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Th Derivative Of Its Integrand*

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# ESTIMATES FOR AN INTEGRAL IN $L^p$ NORM OF THE $(n + 1)$ -TH DERIVATIVE OF ITS INTEGRAND

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ABSTRACT. Basing on Taylor's formula with an integral remainder, an integral is estimated in  $L^p$  norm of the  $(n + 1)$ -th derivative of its integrand, and the Iyengar's inequality and many other useful inequalities are generalized.

## 1. INTRODUCTION

The study of Iyengar's inequality [6] has a rich literature and a long history. For details please refer to references in this article.

In [4, 5] and [11]–[15], making use of mean value theorems for derivative (including Rolle's, Lagarange's, and Taylor's), mean value theorem for integral, the Taylor's formula for functions of several variables, and other technique, the following result was obtained:

**Theorem 1.** *Let the function  $f \in C^n([a, b])$  have derivative of  $(n + 1)$ -th order in  $(a, b)$  satisfying  $N \leq f^{(n+1)}(x) \leq M$ . Denote*

$$(1) \quad \begin{aligned} S_n(u, v, w) &= \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \cdot u^k f^{(k-1)}(v) + \frac{w}{n!} \cdot (-1)^n u^n, \\ \frac{\partial^k S_n}{\partial u^k} &= S_n^{(k)}(u, v, w), \end{aligned}$$

then, for any  $t \in (a, b)$ , when  $n$  is odd we have

$$(2) \quad \begin{aligned} \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left( S_{n+2}^{(i)}(a, a, N) - S_{n+2}^{(i)}(b, b, N) \right) t^i &\leq \int_a^b f(x) \, dx \\ &\leq \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left( S_{n+2}^{(i)}(a, a, M) - S_{n+2}^{(i)}(b, b, M) \right) t^i; \end{aligned}$$

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when  $n$  is even we have

$$(3) \quad \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left( S_{n+2}^{(i)}(a, a, N) - S_{n+2}^{(i)}(b, b, M) \right) t^i \leq \int_a^b f(x) dx \\ \leq \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left( S_{n+2}^{(i)}(a, a, M) - S_{n+2}^{(i)}(b, b, N) \right) t^i.$$

In [1]–[3], [8] and [18], similar problems were investigated by using Hayashi's inequality. For more information, please see [7], [9] and [10].

In this article, by the Taylor's mean value theorem [17] with an integral remainder, we have

**Theorem 2.** *Let the function  $f \in C^n([a, b])$  have derivative of  $(n + 1)$ -th order in  $(a, b)$ , and  $f^{(n+1)} \in L^p([a, b])$  for positive numbers  $p$  and  $q$  satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ . Then, for any  $t \in (a, b)$ , we have*

$$(4) \quad \left| \int_a^b f(x) dx - \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} + \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} \right| \\ \leq \frac{(t-a)^{n+1+\frac{1}{q}} + (b-t)^{n+1+\frac{1}{q}}}{(n+1)! \sqrt[q]{nq+q+1}} \|f^{(n+1)}\|_{L^p([a,b])}.$$

**Corollary 1.** *Let  $f \in C^n([a, b])$ . If  $f^{(i)}(a) = f^{(i)}(b) = 0$  for  $1 \leq i \leq n$ , and  $f^{(n+1)} \in L^p([a, b])$  is not identically zero, then for positive numbers  $p$  and  $q$  satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ , we have*

$$(5) \quad \left| \int_a^b f(x) dx \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{\sqrt[q]{2(q+1)}} \|f^{(n+1)}\|_{L^p([a,b])}.$$

## 2. PROOF OF THEOREM 2

Let  $t$  be a parameter satisfying  $a < t < b$ , and write

$$(6) \quad \int_a^b f(x) dx = \int_a^t f(x) dx + \int_t^b f(x) dx.$$

The well-known Taylor's formula with an integral remainder [17, pp. 4–6] states that

$$(7) \quad f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i + \frac{1}{n!} \int_a^x (x-s)^n f^{(n+1)}(s) ds,$$

$$(8) \quad f(x) = \sum_{i=0}^n \frac{f^{(i)}(b)}{i!} (x-b)^i + \frac{1}{n!} \int_b^x (x-s)^n f^{(n+1)}(s) ds,$$

and

$$(9) \quad \int_a^x \int_a^t (t-s)^n f^{(n+1)}(s) ds dt = \frac{1}{n+1} \int_a^x (x-s)^{n+1} f^{(n+1)}(s) ds.$$

Integrating on both sides of (7) over  $[a, t]$ , we obtain

$$\begin{aligned}
 \int_a^t f(x) dx &= \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} + \frac{1}{n!} \int_a^t \int_a^x (x-s)^n f^{(n+1)}(s) ds dx \\
 (10) \qquad &= \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} + \frac{1}{(n+1)!} \int_a^t (t-s)^{n+1} f^{(n+1)}(s) ds.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 &\left| \int_a^t f(x) dx - \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} \right| \\
 (11) \qquad &\leq \frac{1}{(n+1)!} \int_a^t \left| (t-s)^{n+1} f^{(n+1)}(s) \right| ds \\
 &\leq \frac{1}{(n+1)!} \left( \int_a^t (t-s)^{q(n+1)} ds \right)^{1/q} \left( \int_a^t \left| f^{(n+1)}(s) \right|^p ds \right)^{1/p} \\
 &\leq \frac{1}{(n+1)!} \left( \frac{(t-a)^{nq+q+1}}{nq+q+1} \right)^{1/q} \left\| f^{(n+1)} \right\|_{L^p[a,b]},
 \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

Integrating on both sides of (8) over  $[t, b]$ , we get

$$\begin{aligned}
 \int_t^b f(x) dx &= - \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} + \frac{1}{n!} \int_t^b \int_b^x (x-s)^n f^{(n+1)}(s) ds dx \\
 (12) \qquad &= - \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} + \frac{1}{(n+1)!} \int_t^b (t-s)^{n+1} f^{(n+1)}(s) ds.
 \end{aligned}$$

Thus, from (12), it follows that

$$\begin{aligned}
 &\left| \int_t^b f(x) dx + \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} \right| \\
 (13) \qquad &\leq \frac{1}{(n+1)!} \int_t^b \left| (t-s)^{n+1} f^{(n+1)}(s) \right| ds \\
 &\leq \frac{1}{(n+1)!} \left( \int_t^b (s-t)^{p(n+1)} ds \right)^{1/p} \left( \int_t^b \left| f^{(n+1)}(s) \right|^p ds \right)^{1/p} \\
 &\leq \frac{1}{(n+1)!} \left( \frac{(b-t)^{np+p+1}}{np+p+1} \right)^{1/p} \left\| f^{(n+1)} \right\|_{L^p[a,b]}.
 \end{aligned}$$

From (6), (11) and (13), we have

$$\begin{aligned}
 (14) \quad & \left| \int_a^b f(x) dx - \sum_{i=0}^n \frac{f^{(i)}(a)}{(i+1)!} (t-a)^{i+1} + \sum_{i=0}^n \frac{f^{(i)}(b)}{(i+1)!} (t-b)^{i+1} \right| \\
 & \leq \frac{1}{(n+1)!} \left\{ \left( \frac{(t-a)^{np+p+1}}{np+p+1} \right)^{1/p} + \left( \frac{(b-t)^{np+p+1}}{np+p+1} \right)^{1/p} \right\} \cdot \|f^{(n+1)}\|_{L^p[a,b]} \\
 & = \frac{(t-a)^{n+1+\frac{1}{q}} + (b-t)^{n+1+\frac{1}{q}}}{(n+1)! \sqrt[q]{nq+q+1}} \|f^{(n+1)}\|_{L^p([a,b])}.
 \end{aligned}$$

The proof is complete.

#### REFERENCES

- [1] R. P. Agarwal and S. S. Dragomir, *An application of Hayashi's inequality for differentiable functions*, Computers Math. Appl. **32** (1996), no. 6, 95–99.
- [2] P. Cerone and S. S. Dragomir, *On a weighted generalization of Iyengar type inequalities involving bounded first derivative*, RGMIA Research Report Collection **2** (1999), no. 2, 147–157. <http://rgmia.vu.edu.au/v2n2.html>.
- [3] P. Cerone and S. S. Dragomir, *Lobatto type quadrature rules for functions with bounded derivative*, RGMIA Research Report Collection **2** (1999), no. 2, 133–146. <http://rgmia.vu.edu.au/v2n2.html>.
- [4] Li-Hong Cui and Bai-Ni Guo, *On proofs of an integral inequality and its generalizations*, Journal of Zhengzhou Grain College, **17** (1996), Supplement, 152–154 and 158. (Chinese)
- [5] Bai-Ni Guo and Feng Qi, *Proofs of an integral inequality*, Mathematics and Informatics Quarterly, **7** (1997), no. 4, 182–184.
- [6] K. S. K. Iyengar, *Note on an inequality*, Math. Student **6** (1938), 75–76.
- [7] Ji-Chang Kuang, *Applied Inequalities (Changyong Budengshi)*, 2nd edition, Hunan Education Press, Changsha, China, 1993, Chapter 8, page 584. (Chinese)
- [8] G. V. Milovanović and J. E. Pečarić, *Some considerations on Iyengar's inequality and some related applications*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **544–576** (1976), 166–170.
- [9] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, 1970, page 297.
- [10] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1991, Chapter XV.
- [11] Feng Qi, *Inequalities for an integral*, The Mathematical Gazette **80** (1996), no. 488, 376–377.
- [12] Feng Qi, *Further generalizations of inequalities for an integral*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **8** (1997), 79–83.
- [13] Feng Qi, *Inequalities for a multiple integral*, Acta Mathematica Hungarica **84** (1999), no. 1-2, 19–26.
- [14] Feng Qi, *Inequalities for a weighted integral*, RGMIA Research Report Collection **2** (1999), no. 7, Article 2. <http://rgmia.vu.edu.au/v2n7.html>.
- [15] Feng Qi, *Inequalities for a weighted multiple integral*, Journal of Mathematical Analysis and Applications (2000), in the press. RGMIA Research Report Collection **2** (1999), no. 7, Article 4. <http://rgmia.vu.edu.au/v2n7.html>.

- [16] Feng Qi, *Estimates in  $L^p$  norms of partial derivatives of the  $(n+1)$ -th order for a multiple integral*, submitted.
- [17] Michael E. Taylor, *Partial Differential Equations I—Basic Theory*, Springer-Verlag, New York, 1996. Reprinted in China by Beijing World Publishing Corporation, Beijing, 1999.
- [18] P. M. Vasić and G. V. Milonanović, *On an inequality of Iyengar*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **544–576** (1976), 18–24.

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