SOME MONOTONICITY PROPERTIES OF THE q-GAMMA FUNCTION

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Abstract. We prove some properties of completely monotonic functions and apply them to obtain new results on gamma and q-gamma functions.

1. Introduction

The q-gamma function is defined for positive real numbers x and $q \neq 1$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \ 0<q<1;$$

$$\Gamma_q(x) = (q-1)^{1-x} q^{\frac{1}{2}x(x-1)} \prod_{n=0}^{\infty} \frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \ q>1.$$

We note here[11]

$$\lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x) = \int_0^{\infty} t^x e^{-t} \frac{dt}{t},$$

the well-known Euler's gamma function. From the definition, for x positive and $0<q<1$,

$$\Gamma_{1/q}(x) = q^{(x-1)(1-x/2)} \Gamma_q(x),$$

we see that $\lim_{q \to 1} \Gamma_q(x) = \Gamma(x)$. For historical remarks on gamma and q-gamma functions, we refer the reader to [1], [2] and [11].

There exists an extensive and rich literature on inequalities for the gamma and q-gamma functions. For the recent developments in this area, we refer the reader to the articles [1]-[3], [8], [15] and the references therein. Many of these inequalities follow from the monotonicity properties of functions which are closely related to Γ(resp. Γ_q) and its logarithmic derivative ψ(resp. ψ_q). Here we recall that a function $f(x)$ is said to be absolutely monotonic on (a,b) if it has derivatives of all orders and $f^{(k)}(x) \geq 0, x \in (a,b), k \in \mathbb{N}$. A function $f(x)$ is said to be completely monotonic on (a,b) if it has derivatives of all orders and $(-1)^k f^{(k)}(x) \geq 0, x \in (a,b), k \in \mathbb{N}$.

We note here that $\lim_{q \to 1} \psi_q(x) = \psi(x)$(see [12]) and that ψ' and ψ'_q are completely monotonic functions on $(0,\infty)$(see [3], [9]). Thus, one expects to deduce results on gamma and q-gamma functions from properties of completely monotonic functions, by applying them to functions related to ψ' or ψ'_q. It is our goal in this paper to obtain some results on gamma and q-gamma functions via this approach. Our key tool is Lemma 2.2 below and we first illustrate here three examples which only use the fact that ψ'(resp. ψ'_q) is positive and decreasing on $(0,\infty)$. For instance, for positive numbers a,x,y, Lemma 2.2 implies

$$\psi(a) + \psi(a+x+y) \leq \psi(a+x) + \psi(a+y),$$

which is discussed in [4, p. 59]. Similarly, one checks easily that $\Gamma^{\alpha}(x)$ is convex on $(0,\infty)$ for $\alpha \geq 0$. Hence it follows from Lemma 2.2 that for positive numbers x,y,z, $\alpha \geq 0$,

$$\Gamma^{\alpha}(x+y) + \Gamma^{\alpha}(x+z) \leq \Gamma^{\alpha}(x) + \Gamma^{\alpha}(x+y+z),$$

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which is (4.3) in [5]. As another example, we note Alzer[3, Lemma 2.4] has shown that $\psi(e^x)$ is strictly concave on $\mathbb{R}$. It follows from this and Lemma 2.2 that for positive numbers x, y and real numbers r, s with $r + s \neq 0$,

$$
\begin{aligned}
\psi(x)+\psi(y) &\leq \psi(E(r,s;x,y))+\psi(E(-r,-s;x,y)) \leq 2\psi(\sqrt{xy}), \\
\psi(x)+\psi(y) &\leq \psi(G(r,s;x,y))+\psi(G(-r,-s;x,y)) \leq 2\psi(\sqrt{xy}).
\end{aligned} \tag{1.1}
$$

Other than the cases of equalities, the above is Theorem 3.7 in [3]. We shall only need to use the inequality $\psi(x)+\psi(y) \leq 2\psi(\sqrt{xy})$ in our subsequent discussions, so we will omit the definitions of $E(r,s;x,y)$ and $G(r,s;x,y)$ here and refer the reader to [3].

2. Lemmas

Lemma 2.1. *([1, Lemma 1]) If $f'(x)$ is completely monotonic on $(0,\infty)$, then $\exp(-f(x))$ is also completely monotonic on $(0,\infty)$.*

Lemma 2.2. *Let a_i and b_i $(i = 1,\dots,n)$ be real numbers such that $0 < a_1 \leq \cdots \leq a_n$, $0 < b_1 \leq \cdots \leq b_n$, and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1,\dots,n$. If the function $f(x)$ is decreasing and convex on $(0,\infty)$, then*

$$\sum_{i=1}^n f(b_i) \leq \sum_{i=1}^n f(a_i).$$

If $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, then one only needs $f(x)$ being convex for the above inequality to hold.

The above lemma is similar to Lemma 2 in [1], except here we only assume a_i, b_i's to be positive and $f(x)$ defined on $(0,\infty)$. We leave the proof to the reader by pointing out that it follows from the theory of majorization, for example, see the discussions in Chap. 1, §28 – §30 of [6].

Lemma 2.3. *(Hadamard's inequality) Let $f(x)$ be a convex function on $[a,b]$, then*

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Lemma 2.4. *([2, Lemma 2.1]) Let $a > 0, b > 0$ and r be real numbers with $a \neq b$, and let*

$$
\begin{aligned}
L_r(a,b) &= (\frac{a^r-b^r}{r(a-b)})^{1/(r-1)} \quad (r \neq 0,1), \\
L_0(a,b) &= \frac{a-b}{\log a - \log b}, \\
L_1(a,b) &= \frac{1}{e}(\frac{a^a}{b^b})^{1/(a-b)}.
\end{aligned}
$$

The function $r \mapsto L_r(a,b)$ is strictly increasing on $\mathbb{R}$.

3. Main Results

Theorem 3.1. *Let a_i and b_i $(i = 1,\dots,n)$ be real numbers such that $0 \leq a_1 \leq \cdots \leq a_n$, $0 \leq b_1 \leq \cdots \leq b_n$, and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1,\dots,n$. If $f''(x)$ is completely monotonic on $(0,\infty)$, then*

$$\exp\Big(\sum_{i=1}^n \big(f(x+a_i)-f(x+b_i)\big)\Big)$$

is completely monotonic on $(0,\infty)$.

Proof. By Lemma 2.1, it suffices to show that

$$-\sum_{i=1}^{n}\big(f'(x+a_i)-f'(x+b_i)\big)$$

is completely monotonic on $(0,\infty)$ or for $k\geq 1$,

$$(-1)^k\sum_{i=1}^{n}f^{(k)}(x+a_i)\geq(-1)^k\sum_{i=1}^{n}f^{(k)}(x+b_i).$$

By Lemma 2.2, it suffices to show that $(-1)^kf^{(k)}(x)$ is decreasing and convex on $(0,\infty)$ or equivalently, $(-1)^kf^{(k+1)}(x)\leq 0$ and $(-1)^kf^{(k+2)}(x)\geq 0$ for $k\geq 1$. The last two inequalities hold since we assume that $f''(x)$ is completely monotonic on $(0,\infty)$. This completes the proof. □

As a direct consequence of Theorem 3.1, we now generalize a result of Alzer[1, Theorem 10], we note here one can also put our next result into a form similar to that of Theorem 4.1 in [8], we leave this to the reader.

Corollary 3.1. *Let a_i and b_i $(i=1,\dots,n)$ be real numbers such that $0\leq a_1\leq\cdots\leq a_n$, $0\leq b_1\leq\cdots\leq b_n$, and $\sum_{i=1}^k a_i\leq\sum_{i=1}^k b_i$ for $k=1,\dots,n$. Then,*

$$x\mapsto\prod_{i=1}^{n}\frac{\Gamma_q(x+a_i)}{\Gamma_q(x+b_i)}$$

is completely monotonic on $(0,\infty)$.

Proof. Apply Theorem 3.1 to $f(x)=\log\Gamma_q(x)$ and note that $f''(x)=\psi_q'(x)$ is completely monotonic on $(0,\infty)$ and this completes the proof. □

Theorem 3.2. *Let $f''(x)$ be completely monotonic on $(0,\infty)$, then for $0\leq s\leq 1$, the functions*

$$\begin{aligned} x &\mapsto \exp\Big(-\big(f(x+1)-f(x+s)-(1-s)f'(x+\frac{1+s}{2})\big)\Big),\\ x &\mapsto \exp\big(f(x+1)-f(x+s)-\frac{(1-s)}{2}(f'(x+1)+f'(x+s))\big)\end{aligned}$$

are completely monotonic on $(0,\infty)$.

Proof. We may assume $0\leq s<1$. We will prove the first assertion and the second one can be shown similarly. By Lemma 2.1, it suffices to show that

$$f'(x+1)-f'(x+s)-(1-s)f''(x+\frac{1+s}{2})$$

is completely monotonic on $(0,\infty)$ or for $k\geq 1$,

$$\frac{1}{1-s}\int_{x+s}^{x+1}(-1)^{k+1}f^{(k+1)}(t)dt\geq(-1)^{k+1}f^{(k+1)}(x+\frac{1+s}{2}).$$

The last inequality holds by Lemma 2.3 and our assumption that $f''(x)$ is completely monotonic on $(0,\infty)$. This completes the proof. □

Corollary 3.2. *For $0\leq s\leq 1$, the functions*

$$\begin{aligned} x &\mapsto \frac{\Gamma_q(x+s)}{\Gamma_q(x+1)}\exp\big((1-s)\psi_q(x+\frac{1+s}{2})\big),\\ x &\mapsto \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)}\exp\big(-\frac{(1-s)}{2}(\psi_q(x+1)+\psi_q(x+s))\big)\end{aligned}$$

are completely monotonic on $(0,\infty)$.

Proof. Apply Theorem 3.2 to $f(x) = \log \Gamma_q(x)$ and note that $f''(x) = \psi_q'(x)$ is completely monotonic on $(0, \infty)$ and this completes the proof. □

By applying Lemma 2.3 to $f(x) = -\psi_q(x)$, we obtain

Theorem 3.3. *For positive x and $0 \leq s \leq 1$,*

$$\exp\big(\frac{(1-s)}{2}(\psi_q(x+1)+\psi_q(x+s))\big) \leq \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \leq \exp\big((1-s)\psi_q(x+\frac{1+s}{2})\big).$$

The upper bound in Theorem 3.3 is due to Ismail and Muldoon[8]. Our proof here is similar to that of Corollary 3 in [13]. We further note the following integral analogue of Theorem 3.15 in [3]:

$$\psi(L_0(b,a)) \leq \frac{1}{b-a}\int_a^b \psi(x)dx \leq \psi(L_1(b,a)), \quad b > a > 0.$$

It follows from this that for positive x and $0 \leq s \leq 1$,

$$\exp\big((1-s)\psi(L_0(x+1,x+s))\big) \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\big((1-s)\psi(L_1(x+1,x+s))\big).$$

By Lemma 2.4, observing that $L_{-1}(x+1,x+s) = \sqrt{(x+1)(x+s)}$ and $L_2(x+1,x+s) = x+(1+s)/2$, we obtain

$$\exp\big((1-s)\psi(\sqrt{(x+1)(x+s)})\big) \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\big((1-s)\psi(x+(1+s)/2)\big). \tag{3.1}$$

Note by (1.1),

$$\psi(x+1)+\psi(x+s) \leq 2\psi(\sqrt{(x+1)(x+s)}),$$

also note that $\psi(x)$ is an increasing function on $(0, \infty)$ and $\sqrt{(x+1)(x+s)} \geq x + s^{1/2}$, we see that the inequalities in (3.1) refine the case $q \to 1$ in Theorem 3.3 and the following result of Kershaw[10], which states that for positive x and $0 \leq s \leq 1$,

$$\exp\big(\frac{(1-s)}{2}\psi(x+s^{1/2})\big) \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\big((1-s)\psi(x+\frac{1+s}{2})\big).$$

We now show the lower bound above and the corresponding one of the case $q \to 1$ in Theorem 3.3 are not comparable in general(see p. 856, [7] for a similar discussion). In fact, on letting $x \to 0$ and by Theorem 3.7 of [3], we have

$$\psi(1)+\psi(s) < 2\psi(s^{1/2}), \ 0 < s < 1.$$

On the other hand, using the well-known series representation(see, for example, [8, (1.8)]):

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty}(\frac{1}{n+1} - \frac{1}{x+n})$$

with $\gamma = 0.57721\ldots$ denoting Euler's constant, we obtain for $x > 1$,

$$\psi(x+1)+\psi(x+s)-2\psi(x+s^{1/2}) = \sum_{n=0}^{\infty}\frac{(1-s^{1/2})^2(x+n-s^{1/2})}{(x+n+1)(x+n+s)(x+n+s^{1/2})} > 0.$$

We end our paper by answering a question of Qi in [14], stated as: If $f(x)$ is an absolutely or completely monotonic function on the interval $(-\infty, +\infty)$, then the following inequality holds for $0 \leq x < +\infty$ or reverses for $-\infty < x \leq 0$:

$$E(x;f) := f^2(x)f^{'''}(x) - 3f(x)f^{'}(x)f^{''}(x) + 2(f^{'}(x))^3 \leq 0.$$

We point out here in general the above assertion is not true. As one can check easily that for any constant $a > 0$, $g(x) = e^x + a$ is an absolutely monotonic function on $(-\infty, +\infty)$ while $h(x) = e^{-x} + a$ is a completely monotonic function on $(-\infty, +\infty)$. However, $E(0;g) = -E(0;h) = a(a-1)$ which shows the falsity of the above assertion.

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