma \left( f^{(4)}; a, b \right).
\]

**References**


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