A GENERALISATION OF CHEBYSHEV'S INEQUALITY FOR FUNCTIONS OF SEVERAL VARIABLES

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Abstract. The current note serves to develop generalisations of Chebyshev's inequality for Hölder functions of several variables.

1. Introduction

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f, g; a, b) := \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{(b-a)^2} \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx, \tag{1.1}$$

provided that the involved integrals exist. Results involving (1.1) abound in the literature, see for example [1] – [13].

The following inequality is well known as the Grüss inequality [10]

$$|T(f, g; a, b)| \leq \frac{1}{4} (M - m)(N - n), \tag{1.2}$$

provided that $m \leq f \leq M$ and $n \leq g \leq N$ a.e. on $[a, b]$, where m, M, n, N are real numbers. The constant $\frac{1}{4}$ in (1.2) is the best possible.

Another inequality of this type is due to Chebychev (see for example [13, p. 207]). Namely, if f, g are absolutely continuous on $[a, b]$ and $f', g' \in L_\infty [a, b]$ and $\|f'\|_\infty := ess \sup_{t \in [a,b]} |f'(t)|$, then

$$|T(f, g; a, b)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2 \tag{1.3}$$

and the constant $\frac{1}{12}$ is the best possible.

Cerone and Dragomir [2] have pointed out generalizations of the above results for integrals defined on two different intervals $[a, b]$ and $[c, d]$.

They defined the functional (generalised Chebychev functional)

$$\begin{aligned} T(f, g; a, b, c, d) \quad : &= M(fg; a, b) + M(fg; c, d) \\ &-M(f; a, b) M(g; c, d) - M(f; c, d) M(g; a, b), \end{aligned} \tag{1.4}$$

where the integral mean is defined by

$$M(f; a, b) := \frac{1}{b-a} \int_a^b f(x) \, dx \tag{1.5}$$

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and obtained a variety of bounds using a generalisation of Korkine's identity. Budimir, Cerone and Pečarić [1] obtained the bounds for (1.4) for f and g of Hölder type and also, for a weighted version of (1.4). In particular, they obtained the following result.

Theorem 1. *Let $f,g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a,b]$, $[c,d] \subset I$. Further, suppose that f and g are of Hölder type so that for $x \in [a,b]$, $y \in [c,d]$*

$$|f(x) - f(y)| \leq H_1 |x-y|^r \quad \text{and} \quad |g(x) - g(y)| \leq H_2 |x-y|^s, \tag{1.6}$$

where H_1, $H_2 > 0$ and $r,s \in (0,1]$ are fixed. The following inequality then holds,

$$\begin{aligned} &(\theta+1)(\theta+2)|T(f,g;a,b,c,d)| \\ &\leq \frac{H_1 H_2}{(b-a)(d-c)} \left[|b-c|^{\theta+2} - |b-d|^{\theta+2} + |d-a|^{\theta+2} - |c-a|^{\theta+2}\right], \end{aligned} \tag{1.7}$$

where $\theta = r+s$ and $T(f,g;a,b,c,d)$ is as defined by (1.4) and (1.5).

Hanna, Dragomir and Cerone [11] obtained bounds for a two dimensional Chebychev functional for functions of Hölder type and applied them for perturbed Taylor-like formulae in $\mathbb{R}^2$.

It is the express aim of this note to obtain bounds for a Chebychev functional defined on an $n-$dimensional hypercube where the functions are of Hölder type.

2. Bounds for the Chebychev Functional

If we consider the Chebyshev functional:

$$\begin{aligned} D_n(f,g) :&= \frac{1}{\prod_{k=1}^n (b_k - a_k)} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1,\ldots,x_n)\, g(x_1,\ldots,x_n)\, dx_1 \ldots dx_n \\ &- \frac{1}{\prod_{k=1}^n (b_k - a_k)^2} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1,\ldots,x_n)\, dx_1 \ldots dx_n \\ &\times \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} g(x_1,\ldots,x_n)\, dx_1 \ldots dx_n \\ &= \frac{1}{\nu([\bar{a},\bar{b}])} \int_{\bar{a}}^{\bar{b}} f(\bar{x})\, g(\bar{x})\, d\bar{x} - \frac{1}{\nu([\bar{a},\bar{b}])^2} \int_{\bar{a}}^{\bar{b}} f(\bar{x})\, d\bar{x} \cdot \int_{\bar{a}}^{\bar{b}} g(\bar{x})\, d\bar{x}, \end{aligned}$$

where $\nu([\bar{a},\bar{b}]) := \prod_{k=1}^n (b_k - a_k)$, then we can state the following generalisation of Chebyshev's inequality.

Theorem 2. *Let $f,g : [\bar{a},\bar{b}] \to \mathbb{R}$ be of Hölder type. That is,*

$$|f(\bar{x}) - f(\bar{y})| \leq \sum_{i=1}^n L_i |x_i - y_i|^{p_i}, \quad \bar{x},\bar{y} \in [\bar{a},\bar{b}], \tag{2.1}$$

$$|g(\bar{x}) - g(\bar{y})| \leq \sum_{i=1}^n H_i |x_i - y_i|^{q_i}, \quad \bar{x},\bar{y} \in [\bar{a},\bar{b}], \tag{2.2}$$

where L_i, $H_i > 0$ and p_i, $q_i \in (0,1]$ are fixed for $i = 1, 2, \ldots, n$.
Then we have the inequality:

$$\begin{aligned}|D_n(f,g)| &\leq \sum_{i=1}^{n} L_i H_i \frac{(b_i - a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)} \\ &\quad +2 \sum_{\substack{i\neq j\\ i,j=1}}^{n} L_i H_j \frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)}\end{aligned} \tag{2.3}$$

and the inequality is sharp.

Proof. We have

$$|f(\bar{x}) - f(\bar{y})| \leq \sum_{i=1}^{n} L_i |x_i - y_i|^{p_i} \tag{2.4}$$

and

$$|g(\bar{x}) - g(\bar{y})| \leq \sum_{i=1}^{n} H_i |x_i - y_i|^{q_i} . \tag{2.5}$$

If we multiply (2.4) and (2.5), we may get

$$\begin{aligned}& |(f(\bar{x}) - f(\bar{y}))(g(\bar{x}) - g(\bar{y}))| \\ \leq & \sum_{i,j=1}^{n} L_i H_j |x_i - y_i|^{p_i} |x_j - y_j|^{q_j} \\ = & \sum_{i=1}^{n} L_i H_i |x_i - y_i|^{p_i+q_i} + \sum_{\substack{i\neq j\\ i,j=1}}^{n} L_i H_j |x_i - y_i|^{p_i} |x_j - y_j|^{q_j} .\end{aligned}$$

If we integrate over $\bar{x}, \bar{y} \in [\bar{a}, \bar{b}] = \prod_{i=1}^{n} [a_i, b_i] := [a_1, b_1] \times [a_n, b_n]$ we get from Korkine's identity

$$\begin{aligned}& |D_n(f,g)| \\ \leq & \frac{1}{2\left[\prod_{i=1}^{n}(b_i - a_i)\right]^2} \int_{\bar{a}}^{\bar{b}} \int_{\bar{a}}^{\bar{b}} |(f(\bar{x}) - f(\bar{y}))(g(\bar{x}) - g(\bar{y}))| \, d\bar{x} d\bar{y} \\ \leq & \frac{1}{2\left[\prod_{i=1}^{n}(b_i - a_i)\right]^2} \left[\sum_{i=1}^{n} L_i H_i \int_{\bar{a}}^{\bar{b}} \int_{\bar{a}}^{\bar{b}} |x_i - y_i|^{p_i+q_i} \, d\bar{x} d\bar{y} \right. \\ & \left. + \sum_{\substack{i\neq j\\ i,j=1}}^{n} L_i H_j \int_{\bar{a}}^{\bar{b}} \int_{\bar{a}}^{\bar{b}} |x_i - y_i|^{p_i} |x_j - y_j|^{q_j} \, d\bar{x} d\bar{y} \right] .\end{aligned} \tag{2.6}$$

Now, we have that

$$A_i := \int_{\bar{a}}^{\bar{b}} \int_{\bar{a}}^{\bar{b}} |x_i - y_i|^{p_i+q_i} \, d\bar{x} d\bar{y} = \prod_{\substack{k\neq i\\ k=1}}^{n} (b_k - a_k)^2 \int_{a_i}^{b_i} \int_{a_i}^{b_i} |x_i - y_i|^{p_i+q_i} \, dx_i dy_i$$

and as

$$\int_c^d \int_c^d |x - y|^r \, dx dy = 2 \frac{(d-c)^{r+2}}{(r+1)(r+2)}, \tag{2.7}$$

then we get

$$\begin{aligned} A_i &= \prod_{\substack{k\neq i\\k=1}}^{n}(b_k-a_k)^2\cdot\frac{2(b_i-a_i)^{p_i+q_i+2}}{(p_i+q_i+1)(p_i+q_i+2)} \\ &= 2\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)}. \end{aligned}$$

Also,

$$\begin{aligned} A_{ij} &:= \int_{\bar a}^{\bar b}\int_{\bar a}^{\bar b}|x_i-y_i|^{p_i}|x_j-y_j|^{q_j}\,d\bar x d\bar y \\ &= \prod_{\substack{k\neq i,j\\k=1}}^{n}(b_k-a_k)^2\int_{a_i}^{b_i}\int_{a_i}^{b_i}|x_i-y_i|^{p_i}\,dx_idy_i\cdot\int_{a_j}^{b_j}\int_{a_j}^{b_j}|x_j-y_j|^{q_j}\,dx_jdy_j \\ &= \prod_{\substack{k\neq i,j\\k=1}}^{n}(b_k-a_k)^2\cdot\frac{2(b_i-a_i)^{p_i+2}}{(p_i+1)(p_i+2)}\cdot\frac{2(b_j-a_j)^{q_j+2}}{(q_j+1)(q_j+2)} \\ &= 4\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)} \end{aligned}$$

where we have utilised (2.7).

Further, by (2.6), we have:

$$\begin{aligned} &|D_n(f,g)| \\ \leq\ & \frac{1}{2\left[\prod_{i=1}^{n}(b_i-a_i)\right]^2}\left[\sum_{i=1}^{n}L_iH_i-2\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)}\right. \\ &\left.+4\sum_{\substack{i\neq j\\i,j=1}}^{n}L_iH_j\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)}\right] \\ =\ & \sum_{i=1}^{n}L_iH_i\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)} \\ &+2\sum_{\substack{i\neq j\\i,j=1}}^{n}L_iH_j\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)} \end{aligned}$$

and the result (2.6) is thus verified. The sharpness follows from the sharpness of the Chebychev functional for $n=1$ (see for example [12]). ∎

Corollary 1. *Let $f,g:[\bar a,\bar b]\to\mathbb{R}$ be Lipschitzian with constants L_i, $H_i>0$. That is,*

$$|f(\bar x)-f(\bar y)|\leq\sum_{i=1}^{n}L_i|x_i-y_i|,\quad \bar x,\bar y\in[\bar a,\bar b]$$

and

$$|g(\bar x)-g(\bar y)|\leq\sum_{i=1}^{n}H_i|x_i-y_i|,\quad \bar x,\bar y\in[\bar a,\bar b].$$

Then the inequality

$$|D_n(f,g)| \le \frac{1}{12}\sum_{i=1}^{n} L_i H_i (b_i-a_i)^2 + \frac{1}{18}\sum_{\substack{i\neq j\\ i,j=1}}^{n} L_i H_i (b_i-a_i)(b_j-a_j) \tag{2.8}$$

holds and is sharp.

Proof. Taking $p_i = q_i = 1$ for $i = 1, 2, \ldots, n$ in (2.6) readily produces (2.8). ∎

Remark 1. *Result (2.8) is presented in* [12, p. 305], *however, the coefficients of the sums are interchanged. Further, it is apparent that for $x, y \in [a, b]$ and f absolutely continuous, then*

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le \sup_{z\in[a,b]} |f'(z)| = L,$$

demonstrating that a function satisfying a Lipschitzian condition is a weaker condition than one whose derivative belongs to L_∞.

Remark 2. *If $n = 1$, then for $f, g \in [a_1, b_1] \to \mathbb{R}$*

$$|D_1(f,g)| = |T(f,g,a,b)| \le L_1 H_1 \frac{(b_1-a_1)^{p_1+q_1+1}}{(p_1+q_1+1)(p_1+q_1+2)} \tag{2.9}$$

with

$$|f(x_1) - f(y_1)| \le L_1 |x_1 - y_1|^{p_1}, \ |g(x_1) - g(y_1)| \le H_1 |x_1 - y_1|^{q_1}$$

$x_1, y_1 \in [a_1, b_1]$ and $p_1, q_1 \in (0, 1]$.

If $n = 2$ then for $f, g \in [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$

$$\begin{aligned} &|D_2(f,g)| \\ = &\left|\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)\,g(x_1,x_2)\,dx_1dx_2\right. \\ &-\frac{1}{(b_1-a_1)^2(b_2-a_2)^2}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)\,dx_1dx_2 \\ &\left.\times\int_{a_1}^{b_1}\int_{a_2}^{b_2} g(x_1,x_2)\,dx_1dx_2\right| \\ \le\ & L_1H_1\frac{(b_1-a_1)^{p_1+q_1}}{(p_1+q_1+1)(p_1+q_1+2)} + L_2H_2\frac{(b_2-a_2)^{p_2+q_2}}{(p_2+q_2+1)(p_2+q_2+2)} \\ &+2L_1H_2\frac{(b_1-a_1)^{p_1}(b_2-a_2)^{q_2}}{(p_1+1)(p_1+2)(q_2+1)(q_2+2)} \\ &+2L_2H_1\frac{(b_2-a_2)^{p_2}(b_1-a_1)^{q_1}}{(p_2+1)(p_2+2)(q_1+1)(q_1+2)}, \end{aligned} \tag{2.10}$$

with

$$|f(x_i) - f(y_i)| \le L_i |x_i - y_i|^{p_i}, \ \ x_i, y_i \in [a_i, b_i], \ i = 1, 2,$$

and

$$|g(x_i) - g(y_i)| \le H_i |x_i - y_i|^{q_i}, \ \ x_i, y_i \in [a_i, b_i], \ i = 1, 2.$$

Thus (2.10) recaptures the result of Hanna, Dragomir and Cerone [11].

References

[1] I. BUDIMIR, P. CERONE and J. PEČARIĆ, Inequalities related to the Chebychev Functional involving integrals over different intervals, submitted.

[2] P. CERONE and S.S. DRAGOMIR, Generalisations of the Grüss, Chebychev and Lupaş inequalities for integrals over different intervals, *RGMIA Res. Rep. Coll.,* **3**(2) (2000), Article 6.

[3] S.S. DRAGOMIR, Grüss inequality in inner product spaces, *The Australian Math Soc. Gazette*, **26** (1999), No. 2, 66-70.

[4] S.S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74-82.

[5] S.S. DRAGOMIR, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, **31**(3) (2000), 43-47.

[6] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* **31**(4) (2000), 397-415.

[7] S.S. DRAGOMIR and G.L. BOOTH, On a Grüss-Lupaş type inequality and its applications for the estimation of p-moments of guessing mappings, *Mathematical Communications,* **5** (2000), 117-126.

[8] S.S. DRAGOMIR and I. FEDOTOV, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4) (1998), 286-292.

[9] A.M. FINK, A treatise on Grüss' inequality, *submitted.*

[10] G. GRÜSS, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[11] G. HANNA, S.S. DRAGOMIR and P. CERONE, A Taylor like formula for mappings of two variables defined on a rectangle in the plane, *submitted.*

[12] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[13] J. PEČARIĆ, F. PROCHAN and Y. TONG, *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, San Diego, 1992.

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