A GENERALISATION OF CERONE’S IDENTITY AND APPLICATIONS

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Abstract. An identity due to P. Cerone for the Čebyšev functional is extended for Stieltjes integrals. A sharp inequality and its application in approximating Stieltjes integrals are also given.

1. Introduction

In 2001, P. Cerone [1] established the following identity for the Čebyšev functional:

\[
T(f,g) := \frac{1}{b-a} \int_a^b f(t)g(t)\,dt - \frac{1}{b-a} \int_a^b f(t)\,dt \cdot \frac{1}{b-a} \int_a^b g(t)\,dt = \frac{1}{(b-a)^2} \int_a^b \left( (t-a) \int_t^b g(s)\,ds - (b-t) \int_a^t g(s)\,ds \right) df(t),
\]

provided \(f\) is of bounded variation on \([a,b]\) and \(g\) is continuous on \([a,b]\). He proved (1.1) on utilising the auxiliary function \(\Psi : [a,b] \to \mathbb{R}\),

(1.2) \(\Psi (t) := (t-a) \int_t^b g(s)\,ds - (b-t) \int_a^t g(s)\,ds\)

and integrating by parts in the Stieltjes integral \(\int_a^b \Psi (t) \, df(t)\), which exists, since \(f\) is of bounded variation and \(\Psi\) is differentiable on \((a,b)\).

One may observe that the result remains valid if one assumes that \(g\) is Lebesgue integrable on \([a,b]\) and \(f\) is of bounded variation. This follows by the fact that, in this case \(\Psi\) becomes absolutely continuous on \([a,b]\), the Stieltjes integral \(\int_a^b \Psi (t) \, df(t)\) still exists and the argument will follow as in [1].

The weighted version of this inequality has been obtained in the same paper [1] and can be stated as:

\[
T(f,g;p) := \frac{1}{\int_a^b p(s)\,ds} \int_a^b p(t) f(t) g(t)\,dt - \frac{1}{\int_a^b p(s)\,ds} \int_a^b p(t) f(t)\,dt \cdot \frac{1}{\int_a^b p(s)\,ds} \int_a^b p(t) g(t)\,dt
\]

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\[
\frac{1}{\left( \int_a^b p(s) \, ds \right)^2} \int_a^b \left[ \int_a^t p(s) \, ds \int_t^b p(s) g(s) \, ds \\
- \int_t^b p(s) \, ds \int_a^t p(s) g(s) \, ds \right] df(t),
\]
provided \( f \) is of bounded variation on \([a, b]\) and \( p, g \) are continuous on \([a, b]\) with \( \int_a^b p(s) \, ds > 0 \). The same remark for the extension of the identity in the case that \( p, g \) are Lebesgue integrable on \([a, b]\) so that \( pg \) is also integrable, may apply.

The above two identities have been applied in [1] to obtain some interesting new bounds for the Čebyšev functionals \( T(f, g) \) and \( T(f, g; p) \) from which we only mention the following:

\[(1.4) \quad |T(f, g)| \leq \frac{1}{(b-a)^2} \times \left\{ \begin{array}{ll}
\sup_{t \in [a,b]} |\Psi(t)| \sqrt{a^b(f)} ; & \\
L \int_a^b |\Psi(t)| \, dt & \text{for } f \text{ Lipschitzian}; \\
\int_a^b |\Psi(t)| \, df(t) & \text{for } f \text{ monotonically nondecreasing},
\end{array} \right.\]

where \( \sqrt{a^b(f)} \) is the total variation of \( f \) on \([a, b]\), \( \Psi(t) \) is given by \((1.2)\), and

\[(1.5) \quad |T(f, g; p)| \leq \frac{1}{\left( \int_a^b p(s) \, ds \right)^2} \times \left\{ \begin{array}{ll}
\sup_{t \in [a,b]} |\Psi_p(t)| \sqrt{a^b(f)} ; & \\
L \int_a^b |\Psi_p(t)| \, dt & \text{if } f \text{ Lipschitzian}; \\
\int_a^b |\Psi_p(t)| \, df(t) & \text{for } f \text{ monotonically nondecreasing},
\end{array} \right.\]

where in this case the weighted auxiliary mapping \( \Psi_p \) is defined as \( \Psi_p : [a, b] \to \mathbb{R} \),

\[\Psi_p(t) := \int_a^t p(s) \, ds \int_t^b p(s) g(s) \, ds - \int_t^b p(s) \, ds \int_a^t p(s) g(s) \, ds.\]

For other inequalities and applications for moments, see [1].

For further results, see the follow up paper [2] where various lower and other upper bounds were established.

2. A Related Functional

In [3], the authors have considered the following functional

\[(2.1) \quad D(f; u) := \int_a^b f(x) \, du(x) - [u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) \, dt,\]

provided that the Stieltjes integral \( \int_a^b f(x) \, du(x) \) exists.

This functional plays an important role in approximating the Stieltjes integral \( \int_a^b f(x) \, du(x) \) in terms of the Riemann integral \( \int_a^b f(t) \, dt \) and the divided difference of the integrator \( u \). Therefore, further bounds on \( D(f; u) \) will generate a flow of different error estimates for the approximation of the Stieltjes integral that plays an
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important role in various fields of Analysis, Numerical Analysis, Integral Operator Theory, Probability & Statistics and other fields of Modern Mathematics.

In [4], the following result in estimating the above functional \( D(f; u) \) has been obtained:

\[
|D(f; u)| \leq \frac{1}{2} L (M - m) (b - a),
\]

provided \( u \) is \( L \)-Lipschitzian and \( f \) is Riemann integrable and with the property that there exists the constants \( m, M \in \mathbb{R} \) such that

\[
m \leq f(x) \leq M \quad \text{for any} \quad x \in [a, b].
\]

The constant \( \frac{1}{2} \) is best possible in (2.2) in the sense that it cannot be replaced by a smaller quantity.

If one assumes that \( u \) is of bounded variation and \( f \) is \( K \)-Lipschitzian, then \( D(f, u) \) satisfies the inequality [5]

\[
|D(f; u)| \leq \frac{1}{2} K (b - a) \bigvee_a^b (u).
\]

Here the constant \( \frac{1}{2} \) is also best possible.

The above inequalities have been used in [4] and [5] for obtaining inequalities between special means and on estimating the error in approximating the Stieltjes integral \( \int_a^b f(x) du(x) \) in terms of the Riemann integral for the function \( f \) and the divided difference of \( u \).

Now, for the function \( u : [a, b] \to \mathbb{R} \), consider the following auxiliary mappings \( \Phi, \Gamma \) and \( \Delta \) [3]:

\[
\Phi(t) := \frac{(t - a) u(b) + (b - t) u(a)}{b - a} - u(t), \quad t \in [a, b],
\]

\[
\Gamma(t) := (t - a) \left[ u(b) - u(t) \right] - (b - t) \left[ u(t) - u(a) \right], \quad t \in [a, b],
\]

\[
\Delta(t) := [u; b, t] - [u; t, a], \quad t \in (a, b),
\]

where \([u; \alpha, \beta]\) is the divided difference of \( u \) in \( \alpha, \beta \), i.e.,

\[
[u; \alpha, \beta] := \frac{u(\alpha) - u(\beta)}{\alpha - \beta}.
\]

The following representation of \( D(f, u) \) may be stated.

**Theorem 1.** Let \( f, u : [a, b] \to \mathbb{R} \) be such that the Stieltjes integral \( \int_a^b f(t) du(t) \) and the Riemann integral \( \int_a^b f(t) dt \) exist. Then

\[
D(f, u) = \int_a^b \Phi(t) df(t) = \frac{1}{b - a} \int_a^b \Gamma(t) df(t)
\]

\[
= \frac{1}{b - a} \int_a^b (t - a) (b - t) \Delta(t) df(t).
\]
Proof. Since \( \int_a^b f(t) \, du(t) \) exists, hence \( \int_a^b \Phi(t) \, df(t) \) also exists, and the integration by parts formula for Stieltjes integrals gives that

\[
\int_a^b \Phi(t) \, df(t) = \left[ \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t) \right] \bigg|_a^b df(t)
\]

\[
= \left[ \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t) \right] f(t) \bigg|_a^b
\]

\[
- \int_a^b f(t) \, d \left[ \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t) \right]
\]

\[
= - \int_a^b f(t) \left[ \frac{u(b) - u(a)}{b-a} \right] dt - du(t) = D(f,u),
\]

proving the required identity. \( \square \)

Remark 1. The identity (2.5) has been established in \( [3] \). There were some typographical errors in \( [3] \) that have been corrected above.

Remark 2. If \( u \) is an integral, i.e., \( u(t) = \int_a^t g(s) \, ds, \, t \in [a,b], \) then

\[
\Phi(t) = \frac{t-a}{b-a} \int_a^b g(s) \, ds - \int_a^t g(s) \, ds,
\]

\[
\Gamma(t) = (t-a) \int_a^b g(s) \, ds - (b-t) \int_a^t g(s) \, ds,
\]

\[
\Delta(t) = \frac{\int_a^b g(s) \, ds}{b-t} - \frac{\int_a^t g(s) \, ds}{t-a},
\]

and then, from (2.5), one may recapture Cerone’s identity (1.1) for the Čebyšev functional \( T(f,g) \).

Since it well known that \( u \) is an integral if and only if \( u \) is absolutely continuous, and in this case \( g(s) = u'(s) \) for \( s \in [a,b], \) hence (2.5) is indeed a proper generalisation of (1.1) holding for larger classes of functions than the absolutely continuous functions.

Remark 3. If one chooses \( u: [a,b] \rightarrow \mathbb{R}, \)

\[
u(t) = \frac{\int_a^t p(s) g(s) \, ds}{\int_a^b p(s) \, ds}, \quad t \in [a,b],
\]

where \( p, g \) are Lebesgue integrable with \( pq \) is also integrable and \( \int_a^b p(s) \, ds \neq 0, \) then the identity (2.5) produces the representation:

\[
E(f,g;p) := \int_a^b p(s) f(s) g(s) \, ds \int_a^b p(s) \, ds - \int_a^b p(s) g(s) \, ds \int_a^b p(s) \, ds \cdot \frac{1}{b-a} \int_a^b f(t) \, dt
\]

\[
= \int_a^b \Phi_p(t) \, df(t) = \frac{1}{b-a} \int_a^b \Gamma_p(t) \, dt(t)
\]

\[
= \frac{1}{b-a} \int_a^b (t-a)(b-t) \, \Delta_p(t) \, dt(t),
\]
where
\[ \Phi_p(t) := \frac{t-a}{b-a} \int_a^b p(s) g(s) \, ds - \frac{t^p - a^p}{b^p - a^p} \int_a^b p(s) g(s) \, ds, \]
\[ \Gamma_p(t) := (t-a) \int_a^b p(s) g(s) \, ds - (b-t) \int_a^b p(s) g(s) \, ds, \]
and
\[ \Delta_p(t) := \frac{t^p - a^p}{b-a} \int_a^b p(s) g(s) \, ds - \frac{t^p - a^p}{b-a} \int_a^b p(s) g(s) \, ds. \]

One must observe that the identity (2.6) is not the same as Cerone’s identity for weighted integrals (1.3).

For recent inequalities related to \( D(f;u) \) for various pairs of functions \((f,u)\), see [3, pp. 112-118].

3. A Bound for \( f \) of Bounded Variation and \( u \) Continuous

It is known that if \( u \) is continuous on \([a,b]\) and \( f \) is of bounded variation on \([a,b]\), then the Stieltjes integral \( \int_a^b f(t) \, du(t) \) exists. This integral may exists even for larger classes of integrators \( f \), for instance, piecewise continuous functions for which the discontinuities of the integrand \( f \) do not overlap with those of the integrator \( u \).

The following result may be stated:

**Theorem 2.** Let \( f : [a,b] \rightarrow \mathbb{R} \) be of bounded variation on \([a,b]\) and \( u : [a,b] \rightarrow \mathbb{R} \) such that there exist the constants \( \gamma, \Gamma \in \mathbb{R} \) with:

\[ \gamma \leq u(t) \leq \Gamma \quad \text{for any} \quad t \in [a,b] \]

and the Stieltjes integral \( \int_a^b f(t) \, du(t) \) exists. Then

\[ |D(f;u)| \leq (\Gamma - \gamma) \sup_{t \in [a,b]} |u(t)|. \]

The multiplicative constant 1 in front of \( \Gamma - \gamma \) cannot be replaced by a smaller quantity.

**Proof.** By (1.1), we obviously have:

\[ \gamma (b-t) \leq (b-t) u(a) \leq (b-t) \Gamma, \]
\[ \gamma (t-a) \leq (t-a) u(b) \leq (t-a) \Gamma, \]
\[ -(b-a) \Gamma \leq -(b-a) u(t) \leq -(b-a) \gamma, \]

which gives by addition and division with \( b-a \) that

\[ -(\Gamma - \gamma) \leq \frac{(b-t) u(a) + (t-a) u(b)}{b-a} - u(t) \leq \Gamma - \gamma, \]

showing that \( |\Phi(t)| \leq \Gamma - \gamma \) for any \( t \in [a,b] \).

Taking into account that for \( \varphi \) bounded and \( \psi \) of bounded variation on \([a,b]\) one has

\[ \left| \int_a^b \varphi(t) \, d\psi(t) \right| \leq \sup_{t \in [a,b]} |\varphi(t)| \sup_{t \in [a,b]} |\psi(t)|, \]
provided the Stieltjes integral exists, we have by (2.5) that
\[|D (f; u)| \leq \sup_{t \in [a, b]} |\phi(t)| \int_{a}^{b} (f) \leq (\Gamma - \gamma) \int_{a}^{b} (f),\]
proving the required inequality (3.2).

Now, for the sharpness of the inequality.

Assume that there exists a \( c > 0 \) such that
\[(3.3)\]
\[|D (f; u)| \leq c (\Gamma - \gamma) \int_{a}^{b} (f),\]
where \( u \) and \( f \) are as in the hypothesis of the theorem.

Consider \( u, f : [a, b] \to \mathbb{R} \) with
\[u(t) = \frac{1}{2} \left( t - \frac{a + b}{2} \right)^2, \quad f(t) = \text{sgn} \left( t - \frac{a + b}{2} \right), \quad t \in [a, b].\]

Then \( u \) is continuous, \( f \) is of bounded variation, the integral \( \int_{a}^{b} f(t) \, du(t) \) exists and
\[\int_{a}^{b} (f) = 2, \quad \int_{a}^{b} f(t) \, dt = 0,\]
\[\Gamma = \sup_{t \in [a, b]} u(t) = \frac{(b - a)^2}{8}, \quad \gamma = \inf_{t \in [a, b]} u(t) = 0,\]
\[
\int_{a}^{b} f(t) \, du(t) = \int_{a}^{b} \text{sgn} \left( t - \frac{a + b}{2} \right) \left( t - \frac{a + b}{2} \right) \, dt = \int_{a}^{b} \left| t - \frac{a + b}{2} \right| \, dt = \frac{(b - a)^2}{4}.
\]
Substituting into (3.3) we get \( \frac{(b - a)^2}{4} \leq c (\frac{(b - a)^2}{4}) \), which implies that \( c \geq 1 \). \( \blacksquare \)

**Corollary 1.** Let \( f : [a, b] \to \mathbb{R} \) be of bounded variation and \( u : [a, b] \to \mathbb{R} \) continuous on \([a, b]\). Then:
\[(3.4)\]
\[|D (f; u)| \leq \left[ \max_{t \in [a, b]} u(t) - \min_{t \in [a, b]} u(t) \right] \int_{a}^{b} (f).
\]
The inequality (3.4) is sharp.

If we consider the Čebyshev functional \( T(f, g) \), then we can state the following corollary as well:

**Corollary 2.** Let \( f : [a, b] \to \mathbb{R} \) be of bounded variation and \( g : [a, b] \to \mathbb{R} \) a Lebesgue integrable function such that there exists the constants \( m \) and \( M \) with
\[(3.5)\]
\[m \leq g(s) \leq M \text{ for a.e. } s \in [a, b].\]
Then
\[(3.6)\]
\[|T(f, g)| \leq (b - a) (M - m) \int_{a}^{b} (f).
\]
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Proof. We choose \( u(t) := \int_a^t g(s) \, ds \) which is continuous on \([a, b]\) and satisfies the inequality (3.1) with \( \gamma = (b-a) m \) and \( \Gamma = (b-a) M \) and apply Theorem 2.

**Remark 4.** If we assume that for the Lebesgue integrable function \( g, \int_a^t g(s) \, ds \) satisfies the condition

\[
\gamma \leq \int_a^t g(s) \, ds \leq \Gamma \quad \text{for any} \quad t \in [a, b],
\]

then

\[
|T(f, g)| \leq (\Gamma - \gamma) \left( \frac{b}{a} \right) f(t)
\]

and the inequality is sharp. The equality case is realised for \( g(t) = t - \frac{a+b}{2} \) and \( f(t) = \text{sgn} \left( t - \frac{a+b}{2} \right), \quad t \in [a, b]. \)

It is an open problem whether or not the bound in (3.6) is sharp.

**Remark 5.** If \( p, g \in L[a, b] \) so that \( pg \in L[a, b] \) and \( \int_a^b p(s) \, ds \neq 0 \) and there exists the constants \( \delta, \Delta \) so that

\[
\delta \leq \frac{\int_a^t p(s) g(s) \, ds}{\int_a^b p(s) \, ds} \leq \Delta
\]

for any \( t \in [a, b] \), then

\[
|E(f, g; p)| \leq (\Delta - \delta) \left( \frac{b}{a} \right) f(t).
\]

The last inequality is sharp.

4. APPLICATION FOR APPROXIMATING THE STEILOVS INTEGRAL

Let us consider the partition of the interval \([a, b]\) given by

\[ I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b. \]

Denote \( v(I_n) := \max \{ h_i | i = 0, \ldots, n - 1 \} \), where \( h_i := t_{i+1} - t_i, \quad i = 0, \ldots, n - 1. \)

If \( u : [a, b] \rightarrow \mathbb{R} \) is continuous on \([a, b]\) and if we define

\[
M_i := \sup_{t \in [t_i, t_{i+1}]} u(t), \quad m_i := \inf_{t \in [t_i, t_{i+1}]} u(t)
\]

and

\[
v(u, I_n) := \max_{0 \leq i \leq n-1} (M_i - m_i),
\]

then, obviously, by the continuity of \( u \) on \([a, b]\), for any \( \varepsilon \geq 0 \), there exists a \( \delta > 0 \) and a division \( I_n \) with norm \( v(I_n) < \delta \) such that \( v(u, I_n) < \varepsilon. \)

Consider now the quadrature rule

\[
S_n(f, u, I_n) := \sum_{i=0}^{n-1} \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \cdot \int_{t_i}^{t_{i+1}} \frac{f(t)}{t_{i+1} - t_i} \, dt,
\]

provided \( u \) is continuous on \([a, b]\) and \( f \) is of bounded variation on \([a, b]\).

We may state the following result in approximating the Stieltjes integral:
Theorem 3. Let \( f, u : [a, b] \to \mathbb{R} \) be such that \( f \) is of bounded variation on \([a, b]\) and \( u \) is continuous on \([a, b]\). Then for any division \( I_n \) as above,

\[
\int_a^b f(t) \, du(t) = S_n(f, I_n) + R_n(f, u, I_n),
\]

where the remainder \( R_n(f, u, I_n) \) satisfies the estimate:

\[
|R_n(f, u, I_n)| \leq v(u, I_n) \bigvee_{a}^{b} (f).
\]

Proof. Applying Theorem 2 on the intervals \([t_i, t_{i+1}]\), \( i = 0, \ldots, n - 1 \), we have successively:

\[
|R_n(f, u, I_n)| = \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t) \, du(t) - \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f(t) \, dt \right|
\]

\[
\leq \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} f(t) \, du(t) - \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f(t) \, dt \right|
\]

\[
\leq \sum_{i=0}^{n-1} (M_i - m_i) \bigvee_{t_i}^{t_{i+1}} (f) \leq v(u, I_n) \bigvee_{a}^{b} (f)
\]

and the estimate (4.3) is obtained. \( \square \)

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