

ON SOME DISCRETE INEQUALITIES IN NORMED LINEAR SPACES

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ABSTRACT. Some sharp discrete inequalities in normed linear spaces are obtained. New reverses of the generalised triangle inequality are also given.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . The mapping $f : X \rightarrow \mathbb{R}$, $f(x) = \frac{1}{2} \|x\|^2$ is obviously convex on \mathbb{R} and then there exists the following limits:

$$\langle x, y \rangle_i := \lim_{t \rightarrow 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t},$$

$$\langle x, y \rangle_s := \lim_{\tau \rightarrow 0^-} \frac{\|y + \tau x\|^2 - \|y\|^2}{2\tau}$$

for any two vectors in X . The mapping $\langle \cdot, \cdot \rangle_s$ ($\langle \cdot, \cdot \rangle_i$) will be called the *superior semi-inner product* (*inferior semi-inner product*) associated to the norm $\|\cdot\|$.

The following fundamental calculus rules are valid for these semi-inner products (see for instance [4, p. 27–32]):

- (1.1) $\langle x, x \rangle_p = \|x\|^2$ for $x \in X$;
- (1.2) $\langle \lambda x, y \rangle_p = \lambda \langle x, y \rangle_p$ for $\lambda \geq 0$ and $x, y \in X$;
- (1.3) $\langle x, \lambda y \rangle_p = \lambda \langle x, y \rangle_p$ for $\lambda \geq 0$ and $x, y \in X$;
- (1.4) $\langle \lambda x, y \rangle_p = \lambda \langle x, y \rangle_q$ for $\lambda \leq 0$ and $x, y \in X$;
- (1.5) $\langle \alpha x, \beta y \rangle_p = \alpha \beta \langle x, y \rangle_p$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha \beta \geq 0$ and $x, y \in X$;
- (1.6) $\langle -x, y \rangle_p = \langle x, -y \rangle_p = -\langle x, y \rangle_q$ for $x, y \in X$;

where $p, q \in \{s, i\}$ and $p \neq q$.

The following inequality is valid:

$$(1.7) \quad \frac{\|y + tx\|^2 - \|y\|^2}{2t} \geq \langle x, y \rangle_s \geq \langle x, y \rangle_i$$

$$\geq \frac{\|y + sx\|^2 - \|y\|^2}{2s},$$

for any $x, y \in X$ and $s < 0 < t$.

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An important result is the following *Schwarz inequality*:

$$(1.8) \quad \left| \langle x, y \rangle_p \right| \leq \|x\| \|y\| \quad \text{for each } x, y \in X.$$

Also, the following properties of sub(super)-additivity should be noted:

$$(1.9) \quad \langle x_1 + x_2, y \rangle_{s(i)} \leq (\geq) \langle x_1, y \rangle_{s(i)} + \langle x_2, y \rangle_{s(i)}$$

for each $x_1, x_2, y \in X$.

Another important property of “quasi-linearity” holds as well:

$$(1.10) \quad \langle \alpha x + y, x \rangle_p = \alpha \|x\|^2 + \langle y, x \rangle_p$$

for any $x, y \in X$ and α a real number, where $p = s$ or $p = i$.

Finally, we mention the continuity property:

$$(1.11) \quad \left| \langle y + z, x \rangle_p - \langle z, x \rangle_p \right| \leq \|y\| \|x\|$$

for each $x, y, z \in X$ and $p = s$ or $p = i$.

One of the most used inequalities in normed spaces is the triangle inequality for several vectors, i.e.,

$$(1.12) \quad \left\| \sum_{j=1}^n x_j \right\| \leq \sum_{j=1}^n \|x_j\|$$

for any $x_j \in X$, $j \in \{1, \dots, n\}$.

The main aim of this paper is to point out some inequalities for norms of the vectors x_j and $\sum_{j=1}^n x_j$, including some reverses of the triangle inequality in the multiplicative form, i.e., lower bounds for the quantity

$$\frac{\left\| \sum_{j=1}^n x_j \right\|}{\sum_{j=1}^n \|x_j\|},$$

provided that not all x_j are zero and satisfy some appropriate conditions.

For classical results related to the reverse of the triangle inequality in normed spaces see [3], [7], [9] and [8]. For more recent results, see [5], [6], [1] and [2].

2. THE RESULTS

The following lemma is of interest itself as well.

Lemma 1. *Let $(X, \|\cdot\|)$ be a normed linear space. If $x, a \in X$, then*

$$(2.1) \quad \langle x, a \rangle_i \geq \frac{1}{2} \left(\|a\|^2 - \|x - a\|^2 \right).$$

If $\|a\| > \|x - a\|$, then the constant $\frac{1}{2}$ cannot be replaced by a larger quantity.

Proof. Utilising the semi-inner product properties, we have by (1.7) that

$$\langle x, a \rangle_i = \lim_{s \rightarrow 0^-} \frac{\|a + sx\|^2 - \|a\|^2}{2s} \geq \frac{\|a + (-1)x\|^2 - \|a\|^2}{2(-1)} = \frac{\|a\|^2 - \|x - a\|^2}{2}$$

and the inequality (2.1) is proved.

Now, assume that $\|a\| > \|x - a\|$ and there exists a $C > 0$ with the property that

$$(2.2) \quad \langle x, a \rangle_i \geq C \left(\|a\|^2 - \|x - a\|^2 \right).$$

Obviously $a \neq 0$, and if we choose $x = \varepsilon a$, $\varepsilon \in (0, 1)$, then $\|a\| > \|x - a\|$ since $\|x - a\| = (1 - \varepsilon)\|a\|$. Replacing x in (2.2) we get

$$\varepsilon \|a\|^2 \geq C \left(\|a\|^2 - (1 - \varepsilon)^2 \|a\|^2 \right)$$

giving

$$\varepsilon \geq C (2\varepsilon - \varepsilon^2),$$

for any $\varepsilon \in (0, 1)$. This is in fact $1 \geq C(2 - \varepsilon)$ and if we let $\varepsilon \rightarrow 0+$, we get $C \geq \frac{1}{2}$. ■

Remark 1. *As a coarser, but maybe more useful inequality, we can state that*

$$(2.3) \quad \langle x, a \rangle_i \geq \frac{1}{2} \|x\| (\|a\| - \|x - a\|),$$

provided $\|a\| \geq \|x - a\|$.

We observe that (2.3) follows from (2.1) since, for $\|a\| \geq \|x - a\|$, the triangle inequality gives:

$$\begin{aligned} \frac{1}{2} \left(\|a\|^2 - \|x - a\|^2 \right) &= \frac{1}{2} (\|a\| - \|x - a\|) (\|a\| + \|x - a\|) \\ &\geq \frac{1}{2} (\|a\| - \|x - a\|) \|x\|. \end{aligned}$$

It is an open question whether the constant $\frac{1}{2}$ in (2.3) is sharp.

The following result may be stated.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed space and $x_j \in X$, $j \in \{1, \dots, n\}$, $a \in X \setminus \{0\}$. Then for any $p_j \geq 0$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ we have*

$$(2.4) \quad \left\| \sum_{j=1}^n p_j x_j \right\| \|a\| + \frac{1}{2} \sum_{j=1}^n p_j \|x_j - a\|^2 \geq \frac{1}{2} \|a\|^2.$$

The constant $\frac{1}{2}$ in the right hand side of (2.4) is best possible in the sense that it cannot be replaced by a larger quantity.

Proof. We apply Lemma 1 on stating that

$$\langle x_j, a \rangle_i + \frac{1}{2} \|x_j - a\|^2 \geq \frac{1}{2} \|a\|^2$$

for each $j \in \{1, \dots, n\}$.

Multiplying with $p_j \geq 0$ and summing over j from 1 to n , we get

$$(2.5) \quad \sum_{j=1}^n p_j \langle x_j, a \rangle_i + \frac{1}{2} \sum_{j=1}^n p_j \|x_j - a\|^2 \geq \frac{1}{2} \|a\|^2 \sum_{j=1}^n p_j.$$

Utilising the superadditivity property of the semi-inner product $\langle \cdot, \cdot \rangle_i$ in the first variable (see [4, p. 29]) we have

$$(2.6) \quad \left\langle \sum_{j=1}^n p_j x_j, a \right\rangle_i \geq \sum_{j=1}^n p_j \langle x_j, a \rangle_i.$$

By the Schwarz inequality applied for $\sum_{j=1}^n p_j x_j$ and a , we also have

$$(2.7) \quad \left\| \sum_{j=1}^n p_j x_j \right\| \|a\| \geq \left\langle \sum_{j=1}^n p_j x_j, a \right\rangle_i.$$

Therefore, by (2.5) – (2.7) we deduce the desired inequality (2.4).

Now assume that there exists a $D > 0$ with the property that

$$(2.8) \quad \left\| \sum_{j=1}^n p_j x_j \right\| \|a\| + \frac{1}{2} \sum_{j=1}^n p_j \|x_j - a\|^2 \geq D \|a\|^2,$$

for any $n \geq 1$, $x_j \in X$, $p_j \geq 0$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ and $a \in X \setminus \{0\}$.

If in (2.8) we choose $n = 1$, $p_1 = 1$, $x_1 = \varepsilon a$, $\varepsilon \in (0, 1)$, then we get

$$\varepsilon \|a\|^2 + \frac{1}{2} (1 - \varepsilon)^2 \|a\|^2 \geq D \|a\|^2,$$

giving

$$\varepsilon + \frac{1}{2} (1 - \varepsilon)^2 \geq D,$$

for any $\varepsilon \in (0, 1)$. Letting $\varepsilon \rightarrow 0+$, we deduce $D \leq \frac{1}{2}$ and the proof is complete. ■

The following result may be stated as well:

Proposition 1. *Let $x_j, a \in X$ with $a \neq 0$ and $\|x_j - a\| \leq \|a\|$ for each $j \in \{1, \dots, n\}$. Then for any $p_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ we have*

$$(2.9) \quad \left\| \sum_{j=1}^n p_j x_j \right\| \|a\| + \frac{1}{2} \sum_{j=1}^n p_j \|x_j\| \|x_j - a\| \geq \frac{1}{2} \|a\| \sum_{j=1}^n p_j \|x_j\|.$$

Proof. From (2.3) we have

$$\langle x_j, a \rangle_i + \frac{1}{2} \|x_j\| \|x_j - a\| \geq \frac{1}{2} \|a\| \|x_j\|$$

for any $j \in \{1, \dots, n\}$.

The proof follows in the same manner as in Theorem 1 and we omit the details. ■

The following reverse of the generalised triangle inequality may be stated:

Theorem 2. *Let $x_j \in X \setminus \{0\}$ and $a \in X \setminus \{0\}$ such that $\|a\| \geq \|x_j - a\|$ for each $j \in \{1, \dots, n\}$. Then for any $p_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ we have*

$$(2.10) \quad \frac{\left\| \sum_{j=1}^n p_j x_j \right\|}{\sum_{j=1}^n p_j \|x_j\|} \geq \frac{1}{2} \min_{1 \leq j \leq n} \left\{ \frac{\|a\|^2 - \|a - x_j\|^2}{\|x_j\| \|a\|} \right\} (\geq 0).$$

The constant $\frac{1}{2}$ is best possible in (2.10).

Proof. Let us denote

$$\rho := \min_{1 \leq j \leq n} \left\{ \frac{\|a\|^2 - \|a - x_j\|^2}{\|x_j\| \|a\|} \right\}.$$

From Lemma 1 we have

$$\frac{\langle x_j, a \rangle_i}{\|x_j\|} \geq \frac{1}{2} \cdot \frac{\|a\|^2 - \|x_j - a\|^2}{\|x_j\| \|a\|} \geq \frac{1}{2} \rho$$

for each $j \in \{1, \dots, n\}$. Therefore

$$\langle x_j, a \rangle_i \geq \frac{1}{2} \rho \|x_j\|, \quad j \in \{1, \dots, n\}.$$

Multiplying with p_j and summing over j from 1 to n we obtain

$$(2.11) \quad \sum_{j=1}^n p_j \langle x_j, a \rangle_i \geq \frac{1}{2} \rho \sum_{j=1}^n p_j \|x_j\|,$$

and since:

$$(2.12) \quad \left\| \sum_{j=1}^n p_j x_j \right\| \|a\| \geq \left\langle \sum_{j=1}^n p_j x_j, a \right\rangle_i \geq \sum_{j=1}^n p_j \langle x_j, a \rangle_i,$$

hence by (2.11) and (2.12) we deduce the desired result (2.10).

Now, assume that there exists a constant $E > 0$ such that

$$(2.13) \quad \frac{\left\| \sum_{j=1}^n p_j x_j \right\|}{\sum_{j=1}^n p_j \|x_j\|} \geq E \cdot \min_{1 \leq j \leq n} \left\{ \frac{\|a\|^2 - \|a - x_j\|^2}{\|x_j\| \|a\|} \right\},$$

provided $\|a\| \geq \|x_j - a\|$, $j \in \{1, \dots, n\}$.

If we choose $x_1 = \dots = x_n = \varepsilon a$, $\varepsilon \in (0, 1)$, and $p_1 = \dots = p_n = \frac{1}{n}$, then we get

$$1 \geq E \cdot \frac{\|a\|^2 - (1 - \varepsilon)^2 \|a\|^2}{\varepsilon \|a\|^2},$$

giving

$$1 \geq E(2 - \varepsilon)$$

for any $\varepsilon \in (0, 1)$. Letting $\varepsilon \rightarrow 0+$, we deduce $E \leq \frac{1}{2}$ and the proof is complete. ■

The following result may be stated as well:

Proposition 2. *Let $x_j, a \in X \setminus \{0\}$, $j \in \{1, \dots, n\}$ such that $\|x_j - a\| \leq \|a\|$. Then for any $p_j \geq 0$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ we have*

$$(2.14) \quad \frac{\left\| \sum_{j=1}^n p_j x_j \right\|}{\sum_{j=1}^n p_j \|x_j\|} \geq \frac{(\|a\| - \max_{1 \leq j \leq n} \|x_j - a\|)}{2 \|a\|} \quad (\geq 0).$$

Proof. From (2.3) we have

$$\begin{aligned} \frac{\langle x_j, a \rangle_i}{\|x_j\|} &\geq \frac{1}{2} (\|a\| - \|x_j - a\|) \\ &\geq \frac{1}{2} \min_{1 \leq j \leq n} (\|a\| - \|x_j - a\|) \\ &= \frac{1}{2} \left(\|a\| - \max_{1 \leq j \leq n} \|x_j - a\| \right). \end{aligned}$$

Now the proof follows the same steps as in that of Theorem 1 and the details are omitted. ■

Remark 2. *If $\|a\| = 1$ and $\|x_j - a\| \leq 1$, then (2.10) has a simpler form:*

$$(2.15) \quad \frac{\left\| \sum_{j=1}^n p_j x_j \right\|}{\sum_{j=1}^n p_j \|x_j\|} \geq \frac{1}{2} \min_{1 \leq j \leq n} \left\{ \frac{1 - \|x_j - a\|^2}{\|x_j\|} \right\} (\geq 0),$$

while (2.14) becomes

$$(2.16) \quad \frac{\left\| \sum_{j=1}^n p_j x_j \right\|}{\sum_{j=1}^n p_j \|x_j\|} \geq \frac{1}{2} \left(1 - \max_{1 \leq j \leq n} \|x_j - a\| \right) (\geq 0).$$

A different approach for bounding the semi-inner product is incorporated in the following:

Lemma 2. *Let $(X, \|\cdot\|)$ be a normed space. If $x, a \in X$, then*

$$(2.17) \quad \langle x, a \rangle_i \geq \|a\| (\|a\| - \|x - a\|).$$

The inequality (2.17) is sharp.

Proof. If $a = 0$, then obviously (2.17) holds with equality. For $a \neq 0$, consider

$$\tau_-(x, a) := \lim_{s \rightarrow 0^-} \frac{\|a + sx\| - \|a\|}{s}.$$

Observe that

$$(2.18) \quad \begin{aligned} \langle x, a \rangle_i &= \lim_{s \rightarrow 0^-} \frac{\|a + sx\|^2 - \|a\|^2}{2s} \\ &= \tau_-(x, a) \lim_{s \rightarrow 0^-} \left[\frac{\|a + sx\| + \|a\|}{2} \right] = \tau_-(x, a) \|a\|. \end{aligned}$$

On the other hand, since the function $R \ni s \mapsto \|a + sx\| \in \mathbb{R}_+$ is convex on \mathbb{R} , hence

$$(2.19) \quad \tau_-(x, a) \geq \frac{\|a + (-1)x\| - \|a\|}{(-1)} = \|a\| - \|x - a\|.$$

Consequently, by (2.18) and (2.19) we get (2.17).

Now, let $x = \varepsilon a$, $\varepsilon \in (0, 1)$, $a \neq 0$. Then

$$\langle x, a \rangle_i = \varepsilon \|a\|^2, \quad \|a\| - \|x - a\| = \|a\| - (1 - \varepsilon) \|a\| = \varepsilon \|a\|,$$

which shows that the equality case in (2.17) holds true for the nonzero quantities $\varepsilon \|a\|^2$. The proof is complete. ■

The following reverse of the generalised triangle inequality may be stated.

Theorem 3. *Let $a, x_j \in X \setminus \{0\}$ for $j \in \{1, \dots, n\}$ with the property that $\|a\| \geq \|x_j - a\|$ for $j \in \{1, \dots, n\}$. Then for any $p_j \geq 0$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ we have*

$$(2.20) \quad \frac{\left\| \sum_{j=1}^n p_j x_j \right\|}{\sum_{j=1}^n p_j \|x_j\|} \geq \min_{1 \leq j \leq n} \left\{ \frac{\|a\| - \|x_j - a\|}{\|x_j\|} \right\} (\geq 0).$$

The inequality (2.20) is sharp.

Proof. On making use of Lemma 2, we have:

$$\begin{aligned} \frac{\langle x_j, a \rangle_i}{\|x_j\|} &\geq \|a\| \left(\frac{\|a\| - \|x_j - a\|}{\|x_j\|} \right) \\ &\geq \|a\| \eta, \end{aligned}$$

for each $j \in \{1, \dots, n\}$, where

$$\eta := \min_{1 \leq j \leq n} \left\{ \frac{\|a\| - \|x_j - a\|}{\|x_j\|} \right\}.$$

Now utilising the same argument explained in the proof of Theorem 2, we get the desired inequality (2.20).

If we choose in (2.20) $x_1 = \cdots = x_n = \varepsilon a$, $\varepsilon \in (0, 1)$, $a \neq 0$, and $p_1 = \dots = p_n = 1$ then we have equality, and the proof is complete. ■

Remark 3. *The above result may be stated in a simpler way, i.e., if $\rho \in (0, 1)$, a and $x_j \in X \setminus \{0\}$, $j \in \{1, \dots, n\}$ are such that*

$$(2.21) \quad (\|x_j\| \geq) \|a\| - \|x_j - a\| \geq \rho \|x_j\| \quad (\geq 0)$$

for each $j \in \{1, \dots, n\}$, then

$$(2.22) \quad \left\| \sum_{j=1}^n p_j x_j \right\| \geq \rho \sum_{j=1}^n p_j \|x_j\|.$$

3. OTHER RELATED RESULTS TO THE TRIANGLE INEQUALITY

The following result may be stated:

Theorem 4. *Let $(X, \|\cdot\|)$ be a normed linear space and x_1, \dots, x_n nonzero vectors in X and $p_j \geq 0$ with $\sum_{j=1}^n p_j = 1$. If $\bar{x}_p := \sum_{j=1}^n p_j x_j \neq 0$ and there exists a $r > 0$ with*

$$(3.1) \quad \frac{\langle x_j, \bar{x}_p \rangle_i}{\|x_j\| \|\bar{x}_p\|} \geq r \quad \text{for each } j \in \{1, \dots, n\}$$

then

$$(3.2) \quad \left\| \sum_{j=1}^n p_j x_j \right\| \geq r \sum_{j=1}^n p_j \|x_j\|.$$

If $p_j > 0$ for each $j \in \{1, \dots, n\}$, then the equality holds in (3.2) if and only if the equality case hold in (3.1) for each $j \in \{1, \dots, n\}$.

Proof. From (3.1) on multiplying with $p_i \geq 0$ we have

$$\langle p_j x_j, \bar{x}_p \rangle_i \geq r p_j \|\bar{x}_p\| \|x_j\|$$

for any $j \in \{1, \dots, n\}$.

Summing over j from 1 to n and taking into account the superadditivity property of the interior semi-inner product, we have

$$(3.3) \quad \left\langle \sum_{j=1}^n p_j x_j, \bar{x}_p \right\rangle_i \geq \sum_{j=1}^n \langle p_j x_j, \bar{x}_p \rangle_i \geq r \|\bar{x}_p\| \sum_{j=1}^n p_j \|x_j\|$$

and since

$$\left\langle \sum_{j=1}^n p_j x_j, \bar{x}_p \right\rangle_i = \left\| \sum_{j=1}^n p_j x_j \right\|^2 \neq 0$$

hence by (3.3) we get (3.2).

The equality case is obvious and the proof is complete. ■

For the system of vectors $x_1, \dots, x_k \in X$, we denote by \bar{x} its gravity center, i.e.,

$$\bar{x} := \frac{1}{n} \sum_{i=1}^n x_i.$$

The following corollary is obvious.

Corollary 1. *Let $x_1, \dots, x_n \in X \setminus \{0\}$ be such that $\bar{x} \neq 0$. If there exists a $r > 0$ such that*

$$(3.4) \quad \frac{\langle x_j, \bar{x} \rangle_i}{\|x_j\| \|\bar{x}\|} \geq r \quad \text{for each } j \in \{1, \dots, n\},$$

then the following reverse of the generalised triangle inequality holds:

$$(3.5) \quad \left\| \sum_{j=1}^n x_j \right\| \geq r \sum_{j=1}^n \|x_j\|.$$

The equality holds in (3.5) if and only if the case of equality holds in (3.4) for each $j \in \{1, \dots, n\}$.

The following refinements of the generalised triangle inequality may be stated as well:

Theorem 5. *Let x_i, \bar{x}_p, p_i , $i \in \{1, \dots, n\}$ be as in Theorem 4. If there exists a constant R with $1 > R > 0$ and such that*

$$(3.6) \quad R \geq \frac{\langle x_j, \bar{x}_p \rangle_s}{\|x_j\| \|\bar{x}_p\|} \quad \text{for each } j \in \{1, \dots, n\},$$

then

$$(3.7) \quad R \sum_{j=1}^n p_j \|x_j\| \geq \left\| \sum_{j=1}^n p_j x_j \right\|.$$

If $p_j > 0$ for each $j \in \{1, \dots, n\}$, then the equality holds in (3.7) if and only if the equality case holds in (3.6) for each $j \in \{1, \dots, n\}$.

The proof is similar to the one in Theorem 4 on taking into account that the superior semi-inner product is a subadditive functional in the first variable.

Corollary 2. *Let x_j , $j \in \{1, \dots, n\}$ be as in Corollary 1. If there exists an R with $1 > R > 0$ and*

$$(3.8) \quad R \geq \frac{\langle x_j, \bar{x} \rangle_s}{\|x_j\| \|\bar{x}\|} \quad \text{for each } j \in \{1, \dots, n\},$$

then the following refinement of the generalised triangle inequality holds:

$$(3.9) \quad R \sum_{j=1}^n \|x_j\| \geq \left\| \sum_{j=1}^n x_j \right\|.$$

The equality hold in (3.9) if and only if the case of equality holds in (3.8) for each $j \in \{1, \dots, n\}$.

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