AN IMPROVEMENT OF THE REMAINDER ESTIMATE IN THE GENERALISED TAYLOR FORMULA

S.S. DRAGOMIR

Abstract. In this note we point out an estimate for the remainder in the generalised Taylor formula which improves the recent result by Matić, Pečarić and Ujević [2].

1. Introduction

In the recent paper [2], M. Matić, J. E. Pečarić and N. Ujević proved the following generalised Taylor formula.

Theorem 1. *Let* $\{P_n\}_{n\in\mathbb{N}}$ *be a harmonic sequence of polynomials, that is,* $P_n'(t) = P_{n-1}(t)$ *for* $n \geq 1$, $n \in \mathbb{N}$, $P_0(t) = 1$, $t \in \mathbb{R}$. *Further, let* $I \subset \mathbb{R}$ *be a closed interval and* $a \in I$. *If* $f : I \to \mathbb{R}$ *is a function such that for some* $n \in \mathbb{N}$, $f^{(n)}$ *is absolutely continuous, then*

$$f(x) = \tilde{T}_n(f; a, x) + \tilde{R}_n(f; a, x), \quad x \in I, \tag{1.1}$$

where

$$\tilde{T}_n(f; a, x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] \tag{1.2}$$

and

$$\tilde{R}_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\, dt. \tag{1.3}$$

For some particular instances of harmonic sequences, they obtained the following Taylor-like expansions:

$$f(x) = T_n^{(M)}(f; a, x) + R_n^{(M)}(f; a, x), \quad x \in I, \tag{1.4}$$

where

$$T_n^{(M)}(f; a, x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!} \left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x) \right], \tag{1.5}$$

$$R_n^{(M)}(f; a, x) = \frac{(-1)^n}{n!} \int_a^x \left(t - \frac{a+x}{2} \right)^n f^{(n+1)}(t)\, dt; \tag{1.6}$$

and

$$f(x) = T_n^{(B)}(f; a, x) + R_n^{(B)}(f; a, x), \quad x \in I, \tag{1.7}$$

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where

$$T_n^{(B)}(f;a,x) = f(a) + \frac{x-a}{2}\left[f'(x) + f'(a)\right] - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(x-a)^{2k}}{(2k)!} B_{2k}\left[f^{(2k)}(x) - f^{(2k)}(a)\right], \tag{1.8}$$

and $[r]$ is the integer part of r. Here, B_{2k} are the Bernoulli numbers, and

$$R_n^{(B)}(f;a,x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x B_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)\,dt, \tag{1.9}$$

where $B_n(\cdot)$ are the Bernoulli polynomials, respectively.

In addition, they proved that

$$f(x) = T_n^{(E)}(f;a,x) + R_n^{(E)}(f;a,x), \quad x \in I, \tag{1.10}$$

where

$$T_n^{(E)}(f;a,x) = f(a) + 2\sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(x-a)^{2k-1}\left(4^k-1\right)}{(2k)!} B_{2k}\left[f^{(2k-1)}(x) + f^{(2k-1)}(a)\right] \tag{1.11}$$

and

$$R_n^{(E)}(f;a,x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x E_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)\,dt, \tag{1.12}$$

where $E_n(\cdot)$ are the Euler polynomials.

In [1], S.S. Dragomir was the first author to introduce the perturbed Taylor formula

$$f(x) = T_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!}\left[f^{(n)};a,x\right] + G_n(f;a,x), \tag{1.13}$$

where

$$T_n(f;a,x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{1.14}$$

and

$$\left[f^{(n)};a,x\right] := \frac{f^{(k)}(x) - f^{(k)}(a)}{x-a};$$

and had the idea to estimate the remainder $G_n(f;a,x)$ by using Grüss and Chebychev type inequalities.

In [2], the authors generalised and improved the results from [1]. We mention here the following result obtained via a pre-Grüss inequality (see [2, Theorem 3]).

Theorem 2. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose $f : I \to \mathbb{R}$ is as in Theorem 1. Then for all $x \in I$ we have the perturbed generalised Taylor formula:*

$$f(x) = \tilde{T}_n(f;a,x) + (-1)^n\left[P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n)};a,x\right] + \tilde{G}_n(f;a,x). \tag{1.15}$$

For $x \geq a$, the remainder $\tilde{G}(f;a,x)$ satisfies the estimate

$$\left|\tilde{G}_n(f;a,x)\right| \leq \frac{x-a}{2}\sqrt{T(P_n,P_n)}\left[\Gamma(x)-\gamma(x)\right], \tag{1.16}$$

provided that $f^{(n+1)}$ is bounded and

$$\Gamma(x) := \sup_{t\in[a,x]} f^{(n+1)}(t) < \infty, \quad \gamma(x) := \inf_{t\in[a,x]} f^{(n+1)}(t) > -\infty, \tag{1.17}$$

where $T(\cdot,\cdot)$ is the Chebychev functional on the interval $[a,x]$, that is, we recall

$$T(g,h) := \frac{1}{x-a}\int_a^x g(t)h(t)\,dt - \frac{1}{x-a}\int_a^x g(t)\,dt \cdot \frac{1}{x-a}\int_a^x h(t)\,dt. \tag{1.18}$$

The main aim of the present note is to improve the inequality (1.16) as follows.

2. The Results

The following result holds.

Theorem 3. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f: I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \in L_2(I)$. If $x \geq a$, then we have the inequality*

$$\begin{aligned} &\left|\tilde{G}_n(f;a,x)\right| \\ \leq\ & (x-a)\left[T(P_n,P_n)\right]^{\frac{1}{2}}\left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2 - \left(\left[f^{(n)};a,x\right]\right)^2\right]^{\frac{1}{2}} \\ & \left(\leq \frac{x-a}{2}\left[T(P_n,P_n)\right]^{\frac{1}{2}}\left[\Gamma(x)-\gamma(x)\right], \quad \text{if } f^{(n+1)} \in L_\infty[a,x]\right), \end{aligned} \tag{2.1}$$

where $\|\cdot\|_2$ is the usual Euclidean norm on $[a,x]$, i.e.,

$$\left\|f^{(n+1)}\right\|_2 = \left(\int_a^x \left|f^{(n+1)}(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Proof. Recall Korkine's identity for the mappings h, g, which can be easily proved by direct computation:

$$T(h,g) := \frac{1}{2(x-a)^2}\int_a^x\int_a^x (h(t)-h(s))(g(t)-g(s))\,dtds. \tag{2.2}$$

Using (2.2), we have

$$\begin{aligned} & (-1)^n\int_a^x P_n(t) f^{(n+1)}(t)\,dt - \frac{(-1)^n}{x-a}\int_a^x f^{(n+1)}(t)\,dt \cdot \int_a^x P_n(t)\,dt \\ =\ & \frac{(-1)^n}{2}\cdot\frac{1}{x-a}\int_a^x\int_a^x (P_n(t)-P_n(s))\left(f^{(n+1)}(t)-f^{(n+1)}(s)\right)dtds \end{aligned} \tag{2.3}$$

and then, by the equalities (1.1) and (1.15), we have the following representation of the remainder in the perturbed formula (1.15)

$$\begin{aligned} & \tilde{G}_n(f;a,x) \\ =\ & \frac{(-1)^n}{2}\cdot\frac{1}{x-a}\int_a^x\int_a^x (P_n(t)-P_n(s))\left(f^{(n+1)}(t)-f^{(n+1)}(s)\right)dtds, \end{aligned} \tag{2.4}$$

which is an identity that is interesting in itself as well. Using now the Cauchy-Buniakowsky-Schwartz integral inequality for double integrals, we have

$$\begin{aligned}
&\left|\tilde{G}_n(f;a,x)\right| \\
\leq\ & \frac{1}{2(x-a)}\left[\int_a^x\int_a^x (P_n(t)-P_n(s))^2\,dtds\right]^{\frac{1}{2}} \\
&\times\left[\int_a^x\int_a^x \left(f^{(n+1)}(t)-f^{(n+1)}(s)\right)^2 dtds\right]^{\frac{1}{2}} \\
=\ & (x-a)\left[T(P_n,P_n)\right]^{\frac{1}{2}}\left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2-\left(\frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt\right)^2\right]^{\frac{1}{2}},
\end{aligned}$$

and the first inequality in (2.1) is proved.

The second inequality is obvious by the Grüss inequality

$$\begin{aligned}
&\frac{1}{x-a}\int_a^x\left[f^{(n+1)}(t)\right]^2 dt-\left(\frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt\right)^2 \\
\leq\ & \frac{1}{4}\left[\Gamma(x)-\gamma(x)\right]^2,
\end{aligned} \tag{2.5}$$

and the theorem is proved. ∎

Remark 1. *If $f^{(n+1)}$ is unbounded on (a,x) but $f^{(n+1)}\in L_2(a,x)$, then the first inequality in (2.1) can still be applied, but not the Matić-Pečarić-Ujević result (1.16) which requires the boundedness of the derivative $f^{(n+1)}$.*

The following corollary improves Corollary 3 of [2], which deals with the estimation of the remainder for the particular perturbed Taylor-like formulae (1.4), (1.7) and (1.10).

Corollary 1. *With the assumptions in Theorem 3, we have the following inequalities*

$$\left|\tilde{G}_n^{(M)}(f;a,x)\right| \leq \frac{(x-a)^{n+1}}{n!2^n\sqrt{2n+1}}\times\sigma\left(f^{(n+1)};a,x\right), \tag{2.6}$$

$$\left|\tilde{G}_n^{(B)}(f;a,x)\right| \leq (x-a)^{n+1}\left[\frac{|B_{2n}|}{(2n)!}\right]^{\frac{1}{2}}\times\sigma\left(f^{(n+1)};a,x\right), \tag{2.7}$$

$$\begin{aligned}
&\left|\tilde{G}_n^{(E)}(f;a,x)\right| \\
\leq\ & 2(x-a)^{n+1}\left[\frac{\left(4^{n+1}-1\right)|B_{2n+2}|}{(2n+2)!}-\left[\frac{2\left(2^{n+2}-1\right)B_{n+2}}{(n+1)!}\right]^2\right]^{\frac{1}{2}} \\
&\times\sigma\left(f^{(n+1)};a,x\right),
\end{aligned} \tag{2.8}$$

and

$$\left|G_n(f;a,x)\right| \leq \frac{n(x-a)^{n+1}}{(n+1)!\sqrt{2n+1}}\times\sigma\left(f^{(n+1)};a,x\right), \tag{2.9}$$

where, as in [2],

$$\tilde{G}_n^{(M)}(f;a,x) = f(x) - T_n^M(f;a,x) - \frac{(x-a)^{n+1}[1+(-1)^n]}{(n+1)!2^{n+1}}\left[f^{(n)};a,x\right];$$

$$\tilde{G}_n^{(B)}(f;a,x) = f(x) - T_n^B(f;a,x);$$

$$\tilde{G}_n^{(E)}(f;a,x) = f(x) - \frac{4(-1)^n(x-a)^{n+1}\left(2^{n+2}-1\right)B_{n+2}}{(n+2)!}\left[f^{(n)};a,x\right],$$

$G_n(f;a,x)$ *is as defined by (1.13),*

$$\sigma\left(f^{(n+1)};a,x\right) := \left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2 - \left(\left[f^{(n+1)};a,x\right]\right)^2\right]^{\frac{1}{2}}, \tag{2.10}$$

and $x \geq a$, $f^{(n+1)} \in L_2[a,x]$.

Note that for all the examples considered in [1] and [2] for f, the quantity $\sigma\left(f^{(n+1)};a,x\right)$ can be completely computed and then those particular inequalities may be improved accordingly. We omit the details.

Remark 2. *Theorem 3 is an implicit refinement of Theorem 4 from* [2] *which uses Chebychev's inequality to estimate* $\sigma\left(f^{(n+1)};a,x\right)$ *and of Theorem 5 from* [2] *which uses Lupaş's inequality to upper bound the same term* $\sigma\left(f^{(n+1)};a,x\right)$.

References

[1] S.S. DRAGOMIR, New estimation of the remainder in Taylor's formula using Grüss' type inequalities and applications, *Math. Ineq. Appl.,* **2** (2) (1999), 183-193.

[2] M. MATIĆ, J.E. PEČARIĆ and N. UJEVIĆ, On new estimation of the remainder in generalised Taylor's formula, *Math. Ineq. Appl.,* **2** (3) (1999), 343-361.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html