A NEW PROOF FOR A ROLEWICZ'S TYPE THEOREM: AN EVOLUTION SEMIGROUP APPROACH

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Abstract. Let $\mathbb{R}_+$ be the set of all non-negative real numbers and $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on a Banach space X. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $\varphi(t) > 0$ for all $t > 0$. We prove that if there exists $M_\varphi > 0$ such that

$$\sup_{s \geq 0} \int_s^\infty \varphi(\|U(t,s)x\|)\, dt = M_\varphi < \infty, \text{ for all } x \in X,\ \|x\| \leq 1,$$

then $\mathcal{U}$ is uniformly exponentially stable. For φ continuous, this result is due to S. Rolewicz.

1. Introduction

Let X be a real or complex Banach space and $L(X)$ the Banach algebra of all linear and bounded operators on X. Let $\mathbf{T} = \{T(t) : t \geq 0\} \subset L(X)$ be a strongly continuous semigroup on X and $\omega_0(\mathbf{T}) = \lim_{t\to\infty} \frac{\ln(\|T(t)\|)}{t}$ be its growth bound. The Datko-Pazy theorem ([1], [2]) states that $\omega_0(\mathbf{T}) < 0$ if and only if for all $x \in X$ the maps $t \longmapsto \|T(t)x\|$ belongs to $L^p(\mathbb{R}_+)$ for some $1 \leq p < \infty$.

A family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset L(X)$ is called an *evolution family* of bounded linear operators on X if $U(t,t) = \mathbf{I}$ (the identity operator on X) and $U(t,\tau)U(\tau,s) = U(t,s)$ for all $t \geq \tau \geq s \geq 0$. Such a family is said to be *strongly continuous* if for every $x \in X$, the maps

$$(t,s) \mapsto U(t,s)x : \{(t,s) : t \geq s \geq 0\} \to X$$

are continuous, and *exponentially bounded* if there are $\omega > 0$ and $K_\omega > 0$ such that

$$\|U(t,s)\| \leq K_\omega e^{\omega(t-s)} \text{ for all } t \geq s \geq 0. \tag{1.1}$$

The family $\mathcal{U}$ is called *uniformly exponentially stable* if (1.1) holds for some negative ω. If $\mathbf{T} = \{T(t) : t \geq 0\} \subset L(X)$ is a strongly continuous semigroup on X, then the family $\{U(t,s) : t \geq s \geq 0\}$ given by $U(t,s) = T(t-s)$ is a strongly continuous and exponentially bounded evolution family on X. Conversely, if $\mathcal{U}$ is a strongly continuous evolution family on X and $U(t,s) = U(t-s,0)$ then the family $\mathbf{T} = \{T(t) : t \geq 0\}$ given by $T(t) = U(t,0)$ is a strongly continuous semigroup on X.

The Datko-Pazy theorem can be obtained from the following result given by S. Rolewicz ([3], [4]).

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and nondecreasing function such that $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. If $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset L(X)$ is a strongly

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continuous and exponentially bounded evolution family on the Banach space X such that

$$\sup_{s\geq 0}\int_s^\infty \varphi\left(\|U(t,s)x\|\right)dt = M_\varphi < \infty, \text{ for all } x\in X,\ \|x\|\leq 1, \tag{1.2}$$

then $\mathcal{U}$ is uniformly exponentially stable.

A shorter proof of the Rolewicz theorem was given by Q. Zheng [5] who removed the continuity assumption about φ. Other proofs of (the semigroup case) Rolewicz's theorem were offered by W. Littman [6] and J. van Neervan [7, pp. 81-82]. Some related results have been obtained by K.M. Przyłuski [8], G. Weiss [13] and J. Zabczyk [9].

In this note we prove the following:

Theorem 1. *Let $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing function such that $\varphi(t)>0$ for all $t>0$. If $\mathcal{U}=\{U(t,s): t\geq s\geq 0\}\subset L(X)$ is a strongly continuous and exponentially bounded evolution family of operators on X such that (1.2) holds, then $\mathcal{U}$ is uniformly exponentially stable.*

Our proof of Theorem 1 is very simple. In fact, we apply a result of Neerven (see below) for the evolution semigroup associated to $\mathcal{U}$ on $C_{00}(\mathbb{R}_+,X)$, the space of all continuous, $X-$valued functions defined on $\mathbb{R}_+$ such that $f(0)=\lim_{t\to\infty} f(t)=0$.

Lemma 1. *Let $\mathcal{U}$ be a strongly continuous and exponentially bounded evolution family of operators on X such that*

$$\sup_{s\geq 0}\int_s^\infty \varphi\left(\|U(t,s)x\|\right)dt = M_\varphi(x) < \infty, \text{ for all } x\in X. \tag{1.3}$$

Then $\mathcal{U}$ is uniformly bounded, that is,

$$\sup_{t\geq\xi\geq 0}\|U(t,\xi)\|<\infty.$$

Proof of Lemma 1. Let $x\in X$ and $N(x)$ be a positive integer such that $M_\varphi(x)<N(x)$ and let $s\geq 0$, $t\geq s+N$. For each $\tau\in[t-N,t]$, we have

$$\begin{aligned} e^{-\omega N}1_{[t-N,t]}(u)\|U(t,s)x\| &\leq e^{-\omega(t-\tau)}1_{[t-N,t]}(u)\|U(t,\tau)U(\tau,s)x\| \\ &\leq K_\omega\|U(u,s)x\|, \end{aligned} \tag{1.4}$$

for all $u\geq s$. Here K_ω and ω are as in (1.1) and $\omega>0$.

If we choose $x=0$ in (1.3), then we get $\varphi(0)=0$, and thus from (1.4) we obtain

$$\begin{aligned} N(x)\varphi\left(\frac{\|U(t,s)x\|}{K_\omega e^{\omega N}}\right) &= \int_s^\infty \varphi\left(\frac{1_{[t-N,t]}(u)\|U(t,s)x\|}{K_\omega e^{\omega N}}\right)du \\ &\leq \int_s^\infty \varphi\left(\|U(u,s)x\|\right)du \leq M_\varphi(x). \end{aligned} \tag{1.5}$$

We assume that $\varphi(1)=1$ (if not, we replace φ be some multiple of itself). Moreover, we may assume that φ is a strictly increasing map. Indeed if $\varphi(1)=1$ and $a:=\int_0^1\varphi(t)\,dt$, then the function given by

$$\bar{\varphi}(t)=\begin{cases}\int_0^t\varphi(u)\,du, & \text{if } 0\leq t\leq 1\\[2ex] \dfrac{at}{at+1-a}, & \text{if } t>1\end{cases}$$

is strictly increasing and $\bar{\varphi} \leq \varphi$. Now φ can be replaced by some multiple of $\bar{\varphi}$. From (1.5) it follows that if $t \geq s + N(x)$ and $x \in X$, then

$$\|U(t,s)\| \leq K_\omega e^{\omega N(x)}, \quad \text{for all } x \in X.$$

Using this inequality and the exponential boundedness of the evolution family, we have that

$$\sup_{t \geq \xi \geq 0} \|U(t,\xi)x\| \leq K_\omega e^{\omega N(x)}, \quad \text{for each } x \in X. \tag{1.6}$$

The conclusion of Lemma 1 follows from (1.6) and the Uniform Boundedness Theorem. ∎

Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators on X. We consider the strongly continuous evolution semigroup associated to $\mathcal{U}$ on $C_{00}(\mathbb{R}_+, X)$. This semigroup is defined by

$$(\mathfrak{T}(t)f)(s) := \begin{cases} U(s,s-t)f(s-t), & \text{if} \quad s \geq t \\ 0, & \text{if} \quad 0 \leq s \leq t \end{cases}, \ t \geq 0 \tag{1.7}$$

for all $f \in C_{00}(\mathbb{R}_+, X)$. It is known that $\mathfrak{T} = \{\mathfrak{T}(t) : t \geq 0\}$ is a strongly continuous semigroup and in addition $\omega_0(\mathfrak{T}) < 0$ if and only if $\mathcal{U}$ is uniformly exponentially stable ([10], [11], [12]).

Proof of Theorem 1. Let φ be as in Theorem 1. We assume that $\varphi(1) = 1$. Then

$$\Phi(t) := \int_0^t \varphi(u)\,du \leq \varphi(t) \quad \text{for all } t \in [0,1].$$

Without loss of generality we may assume that

$$\sup_{t \geq 0} \|\mathfrak{T}(t)\| \leq 1,$$

where $\mathfrak{T}$ is the semigroup defined in (1.7). Then for all $f \in C_{00}(\mathbb{R}_+, X)$ with $\|f\|_\infty \leq 1$, one has

$$\begin{aligned}
&\int_0^\infty \Phi\left(\|\mathfrak{T}(t)f\|_{C_{00}(\mathbb{R}_+,X)}\right)dt \\
= &\int_0^\infty \Phi\left(\sup_{s \geq t}\|U(s,s-t)f(s-t)\|\right)dt = \int_0^\infty \Phi\left(\sup_{\xi \geq 0}\|U(t+\xi,\xi)f(\xi)\|\right)dt \\
= &\int_0^\infty \left(\int_0^\infty 1_{\left[0,\sup\limits_{\xi \geq 0}\|U(t+\xi,\xi)f(\xi)\|\right]}(u)\,\varphi(u)\,du\right)dt \\
= &\sup_{\xi \geq 0}\int_0^\infty \left(\int_0^\infty 1_{[0,\|U(t+\xi,\xi)f(\xi)\|]}(u)\,\varphi(u)\,du\right)dt \\
= &\sup_{\xi \geq 0}\int_0^\infty \Phi(\|U(t+\xi,\xi)f(\xi)\|)\,dt \leq \sup_{\xi \geq 0}\int_0^\infty \varphi(\|U(t+\xi,\xi)f(\xi)\|)\,dt \\
= &\sup_{\xi \geq 0}\int_\xi^\infty \varphi(\|U(\tau,\xi)f(\xi)\|)\,d\tau \leq M_\varphi < \infty,
\end{aligned}$$

where $1_{[0,h]}$ denotes the characteristic function of the interval $[0,h]$, $h > 0$.

Now, from [7, Theorem 3.2.2], it follows that $\omega_0(\mathfrak{T}) < 0$, hence $\mathcal{U}$ is uniformly exponentially stable. ∎

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