SOME NEW INEQUALITIES FOR JEFFREYS DIVERGENCE MEASURE IN INFORMATION THEORY

S.S. DRAGOMIR, J. ŠUNDE, AND C. BUŞE

Abstract. Some new inequalities for the well-known Jeffreys divergence measure in Information Theory are given.

1. Introduction

One of the important issues in many applications of Probability Theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [22], Kullback and Leibler [31], Rényi [42], Havrda and Charvat [20], Kapur [25], Sharma and Mittal [44], Burbea and Rao [5], Rao [41], Lin [34], Csiszár [10], Ali and Silvey [1], Vajda [52], Shioya and Da-te [45] and others (see for example [25] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [41], genetics [37], finance, economics, and political science [43], [47], [48], biology [39], the analysis of contingency tables [19], approximation of probability distributions [9], [26], signal processing [23], [24] and pattern recognition [3], [8].

Assume that a set $\chi$ and the $\sigma$-finite measure $\mu$ are given. Consider the set of all probability densities on $\mu$ to be $\Omega := \{p : \chi \rightarrow \mathbb{R}, p(x) \geq 0, \int_\chi p(x) \, d\mu(x) = 1\}$.

The Kullback-Leibler divergence [31] is well known among the information divergences. It is defined as:

$$D_{KL}(p, q) := \int_\chi p(x) \log \left( \frac{p(x)}{q(x)} \right) \, d\mu(x), \quad p, q \in \Omega,$$

where log is to base 2.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: variation distance $D_v$, Hellinger distance $D_H$ [21], $\chi^2$-divergence $D_{\chi^2}$, $\alpha$-divergence $D_\alpha$, Bhattacharyya distance $D_B$ [4], Harmonic distance $D_{Ha}$, Jeffreys distance $D_J$ [22], triangular discrimination $D_\Delta$ [49], etc. They are defined as follows:

$$D_v(p, q) := \int_\chi |p(x) - q(x)| \, d\mu(x), \quad p, q \in \Omega;$$

$$D_H(p, q) := \int_\chi \left[ \sqrt{p(x)} - \sqrt{q(x)} \right]^2 \, d\mu(x), \quad p, q \in \Omega;$$

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\(D_{\chi}^2(p, q) := \int_x p(x) \left[ \frac{q(x)}{p(x)} \right]^2 - 1 \, d\mu(x), \ p, q \in \Omega; \tag{1.4}\)

\(D_\alpha(p, q) := \frac{4}{1 - \alpha^2} \left[ 1 - \int_x [p(x)]^{\frac{1-\alpha}{2}} [q(x)]^{\frac{1+\alpha}{2}} \, d\mu(x) \right], \ p, q \in \Omega; \tag{1.5}\)

\(D_B(p, q) := \int_x \sqrt{p(x)q(x)} \, d\mu(x), \ p, q \in \Omega; \tag{1.6}\)

\(D_{Ha}(p, q) := \int_x \frac{p(x)q(x)}{p(x) + q(x)} \, d\mu(x), \ p, q \in \Omega; \tag{1.7}\)

\(D_J(p, q) := \int_x [p(x) - q(x)] \ln \left[ \frac{p(x)}{q(x)} \right] \, d\mu(x), \ p, q \in \Omega; \tag{1.8}\)

\(D_\Delta(p, q) := \int_x \frac{[p(x) - q(x)]^2}{p(x) + q(x)} \, d\mu(x), \ p, q \in \Omega. \tag{1.9}\)

For other divergence measures, see the paper [25] by Kapur or the book on line [46] by Taneja. For a comprehensive collection of preprints available on line, see the RGMIA web site \texttt{http://rgmia.vu.edu.au/papersinfth.html}

In [35], Lin and Wong (see also [34]) introduced the following divergence

\(D_{LW}(p, q) := \int_x p(x) \log \left[ \frac{p(x)}{\frac{1}{2}p(x) + \frac{1}{2}q(x)} \right] \, d\mu(x), \ p, q \in \Omega. \tag{1.10}\)

In other words, Lin-Wong divergence is represented as follows, using the Kullback-Leibler divergence:

\(D_{LW}(p, q) = D_{KL} \left( p, \frac{1}{2}p + \frac{1}{2}q \right). \tag{1.11}\)

Lin and Wong have shown various inequalities as follows

\(D_{LW}(p, q) \leq \frac{1}{2} D_{KL}(p, q); \tag{1.12}\)

\(D_{LW}(p, q) + D_{LW}(q, p) \leq D_v(p, q) \leq 2; \tag{1.13}\)

\(D_{LW}(p, q) \leq 1. \tag{1.14}\)

In [45], Shioya and Da-te improved (1.13) – (1.14) by showing that

\(D_{LW}(p, q) \leq \frac{1}{2} D_v(p, q) \leq 1. \tag{1.15}\)

For classical and new results in comparing different kinds of divergence measures, see the papers [22]-[45] where further references are given.

In [18], Dragomir and Wang proved, amongst others, the following midpoint inequality

\(\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{4} (b - a) (\Gamma - \gamma), \tag{1.16}\)
provided that $f$ is absolutely continuous and the derivative $f': [a, b] \to \mathbb{R}$ satisfies the condition

\[ \gamma \leq f'(t) \leq \Gamma \text{ for a.e. } t \in [a, b]. \]

With the same assumptions for the mapping $f$, but using a finer argument based in a “pre-Grüss” inequality, the authors of [38] improved (1.1) as follows

\[ \left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{4\sqrt{3}} (b-a) (\Gamma - \gamma). \]  

For other results concerning the midpoint and trapezoid inequality, see the recent papers [6]-[7] and the website http://rgmia.vu.edu.au/.

The main aim of this paper is to point out some new midpoint and trapezoid type inequalities and apply them for the Jeffreys divergence measure $D_J$.

2. Some Analytic Inequalities

**Lemma 1.** Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$ with $f' \in L^2[a, b]$. Then we have the inequality:

\[ \left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{b-a} \left( \frac{1}{2\sqrt{3}} \| f' \|^2_2 - \frac{1}{(b-a)^2} \right)^{\frac{1}{2}}, \]

where

\[ [f; a, b] := \frac{f(b) - f(a)}{b-a}. \]

**Proof.** Start with the following identity which can be easily proved by the integration by parts formula

\[ f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt = \frac{1}{b-a} \int_a^b m(t) f'(t) \, dt, \]

where

\[ m(t) := \begin{cases} t - a & \text{if } t \in [a, \frac{a+b}{2}] \\ t - b & \text{if } t \in (\frac{a+b}{2}, b] \end{cases}. \]

Using Korkine’s identity, i.e., we recall it

\[ \frac{1}{b-a} \int_a^b u(t) v(t) \, dt - \frac{1}{b-a} \int_a^b u(t) \, dt \cdot \frac{1}{b-a} \int_a^b v(t) \, dt = \frac{1}{2 (b-a)^2} \int_a^b \int_a^b (u(t) - u(s)) (v(t) - v(s)) \, dtds, \]

and this identity can be proved by direct computation, we may write that

\[ \frac{1}{b-a} \int_a^b m(t) f'(t) \, dt - \frac{1}{b-a} \int_a^b m(t) \, dt \cdot \frac{1}{b-a} \int_a^b f'(t) \, dt = \frac{1}{2 (b-a)^2} \int_a^b \int_a^b (m(t) - m(s)) (f'(t) - f'(s)) \, dtds. \]
However,
\[
\int_a^b m(t) \, dt = 0
\]
and then, by (2.2) and (2.4), we have the representation:

\[
\begin{align*}
\quad f \left( \frac{a + b}{2} \right) &- \frac{1}{b-a} \int_a^b f(t) \, dt \\
= &\quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (m(t) - m(s)) \left( f'(t) - f'(s) \right) \, dt \, ds.
\end{align*}
\]

Using the Cauchy-Buniakowski-Schwartz integral inequality for double integrals, we have

\[
\begin{align*}
\quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b |(m(t) - m(s)) \left( f'(t) - f'(s) \right)| \, dt \, ds \\
\leq &\quad \left[ \frac{1}{2(b-a)^2} \int_a^b \int_a^b (m(t) - m(s))^2 \, dt \, ds \right]^{\frac{1}{2}} \\
&\quad \times \left[ \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 \, dt \, ds \right]^{\frac{3}{2}}
\end{align*}
\]

and as
\[
\begin{align*}
\quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (m(t) - m(s))^2 \, dt \, ds \\
= &\quad \frac{1}{b-a} \int_a^b m^2(t) \, dt - \left( \frac{1}{b-a} \int_a^b m(t) \, dt \right)^2 = \frac{(b-a)^2}{12}
\end{align*}
\]
and
\[
\begin{align*}
\quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 \, dt \, ds \\
= &\quad \frac{1}{b-a} \int_a^b (f'(t))^2 \, dt - \left( \frac{1}{b-a} \int_a^b f'(t) \, dt \right)^2,
\end{align*}
\]
then, by (2.5) and (2.6), we deduce (2.1).

**Remark 1.** For another proof of this inequality, see [2].

**Remark 2.** Taking into account, by the Grüss inequality, we have that

\[
0 \leq \frac{1}{b-a} \|f'\|_2^2 - (f; a, b)^2 \leq \frac{1}{4} (\Gamma - \gamma),
\]
then (2.1) is an improvement of (1.18) in the case when \( f' \in L_\infty [a, b] \) and satisfies (1.17).

**Corollary 1.** For any \( a, b > 0 \), we have the inequality

\[
0 \leq (b-a) (\ln b - \ln a) - 2 \cdot \frac{(b-a)^2}{a+b} \leq \frac{(b-a)^4}{6\sqrt{a^3}b^2}.
\]
Proof. Choose $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Then

$$f\left(\frac{a + b}{2}\right) = \frac{2}{a + b},$$

$$\frac{1}{b-a} \int_a^b f(t) \, dt = \frac{\ln b - \ln a}{b-a},$$

$$\frac{1}{b-a} \|f'\|^2 - ([f; a, b])^2 = \frac{(b-a)^2}{3a^3b^3},$$

and then, by (2.1), we get (by the convexity of $f$) that

$$0 \leq \ln b - \ln a - \frac{2}{a+b} \leq \frac{2}{a+b} \leq \frac{(b-a)^2}{6\sqrt{a^3b^3}},$$

which is clearly equivalent to (2.8).

The following lemma also holds.

**Lemma 2.** Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$ with $f' \in L^2[a, b]$. Then we have the inequality:

$$\left(\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt\right)^2 \leq \frac{b-a}{2\sqrt{3}} \left[ \frac{1}{b-a} \|f'\|^2 - ([f; a, b])^2 \right]^\frac{1}{2}.$$

Proof. In the recent paper [17], Dragomir and Mabizela proved the following identity which can be easily verified by direct computation:

$$\left(\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt\right) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))(t-s) \, dt \, ds.$$

Using (2.10) and the Cauchy-Buniakowski-Schwartz integral inequality for double integrals, we have

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt\right| \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |(f'(t) - f'(s))(t-s)| \, dt \, ds \leq \left[ \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 \, dt \, ds \right]^\frac{1}{2} \times \left[ \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s)^2 \, dt \, ds \right]^\frac{1}{2}$$

and as

$$\frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 \, dt \, ds \leq \frac{1}{b-a} \int_a^b (f'(t))^2 \, dt - \left(\frac{1}{b-a} \int_a^b f'(t) \, dt\right)^2,$$
and
\[
\frac{1}{2} \int_a^b \frac{(t-s)^2}{(b-a)^2} dt ds = \frac{1}{b-a} \int_a^b t^2 dt - \left( \frac{1}{b-a} \int_a^b t dt \right)^2 = \frac{(b-a)^2}{12},
\]
then, from (2.11), we deduce the desired inequality (2.9).

**Remark 3.** If we assume that \( f' \) satisfies (1.17), then by (2.7), we can deduce the inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4\sqrt{3}} (b-a) (\Gamma - \gamma),
\] which improves a similar result from [38] with the constant \( \frac{1}{4} \).

The following corollary also holds.

**Corollary 2.** For any \( a, b > 0 \), we have the inequality
\[
0 \leq \frac{a+b}{2ab} (b-a)^2 - (\ln b - \ln a) (b-a) \leq \frac{(b-a)^4}{(a^3b^3,}
\]

### 3. Some New Inequalities for Jeffreys Divergence

The following inequalities involving the Jeffreys divergence are known (see for example the book on line by Taneja [46])

\[
\begin{align*}
D_{Ha}(p,q) & \geq \exp \left[ -\frac{1}{2} D_J(p,q) \right], \quad p, q \in \Omega, \\
D_{Ha}(p,q) & \geq 1 - \frac{1}{4} D_J(p,q), \quad p, q \in \Omega
\end{align*}
\]
and
\[
D_J(p,q) \geq 4 \left[ 1 - D_B(p,q) \right], \quad p, q \in \Omega,
\]
where \( D_{Ha}(\cdot, \cdot) \) is the Harmonic distance and \( D_B(\cdot, \cdot) \) is the Bhattacharyya distance.

The following result holds.

**Theorem 1.** We have the inequality
\[
2D_\Delta(p,q) \leq D_J(p,q) \leq \frac{1}{2} \left[ D_{\chi^2}(p,q) + D_{\chi^2}(q,p) \right], \quad p, q \in \Omega,
\]
where \( D_{\chi^2} \) is the chi-square distance and \( D_\Delta \) is the triangular discrimination.

**Proof.** We use the celebrated Hermite-Hadamard inequality for convex functions
\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}
\]
and choose \( f(t) = \frac{1}{t} \) to get
\[
\frac{2}{a+b} \leq \frac{\ln b - \ln a}{b-a} \leq \frac{a+b}{2ab},
\]
which is equivalent to
\[
\frac{2}{a+b} (b-a) \leq (\ln b - \ln a) \leq \frac{a+b}{2ab} (b-a)^2.
\]
If we choose in (3.6) $b = q(x)$, $a = q(x)$, $x \in \chi$, then we get
\[
\frac{2(q(x) - p(x))^2}{p(x) + q(x)} \leq \frac{(q(x) - p(x))(\ln q(x) - \ln p(x))}{p(x) + q(x)} \leq \frac{p(x) + q(x)}{2p(x)q(x)} (q(x) - p(x))^2
\]
and integrating over $x$ on $\chi$, we deduce
\[
2D_{\Delta}(p, q) \leq D_J(p, q) \leq \frac{1}{6} D^*(p, q),
\]
and the inequality (3.4) is deduced.

Using the analytic inequalities established in Section 2, we can prove the following counterpart results as well.

**Theorem 2.** For all $p, q \in \Omega$, we have
\[
(3.7) \quad 0 \leq D_J(p, q) - 2D_{\Delta}(p, q) \leq \frac{1}{6} D^*(p, q),
\]
where
\[
D^*(p, q) := \int_\chi \frac{(p(x) - q(x))^4}{\sqrt{p^4(x)q^4(x)}} d\mu(x).
\]

The proof follows by the inequality (2.8) by a similar procedure as in the proof of Theorem 1 and we omit the details.

By the use of the analytic inequality (2.13), we may state the following theorem.

**Theorem 3.** For each $p, q \in \Omega$, we have
\[
(3.8) \quad 0 \leq \frac{1}{2} [D_{\chi^2}(p, q) + D_{\chi^2}(q, p)] - D_J(p, q) \leq \frac{1}{6} D^*(p, q).
\]

**References**


School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
E-mail address: Jadranka.Sunde@dsto.defence.gov.au

Communication Division, DSTO, PO Box 1500, Salisbury, SA 5108, Australia.
E-mail address: buse@hilbert.math.uvt.ro

Department of Mathematics, West Timisoara University, B-dul V. Parvan No. 4, RO-1900, Timisoara, Romania