A NEW PROOF FOR A ROLEWICZ'S TYPE THEOREM: AN EVOLUTION SEMIGROUP APPROACH

C. BUSE AND S.S. DRAGOMIR

ABSTRACT. Let \( \mathbb{R}_+ \) be the set of all non-negative real numbers and \( \mathcal{U} = \{ U(t, s) : t \geq s \geq 0 \} \) be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on a Banach space \( X \). Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a non-decreasing function such that \( \varphi(t) > 0 \) for all \( t > 0 \). We prove that if there exists \( M_\varphi > 0 \) such that

\[
\sup_{s \geq 0} \int_s^\infty \varphi(\|U(t, s)x\|) \, dt = M_\varphi < \infty, \quad \text{for all } x \in X, \|x\| \leq 1,
\]

then \( \mathcal{U} \) is uniformly exponentially stable. For \( \varphi \) continuous, this result is due to S. Rolewicz.

1. INTRODUCTION

Let \( X \) be a real or complex Banach space and \( L(X) \) the Banach algebra of all linear and bounded operators on \( X \). Let \( \mathbf{T} = \{ T(t) : t \geq 0 \} \subset L(X) \) be a strongly continuous semigroup on \( X \) and \( \omega_0(\mathbf{T}) = \lim_{t \to \infty} \frac{\ln(\|T(t)\|)}{t} \) be its growth bound. The Datko-Pazy theorem ([1], [2]) states that \( \omega_0(\mathbf{T}) < 0 \) if and only if for all \( x \in X \) the maps \( t \mapsto \|T(t)x\| \) belongs to \( L^p(\mathbb{R}_+) \) for some \( 1 \leq p < \infty \).

A family \( \mathcal{U} = \{ U(t, s) : t \geq s \geq 0 \} \subset L(X) \) is called an evolution family of bounded linear operators on \( X \) if \( U(t, t) = I \) (the identity operator on \( X \)) and

\[
U(t, \tau)U(\tau, s) = U(t, s) \quad \text{for all } t \geq \tau \geq s \geq 0.
\]

Such a family is said to be strongly continuous if for every \( x \in X \), the maps

\[
(t, s) \mapsto U(t, s)x : \{ (t, s) : t \geq s \geq 0 \} \to X
\]

are continuous, and exponentially bounded if there are \( \omega > 0 \) and \( K_\omega > 0 \) such that

\[
\|U(t, s)x\| \leq K_\omega e^{\omega(t-s)} \quad \text{for all } t \geq s \geq 0.
\]

The family \( \mathcal{U} \) is called uniformly exponentially stable if (1.1) holds for some negative \( \omega \). If \( \mathbf{T} = \{ T(t) : t \geq 0 \} \subset L(X) \) is a strongly continuous semigroup on \( X \), then the family \( \{ U(t, s) : t \geq s \geq 0 \} \) given by \( U(t, s) = T(t-s) \) is a strongly continuous and exponentially bounded evolution family on \( X \). Conversely, if \( \mathcal{U} \) is a strongly continuous evolution family on \( X \) and \( U(t, s) = U(t-s, 0) \) then the family \( \mathbf{T} = \{ T(t) : t \geq 0 \} \) given by \( T(t) = U(t, 0) \) is a strongly continuous semigroup on \( X \).

The Datko-Pazy theorem can be obtained from the following result given by S. Rolewicz ([3], [4]).

Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuous and nondecreasing function such that \( \varphi(0) = 0 \) and \( \varphi(t) > 0 \) for all \( t > 0 \). If \( \mathcal{U} = \{ U(t, s) : t \geq s \geq 0 \} \subset L(X) \) is a strongly continuous evolution family of bounded linear operators acting on \( X \) then

\[
\sup_{s \geq 0} \int_s^\infty \varphi(\|U(t, s)x\|) \, dt = M_\varphi < \infty, \quad \text{for all } x \in X, \|x\| \leq 1,
\]

then \( \mathcal{U} \) is uniformly exponentially stable. For \( \varphi \) continuous, this result is due to S. Rolewicz.

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Our proof of Theorem 1 is very simple. In fact, we apply a result of Neerven (see below) for the evolution semigroup associated to $U$ on $C_0$ $(\mathbb{R}_+, X)$, the space of all continuous, $X$-valued functions defined on $\mathbb{R}_+$ such that $f(0) = \lim_{t \to -\infty} f(t) = 0$.

Lemma 1. Let $\mathcal{U}$ be a strongly continuous and exponentially bounded evolution family of operators on $X$ such that (1.2) holds, then $\mathcal{U}$ is uniformly exponentially stable.

Proof of Lemma 1. Let $x \in X$ and $N(x)$ be a positive integer such that $M_{\varphi}(x) < N(x)$ and let $s \geq 0, t \geq s + N$. For each $\tau \in [t - N, t]$, we have

$$e^{-\omega N} 1_{[t-N,t]}(u) \|u(t,s)\| \leq e^{-\omega(t-\tau)} 1_{[t-N,t]}(u) \|u(t,\tau)\| \leq K_\omega \|u(u,s)\|,$$

for all $u \geq s$. Here $K_\omega$ and $\omega$ are as in (1.1) and $\omega > 0$.

If we choose $x = 0$ in (1.3), then we get $\varphi(0) = 0$, and thus from (1.4) we obtain

$$N(x) \varphi \left( \frac{\|u(t,s)\|}{K_\omega e^{-\omega N}} \right) = \int_s^\infty \varphi \left( \frac{1_{[t-N,t]}(u) \|u(t,s)\|}{K_\omega e^{-\omega N}} \right) du \leq \int_s^\infty \varphi \left( \|u(u,s)\| \right) du \leq M_{\varphi}(x).$$

We assume that $\varphi(1) = 1$ (if not, we replace $\varphi$ by some multiple of itself). Moreover, we may assume that $\varphi$ is a strictly increasing map. Indeed if $\varphi(1) = 1$ and $a := \int_0^1 \varphi(t) dt$, then the function given by

$$\varphi(t) = \begin{cases} \int_0^t \varphi(u) du, & \text{if } 0 \leq t \leq 1 \\ \frac{at}{at + 1 - a}, & \text{if } t > 1 \end{cases}$$
is strictly increasing and $\bar{\varphi} \leq \varphi$. Now $\varphi$ can be replaced by some multiple of $\bar{\varphi}$. From (1.5) it follows that if $t \geq s + N(x)$ and $x \in X$, then
\[ \|U(t, s)\| \leq K_{\omega} e^{\omega N(x)}, \]  
for all $x \in X$.

Using this inequality and the exponential boundedness of the evolution family, we have that
\[ (1.6) \sup_{t \geq 0, x} \|U(t, s)x\| \leq K_{\omega} e^{\omega N(x)}, \]  
for each $x \in X$.

The conclusion of Lemma 1 follows from (1.6) and the Uniform Boundedness Theorem.  

Let $\mathcal{U} = \{U(t, s) : t \geq s \geq 0\}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators on $X$. We consider the strongly continuous evolution semigroup associated to $\mathcal{U}$ on $C_{00}(\mathbb{R}_+, X)$. This semigroup is defined by
\[ (1.7) \Phi(t) := \begin{cases} U(s, s-t)f(s-t), & \text{if } s \geq t \\ 0, & \text{if } 0 \leq s \leq t \end{cases}, \]  
for all $f \in C_{00}(\mathbb{R}_+, X)$. It is known that $\Phi = \{\Phi(t) : t \geq 0\}$ is a strongly continuous semigroup and in addition $\omega_0(\Phi) < 0$ if and only if $\mathcal{U}$ is uniformly exponentially stable ([10], [11], [12]).

**Proof of Theorem 1.** Let $\varphi$ be as in Theorem 1. We assume that $\varphi(1) = 1$. Then
\[ \Phi(t) := \int_0^t \varphi(u) \, du \leq \varphi(t) \]  
for all $t \in [0, 1]$.

Without loss of generality we may assume that
\[ \sup_{t \geq 0} \|\Phi(t)\| \leq 1, \]  
where $\Phi$ is the semigroup defined in (1.7). Then for all $f \in C_{00}(\mathbb{R}_+, X)$ with $\|f\|_{C_{00}} \leq 1$, one has
\[ \Phi \left( \|\Phi(t)f\|_{C_{00}(\mathbb{R}_+, X)} \right) dt = \int_0^\infty \Phi \left( \sup_{s \geq t} \|U(s, s-t)f(s-t)\| \right) dt = \int_0^\infty \Phi \left( \sup_{\xi \geq 0} \|U(t + \xi, \xi)f(\xi)\| \right) dt \]
\[ = \int_0^\infty \left( \int_0^\infty \left[ \sup_{\xi \geq 0} \|U(t + \xi, \xi)f(\xi)\| \right] (u) \varphi(u) \, du \right) dt \]
\[ = \sup_{\xi \geq 0} \int_0^\infty \int_0^\infty \varphi \left( \|U(t + \xi, \xi)f(\xi)\| \right) dt \leq \sup_{\xi \geq 0} \int_0^\infty \varphi \left( \|U(t + \xi, \xi)f(\xi)\| \right) dt \]
\[ = \sup_{\xi \geq 0} \int_0^\infty \varphi \left( \|U(\tau, \xi)f(\xi)\| \right) d\tau \leq M_{\varphi} < \infty, \]  
where $1_{[0,h]}$ denotes the characteristic function of the interval $[0, h]$, $h > 0$. 
Now, from [7, Theorem 3.2.2], it follows that \( \omega_0(\mathcal{I}) < 0 \), hence \( \mathcal{U} \) is uniformly exponentially stable.

References


Department of Mathematics, West University of Timișoara, Bd. V. Parvan 4, 1900 Timișoara, România.
E-mail address: buse@hilbert.math.uvt.ro

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC 8001,, Victoria, Australia.
E-mail address: sever@matilda.vu.edu.au