BOUNDS FOR THE $r$–WEIGHTED GINI MEAN DIFFERENCE OF AN EMPIRICAL DISTRIBUTION

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Abstract. Various bounds for the $r$–weighted Gini mean difference of an empirical distribution are established.

1. Introduction

The Gini mean difference of the sample $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is defined by

$$G(a) = \frac{1}{2n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} |a_i - a_j| = \frac{1}{2} \sum_{1 \leq i < j \leq n} |a_i - a_j|$$

and

$$R(a) = \frac{1}{\bar{a}} G(a)$$

is the Gini index of $a$, provided the sample mean $\bar{a}$ is not zero [6, p. 257].

The Gini index of $a$ equals the Gini mean difference of the “scaled down” sample $\tilde{a} = \left( \frac{a_1}{\bar{\tilde{a}}}, \ldots, \frac{a_n}{\bar{\tilde{a}}} \right)$ ($\bar{\tilde{a}} \neq 0$)

$$R(a_1, \ldots, a_n) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{a_i}{\bar{\tilde{a}}} - \frac{a_j}{\bar{\tilde{a}}} \right|.$$ 

The following elementary properties of the Gini index for an empirical distribution of nonnegative data hold [6, p. 257]:

(i) Let $(a_1, \ldots, a_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^{n} a_i > 0$. Then

$$0 = R(\bar{a}, \ldots, \bar{a}) \leq R(a_1, \ldots, a_n) \leq R\left(0, \ldots, 0, \sum_{i=1}^{n} a_i\right) = 1 - \frac{1}{n} < 1,$$

$$R(\beta a_1, \ldots, \beta a_n) = R(a_1, \ldots, a_n) \quad \text{for every} \quad \beta > 0$$

and

$$R(a_1 + \lambda, \ldots, a_n + \lambda) = \frac{\bar{\tilde{a}}}{\bar{\tilde{a}} + \lambda} R(a_1, \ldots, a_n) \quad \text{for} \lambda > 0.$$ 

(ii) $R$ is a continuous function on $\mathbb{R}^n_+$. 

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These and other properties have been investigated in [6], [3] and [4].

For \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \) and \( \mathbf{p} = (p_1, \ldots, p_n) \) a probability sequence, meaning that \( p_i \geq 0 \) (\( i \in \{1, \ldots, n\} \)) and \( \sum_{i=1}^{n} p_i = 1 \), we considered in [1] the weighted Gini mean difference defined by formula

\[
G(\mathbf{p}, \mathbf{a}) = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} p_i p_j |a_i - a_j| = \sum_{1 \leq i < j \leq n} p_i p_j |a_i - a_j|,
\]

and proved that

\[
\frac{1}{2} K(\mathbf{p}, \mathbf{a}) \leq G(\mathbf{p}, \mathbf{a}) \leq \inf_{\gamma \in \mathbb{R}} \left[ \sum_{i=1}^{n} p_i |a_i - \gamma| \right] \leq K(\mathbf{p}, \mathbf{a}),
\]

where \( K(\mathbf{p}, \mathbf{a}) \) is the mean absolute deviation, namely

\[
K(\mathbf{p}, \mathbf{a}) := \sum_{i=1}^{n} p_i \left| a_i - \sum_{j=1}^{n} p_j a_j \right|.
\]

We have also shown that if more information on the sampling data \( \mathbf{a} = (a_1, \ldots, a_n) \) is available, i.e., there exists the real numbers \( a \) and \( A \) such that \( a \leq a_i \leq A \) for each \( i \in \{1, \ldots, n\} \), then

\[
G(\mathbf{p}, \mathbf{a}) \leq (A - a) \max_{J \subseteq \{1, \ldots, n\}} [P_J (1 - P_J)] \left( \leq \frac{1}{4} (A - a) \right),
\]

where \( P_J := \sum_{j \in J} p_j \). Also, we have shown that

\[
G(\mathbf{p}, \mathbf{a}) \leq \sum_{i=1}^{n} p_i \left| a_i - \frac{A + a}{2} \right| \left( \leq \frac{1}{2} (A - a) \right).
\]

Notice that in general the bounds for the weighted Gini mean difference \( G(\mathbf{p}, \mathbf{a}) \) provided by (1.4) and (1.5) cannot be compared to conclude that one is always better than the other [1].

The main aim of this paper is to continue the study begun in [1] and provide various bounds for the more general \( r \)-weighted Gini mean difference that has been introduced in [1].

2. Bounds for the \( r \)-weighted Gini Mean Difference

For \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \) and \( \mathbf{p} = (p_1, \ldots, p_n) \) a probability sequence, meaning that \( p_i \geq 0 \) (\( i \in \{1, \ldots, n\} \)) and \( \sum_{i=1}^{n} p_i = 1 \), define the \( r \)-weighted Gini mean difference, for \( r \in [1, \infty) \), by the formula [1, 291]:

\[
G_r(\mathbf{p}, \mathbf{a}) := \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} p_i p_j |a_i - a_j|^r = \sum_{1 \leq i < j \leq n} p_i p_j |a_i - a_j|^r.
\]

For \( r = 1 \) we have the weighted Gini mean difference \( G(\mathbf{p}, \mathbf{a}) \) of (1.1) which becomes, for the uniform probability distribution \( \mathbf{p} = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \) the Gini mean difference

\[
G(\mathbf{a}) := \frac{1}{2n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} |a_i - a_j| = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|.
\]
Now, if we define $\Delta := \{(i,j)| i, j \in \{1, \ldots, n\}\}$, then we can simply write from (2.1)
\[
G_r(p, a) = \frac{1}{2} \sum_{(i,j) \in \Delta} p_i p_j |a_i - a_j|^r, \quad r \geq 1.
\]

The following result concerning upper and lower bounds for $G_r(p, a)$ may be stated:

**Theorem 1.** For any $p_i \in (0,1)$, $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, we have the inequalities
\[
\frac{1}{2} \max_{(i,j) \in \Delta} \left\{ p_i p_j + p_i p_j (1 - p_i p_j)^{r-1} |a_i - a_j|^r \right\} 
\leq G_r(p, a) 
\leq \frac{1}{2} \max_{(i,j) \in \Delta} |a_i - a_j|^r,
\]
where $r \in (0, \infty)$.

*Proof.* Observe that
\[
\sum_{(i,j) \in \Delta} p_i p_j (a_i - a_j) = 0.
\]

Then, for any fixed $(i,j) \in \Delta$ we have
\[
p_i p_j (a_i - a_j) = - \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l).
\]

Taking the modulus in (2.4) and utilising the Hölder discrete inequality for multiple indices and $r > 1$, $\frac{1}{r} + \frac{1}{q} = 1$ \(q = \frac{r}{r-1}\), we have successively:
\[
p_i p_j |a_i - a_j| 
= \left| \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l) \right| 
\leq \left( \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l \right)^{\frac{1}{q}} \left( \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l |a_k - a_l|^r \right)^{\frac{1}{r}} 
\leq \left( \sum_{(k,l) \in \Delta} p_k p_l - p_i p_j \right)^{\frac{1}{q}} \times \left( \sum_{(k,l) \in \Delta} p_k p_l |a_k - a_l|^r - p_i p_j |a_i - a_j|^r \right)^{\frac{1}{r}} 
= (1 - p_i p_j)^{\frac{r-1}{r}} (2G_r(p, a) - p_i p_j |a_i - a_j|^r)^{\frac{1}{r}}
\]
for each $(i,j) \in \Delta$. 

Gini Mean Difference 3
Taking the power \( r \) in (2.5) we have
\[
p_i^r p_j^r |a_i - a_j|^r \leq (1 - p_i p_j)^r (2G_r (p, a) - p_i p_j |a_i - a_j|^r),
\]
giving
\[
\left[ p_i^r p_j^r + p_i p_j (1 - p_i p_j)^r \right] |a_i - a_j|^r \leq 2 (1 - p_i p_j)^r G_r (p, a),
\]
so that
\[
(2.6) \quad \frac{1}{2} \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^r}{(1 - p_i p_j)^r} |a_i - a_j|^r \leq G_r (p, a)
\]
for each \((i, j) \in \Delta\).

Taking the maximum over \((i, j) \in \Delta\) in (2.6), we deduce the first inequality in (2.3).

The second inequality is obvious on observing that
\[
G_r (p, a) \leq \frac{1}{2} \sum_{(i, j) \in \Delta} p_i p_j \max_{(i, j) \in \Delta} |a_i - a_j|^r = \frac{1}{2} \max_{(i, j) \in \Delta} |a_i - a_j|^r.
\]
The proof is complete. □

**Remark 1.** The case \( r = 2 \) is of interest, since
\[
G_2 (p, a) = \frac{1}{2} \sum_{(i, j) \in \Delta} p_i p_j |a_i - a_j|^2 = \sum_{i=1}^{n} p_i a_i^2 - \left( \sum_{i=1}^{n} p_i a_i \right)^2,
\]
for which we can obtain from Theorem 1 the following bounds:
\[
(2.7) \quad \frac{1}{2} \max_{(i, j) \in \Delta} \left\{ \frac{p_i p_j}{1 - p_i p_j} (a_i - a_j)^2 \right\} \leq G_2 (p, a) \leq \frac{1}{2} \max_{(i, j) \in \Delta} (a_i - a_j)^2.
\]

**Remark 2.** Consider the function
\[
h_r (t) := \frac{t^r + t (1 - t)^{r-1}}{(1 - t)^{r-1}} = t + t^r (1 - t)^{1-r}
\]
defined for \( t \in [0, 1) \) and \( r > 1 \). Then
\[
h_r' (t) = 1 + r t^{r-1} (1 - t)^{1-r} + (r - 1) t^{r} (1 - t)^{-r}
\]
which shows that \( h_r \) is strictly increasing on \([0, 1)\).

Therefore
\[
\min_{(i, j) \in \Delta} \left\{ \frac{p_i^r p_j^r + p_i p_j (1 - p_i p_j)^r}{(1 - p_i p_j)^r} \right\} = \min_{(i, j) \in \Delta} h_r (p_i p_j)
\]
\[
\geq h_r \left[ \min_{(i, j) \in \Delta} (p_i p_j) \right]
\]
\[
\geq h_r \left( \min_{i \in \{1, \ldots, n\}} p_i \cdot \min_{j \in \{1, \ldots, n\}} p_j \right)
\]
\[
= h_r \left( p_m^2 \right)
\]
\[
= \frac{p_m^{2r} + p_m^2 (1 - p_m^2)^{r-1}}{(1 - p_m^2)^{r-1}},
\]
where \( p_m := \min_{i \in \{1, \ldots, n\}} p_i > 0 \).
In conclusion, from Theorem 1 we can obtain a coarser but, perhaps, a more useful lower bound for the $r$–weighted Gini mean difference, namely:

\begin{equation}
G_r(p, a) \geq \frac{1}{2} \cdot \frac{p_m^2 + p_m^2 (1 - p_m^2)^{r-1}}{(1 - p_m^2)^{r-1}} \cdot \max_{(i,j) \in \Delta} |a_i - a_j|^r,
\end{equation}

where $p_m$ is defined above.

For $r = 2$, we then have:

\begin{equation}
G_2(p, a) \geq \frac{1}{2} \cdot \frac{p_m^2}{1 - p_m^2} \cdot \max_{(i,j) \in \Delta} |a_i - a_j|^2.
\end{equation}

The following result for the weighted Gini mean difference can be stated:

**Theorem 2.** For any $p_i \in (0, 1)$, $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, we have the bounds:

\begin{equation}
\frac{1}{2} \max_{(i,j) \in \Delta} \left\{ p_i p_j \left[ 1 + \frac{1}{\max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\}} \right] \right\} \cdot |a_i - a_j| \leq G(p, a) \leq \frac{1}{2} \max_{(i,j) \in \Delta} |a_i - a_j|.
\end{equation}

**Proof.** As in the proof of Theorem 1 we have

\[
p_i p_j |a_i - a_j| = \left| \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l) \right| \\
\leq \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \cdot \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l |a_k - a_l| \\
= \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \left[ \sum_{(k,l) \in \Delta} p_k p_l |a_k - a_l| - p_i p_j |a_i - a_j| \right]
\]

which gives:

\[
p_i p_j \left[ 1 + \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \right] |a_i - a_j| \leq \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \cdot \sum_{(k,l) \in \Delta} p_k p_l |a_k - a_l|.
\]

That is

\[
p_i p_j \left[ 1 + \max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \right] |a_i - a_j| \leq \sum_{(k,l) \in \Delta} p_k p_l |a_k - a_l|,
\]

which, by taking the maximum over $(i,j) \in \Delta$ implies the first part of (2.10).

The second part is obvious.

**Remark 3.** Since

\[
\max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\} \leq \max_{(k,l) \in \Delta} \{p_k p_l\} = p_m^2,
\]

where $p_M := \max_{k \in \{1, \ldots, n\}} p_k$, hence

\[
1 + \frac{1}{\max_{(k,l) \in \Delta \setminus \{(i,j)\}} \{p_k p_l\}} \geq 1 + \frac{1}{p_m^2}.
\]
and we get from Theorem 2 the following lower bounds for \( G(p,a) \)

\[
G(p,a) \geq \frac{1}{2} \left( \frac{p^2_M + 1}{p^2_M} \right) \max_{(i,j) \in \Delta} \{ p_ip_j | a_i - a_j | \}
\]

\[
\geq \frac{1}{2} p^2_m \left( \frac{p^2_M + 1}{p^2_M} \right) \max_{(i,j) \in \Delta} | a_i - a_j |,
\]

where \( p_m := \min_{k \in \{1, \ldots, n\}} p_k \) and \( p_M := \max_{k \in \{1, \ldots, n\}} p_k \).

3. Related Results

The following result is due to Izumino and Pečarić [5] (see also [2, p. 174 - 175]):

**Lemma 1.** Let \( f \) be a convex even function defined on \([m - M, M - m]\) \((0 < m < M)\) with \( f(0) = 0 \). Then for each \( n \)-tuple \( x = (x_1, \ldots, x_n) \) satisfying the condition \( m \leq x_k \leq M \) \((k = 1, \ldots, n)\) and for each positive weight \( q = (q_1, \ldots, q_n) \) we have

\[
\sum_{1 \leq i < j \leq n} q_i q_j f(x_i - x_j) \leq f(M - m) \max_{J \subseteq \{1, \ldots, n\}} (Q_J (1 - Q_J))
\]

\[
\leq \frac{1}{4} f(M - m),
\]

where \( Q_j := \sum_{j \in J} q_j \).

The following result holds concerning upper bounds for the \( r \)-weighted Gini mean difference when some information on the size of the elements \( a_i, i \in \{1, \ldots, n\} \) are available.

**Theorem 3.** For any \( p_i \in (0, 1), i \in \{1, \ldots, n\} \) with \( \sum_{i=1}^{n} p_i = 1 \) and \( a_i \in \mathbb{R}, i \in \{1, \ldots, n\} \) with the property that

\[
-\infty < a_i \leq A < \infty \text{ for each } i \in \{1, \ldots, n\},
\]

we have the inequality:

\[
G_r(p,a) \leq (A - a)^r \max_{J \subseteq \{1, \ldots, n\}} [P_J (1 - P_J)] \left( \leq \frac{1}{4} (A - a)^r \right),
\]

for \( r \geq 1 \).

**Proof.** Without loss of generality, we may assume that \( a \geq 0 \).

Now, if we apply Lemma 1 for \( f(x) = |x|^r, x_i = a_i \) and \( q_i = p_i, i \in \{1, \ldots, n\} \), we get

\[
G_r(p,a) = \frac{1}{2} \sum_{i,j=1}^{n} p_ip_j |a_i - a_j|^r \leq |A - a|^r \max_{J \subseteq \{1, \ldots, n\}} [P_J (1 - P_J)]
\]

and the result is proved. \( \Box \)

Finally, the following result that provides a connection between

\[
G_2(p,a) = \sum_{i=1}^{n} p_i a_i^2 - \left( \sum_{i=1}^{n} p_i a_i \right)^2,
\]

and

\[
G_2(a) = \frac{1}{n} \sum_{i=1}^{n} a_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} a_i \right)^2,
\]

can be stated.
Theorem 4. If \( p_i \in (0,1) \) for \( i \in \{1, \ldots, n\} \) with \( \sum_{i=1}^{n} p_i = 1 \), then for any \( a_i \in \mathbb{R} \) \( i \in \{1, \ldots, n\} \) we have the inequality:

\[
G_2(p, a) \leq n^2 \left[ 1 - \left( \frac{\sum_{i=1}^{n} p_i^2}{\left( \sum_{i=1}^{n} p_i^2 \right)^2} \right) \right] G_2(a).
\]

Proof. Utilising the Cauchy-Bunyakovsky-Schwarz inequality, we have that:

\[
p_i p_j |a_i - a_j| = \left| \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l) \right|
\]

\[
\leq \left( \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k^2 p_l^2 \right)^{\frac{1}{2}} \left( \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} |a_k - a_l|^2 \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{(k,l) \in \Delta} p_k^2 p_l^2 - p_i^2 p_j^2 \right)^{\frac{1}{2}} \left( \sum_{(k,l) \in \Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{i=1}^{n} p_k^2 \right)^{\frac{1}{2}} \left( \sum_{(k,l) \in \Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right)^{\frac{1}{2}}
\]

The square of (3.5) produces

\[
p_i^2 p_j^2 |a_i - a_j|^2 \leq \left( \sum_{k=1}^{n} p_k^2 \right)^2 - p_i^2 p_j^2 \left[ \sum_{(k,l) \in \Delta} |a_k - a_l|^2 - |a_i - a_j|^2 \right],
\]

giving

\[
\left[ p_i^2 p_j^2 + \left( \sum_{k=1}^{n} p_k^2 \right)^2 - p_i^2 p_j^2 \right] |a_i - a_j|^2 \leq \left[ \sum_{k=1}^{n} p_k^2 \right]^2 - p_i^2 p_j^2 \left[ \sum_{(k,l) \in \Delta} |a_k - a_l|^2 \right]
\]

from which we get

\[
|a_i - a_j|^2 \leq \left[ 1 - \frac{p_i^2 p_j^2}{\left( \sum_{k=1}^{n} p_k^2 \right)^2} \right] \sum_{(k,l) \in \Delta} |a_k - a_l|^2.
\]

Now, if we multiply (3.6) with \( p_i p_j \geq 0 \) and sum over \( (i,j) \in \Delta \) then we get

\[
G_2(p, a) \leq n^2 \left[ 1 - \left( \frac{\sum_{i=1}^{n} p_i^3}{\left( \sum_{i=1}^{n} p_i^2 \right)^2} \right) \right] G_2(a),
\]

and the result is proved. \( \blacksquare \)
Remark 4. It is obvious, by the definition of \( G_r (p, a) \) in (2.2) that for \( r = 2 \)
\[
G_2 (p, a) = \frac{1}{2} \sum_{(i,j) \in \Delta} p_ip_j |a_i - a_j|^2 \leq \frac{1}{2} \max_{(i,j) \in \Delta} \{ p_ip_j \} \sum_{(i,j) \in \Delta} |a_i - a_j|^2
\]
\[
= n^2 \max_{(i,j) \in \Delta} \{ p_ip_j \} G_2 (a) .
\]
Then, it is natural to ask when comparing (3.7) and (3.8) the question, when is the bound
\[
B_1 (p) := 1 - \left[ \frac{\sum_{i=1}^{n} p_i^3}{\sum_{i=1}^{n} p_i^2} \right]^2
\]
better than
\[
B_2 (p) := \max_{(i,j) \in \Delta} \{ p_ip_j \} .
\]
If we take \( n = 2 \) and \( p_1 = p, p_2 = 1 - p, p \in (0,1) \) then
\[
B_1 (p) = 1 - \left[ \frac{p^3 + (1 - p)^3}{p^2 + (1 - p)^2} \right]^2
\]
and
\[
B_2 (p) = \max \left\{ p^2, p(1 - p), (1 - p)^2 \right\} .
\]
The variation of the bounds \( B_1 (p) \) and \( B_2 (p) \) are depicted in Figure 1 and Figure 2, respectively. The plot of the difference \( D (p) := B_1 (p) - B_2 (p) \) shows that one bound is not always better than the other (see Figure 3).

Figure 1. The plot of \( B_1 (p) \).
Finally, the following result in comparing the weighted Gini mean difference $G(p,a)$ with the unweighted means $G_r(a)$ may be stated:

**Theorem 5.** If $p_i \in (0,1)$ for $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} p_i = 1$, and $q, r > 1$ with $\frac{1}{q} + \frac{1}{r} = 1$, then for any $a_i \in \mathbb{R}$ $i \in \{1, \ldots, n\}$ we have the inequality:

$$(3.9) \quad G(p,a) \leq 2^{1/r-2} n^{2/r+2} \left( \sum_{i=1}^{n} p_i^q \right)^{2/q} \left[ G_r(a) \right]^{1/r}.$$ 

**Proof.** We use Hölder’s inequality for double sums to get

$$(3.10) \quad p_i p_j |a_i - a_j| = \left| \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k p_l (a_k - a_l) \right|$$

$$\leq \left( \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} p_k^q p_l^q \right)^{1/q} \left( \sum_{(k,l) \in \Delta \setminus \{(i,j)\}} |a_k - a_l|^r \right)^{1/r}$$

$$\leq \left( \sum_{(k,l) \in \Delta} p_k^q p_l^q - p_i^q p_j^q \right)^{1/q} \left( \sum_{(k,l) \in \Delta} |a_k - a_l|^r - |a_i - a_j|^r \right)^{1/r}$$

$$= \left[ \left( \sum_{k=1}^{n} p_k^q \right)^{2} - p_i^q p_j^q \right]^{1/q} (2n^2 G_r(a) - |a_i - a_j|^r)^{1/r}$$

for each $(i,j) \in \Delta$. 

**Figure 2.** The plot of $B_2(p)$. 


Utilising the elementary inequality
\[(\alpha^r - \beta^r)^{1/r} (\gamma^q - \delta^q)^{1/q} \leq \alpha\gamma - \beta\delta\]
provided \(\alpha \geq \beta, \gamma \geq \delta\) and \(q, r > 1\) with \(\frac{1}{q} + \frac{1}{r} = 1\), we can get that
\[p_i p_j |a_i - a_j| \leq \left( \sum_{i=1}^{n} p_i^q \right)^{2/q} \left[ 2n^2 G_r (a) \right]^{1/r} - p_i p_j |a_i - a_j|\]
which gives
\[2p_i p_j |a_i - a_j| \leq \left( \sum_{i=1}^{n} p_i^q \right)^{2/q} \left[ 2n^2 G_r (a) \right]^{1/r},\]
for each \((i, j) \in \Delta\).

Summing in the inequality (3.11) over \((i, j) \in \Delta\) we deduce the desired result (3.9).

**Remark 5.** The particular case \(q = r = 2\) provides the following simple inequality
\[G(p, a) \leq 2^{-3/2} n^3 \left( \sum_{i=1}^{n} p_i^2 \right) [G_2 (a)]^{1/2}.\]

**References**


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