ON SOME INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS IN NORMED SPACES

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Abstract. Some inequalities for convex functions defined on convex subsets in linear spaces with applications for the $p$–mean absolute deviation of a sequence of vectors are given, in a normed linear space.

1. Introduction

Jensen’s inequality is pivotal in the Theory of Inequalities because it implies at once many other classical inequalities including the Hölder, Minkowski, Beckenbach-Dresher and Young inequalities, the arithmetic mean - geometric mean inequality, the generalised triangle inequality.

Let $C$ be a convex subset of the real linear space $X$ and $f : C \to \mathbb{R}$ a convex function on $C$. If $x_i \in C$ and $p_i \in (0,1)$ with $\sum_{i=1}^{n} p_i = 1$, then the following well-known form of Jensen’s discrete inequality holds:

$$f \left( \sum_{i=1}^{n} p_i x_i \right) \leq \sum_{i=1}^{n} p_i f(x_i) .$$

In [2], the authors proved, amongst other results, the following refinement of Jensen’s inequality in the general setting of linear spaces:

$$\sum_{i=1}^{n} p_i f(x_i) - f \left( \sum_{i=1}^{n} p_i x_i \right) \geq \max_{1 \leq i < j \leq n} \left\{ p_i f(x_i) + p_j f(x_j) - (p_i + p_j) f \left( \frac{p_i x_i + p_j x_j}{p_i + p_j} \right) \right\} \geq 0 .$$

In particular, if $p_i = \frac{1}{n}$, $i \in \{1, \ldots, n\}$, then we get the following refinement of the unweighted Jensen inequality:

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) - f \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \geq \frac{1}{n} \max_{1 \leq i < j \leq n} \left\{ f(x_i) + f(x_j) - 2 f \left( \frac{x_i + x_j}{2} \right) \right\} \geq 0 .$$

As a natural and important application of the above result (1.2), the authors of [2] considered the case of normed linear spaces $(X, \| \cdot \|)$ and the convex function
\[ f(x) = \|x\|^r, \quad r \geq 1, \] obtaining refinements of the generalised triangle inequality:

\[ \sum_{i=1}^{n} p_i \|x_i\|^r - \left\| \sum_{i=1}^{n} p_i x_i \right\|^r \]

\[ \geq \max_{1 \leq i < j \leq n} \left\{ p_i \|x_i\|^r + p_j \|x_j\|^r - (p_i + p_j)^{1-r} \|p_i x_i + p_j x_j\|^r \right\} \geq 0. \]

and

\[ \frac{1}{n} \sum_{i=1}^{n} \|x_i\|^r - \left\| \frac{1}{n} \sum_{i=1}^{n} x_i \right\|^r \]

\[ \geq \frac{1}{n} \max_{1 \leq i < j \leq n} \left\{ \|x_i\|^r + \|x_j\|^r - 2^{1-r} \|x_i + x_j\|^r \right\} \geq 0. \]

More recently, the second author [1] has proved the following result:

\[ \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[ \sum_{j=1}^{n} q_j f(x_j) - f \left( \sum_{j=1}^{n} q_j x_j \right) \right] \]

\[ \geq \sum_{j=1}^{n} p_j f(x_j) - f \left( \sum_{j=1}^{n} p_j x_j \right) \]

\[ \geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[ \sum_{j=1}^{n} q_j f(x_j) - f \left( \sum_{j=1}^{n} q_j x_j \right) \right], \]

provided \( f : C \to \mathbb{R} \) is convex on the convex subset \( C \) of the linear space \( X \) and \( p_i, q_i, i \in \{1, \ldots, n\} \) are probability sequences with \( q_i > 0 \) for each \( i \in \{1, \ldots, n\} \).

In particular, from (1.6) the following is obtained that compares the weighted and unweighted Jensen differences:

\[ n \max_{1 \leq i \leq n} \left\{ p_i \right\} \left[ \frac{1}{n} \sum_{j=1}^{n} f(x_j) - f \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right] \]

\[ \geq \sum_{j=1}^{n} p_j f(x_j) - f \left( \sum_{j=1}^{n} p_j x_j \right) \]

\[ \geq n \min_{1 \leq i \leq n} \left\{ p_i \right\} \left[ \frac{1}{n} \sum_{j=1}^{n} f(x_j) - f \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right]. \]

The above inequalities (1.6) and (1.7) have some nice applications for the generalised triangle inequality in normed linear spaces:

\[ \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[ \sum_{j=1}^{n} q_j \|x_j\|^r - \left\| \sum_{j=1}^{n} q_j x_j \right\|^r \right] \]

\[ \geq \sum_{j=1}^{n} p_j \|x_j\|^r - \left\| \sum_{j=1}^{n} p_j x_j \right\|^r \]
\[ \begin{align*}
\geq & \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[ \sum_{j=1}^{n} q_j \|x_j\|^r - \left\| \sum_{j=1}^{n} q_j x_j \right\|^r \right] \geq 0, \\
\text{and} \\
\max_{1 \leq i \leq n} \left\{ p_i \right\} \left[ \sum_{j=1}^{n} \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^{n} x_j \right\|^p \right] & \geq \sum_{j=1}^{n} p_j \|x_j\|^p - \left\| \sum_{j=1}^{n} p_j x_j \right\|^p \\
& \geq \min_{1 \leq i \leq n} \left\{ p_i \right\} \left[ \sum_{j=1}^{n} \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^{n} x_j \right\|^p \right] \geq 0, 
\end{align*} \]

respectively.

In this paper some new inequalities for convex functions defined on linear spaces are given. Applications for the \(p\)-mean absolute deviation of a sequence of vectors in a normed linear space with given probabilities are also provided.

### 2. The Main Results

**Theorem 1.** Let \( C \) be a convex subset in the linear space \( X \), \( f : C \to \mathbb{R} \) a convex function on \( C \), \( x_j \in C \), \( p_j \in (0,1) \), \( j \in \{1, \ldots, n\} \), \( n \geq 2 \) and \( \sum_{j=1}^{n} p_j = 1 \). If

\[ \sum_{j=1}^{n} p_j x_j = 0 \quad \text{and} \quad \frac{p_k}{p_k - 1} \cdot x_k \in C \quad \text{for each} \quad k \in \{1, \ldots, n\}, \]

then,

\[ \sum_{j=1}^{n} p_j f(x_j) \geq \max_{k \in \{1, \ldots, n\}} \left[ p_k f(x_k) + (1 - p_k) f \left( \frac{p_k}{p_k - 1} \cdot x_k \right) \right] \geq f(0). \]

In particular, if

\[ \sum_{j=1}^{n} x_j = 0 \quad \text{and} \quad \frac{n}{n - 1} \cdot x_k \in C \quad \text{for each} \quad k \in \{1, \ldots, n\}, \]

then,

\[ \frac{1}{n} \sum_{j=1}^{n} f(x_j) \geq \frac{1}{n} \max_{k \in \{1, \ldots, n\}} \left[ f(x_k) + (n - 1) f \left( \frac{n}{n - 1} \cdot x_k \right) \right] \geq f(0). \]

**Proof.** Firstly, since \( C \) is convex and \( x_k, \frac{p_k}{p_k - 1} \cdot x_k \in C \) for \( k \in \{1, \ldots, n\} \), then,

\[ p_k x_k + (1 - p_k) \left( \frac{p_k}{p_k - 1} \cdot x_k \right) = 0 \in C, \]

and by the convexity of \( f \),

\[ p_k f(x_k) + (1 - p_k) f \left( \frac{p_k}{p_k - 1} \cdot x_k \right) \geq f(0) \]
for each $k \in \{1, \ldots, n\}$, which proves the last part of (2.2).

Since $\sum_{j=1}^{n} p_j x_j = 0$, we have,

$$p_k x_k = -\frac{n}{\sum_{j=1}^{n} p_j j \neq k} \cdot \frac{1}{\sum_{j=1}^{n} p_j j \neq k} \cdot p_j x_j$$

for each $k \in \{1, \ldots, n\}$, which implies,

$$\frac{p_k}{p_k - 1} \cdot x_k = \frac{1}{\sum_{j=1}^{n} p_j j \neq k} \cdot \sum_{j=1}^{n} p_j x_j,$$

(2.5)

for each $k \in \{1, \ldots, n\}$.

Applying the Jensen inequality, we have from (2.5) that,

$$f \left( \frac{p_k}{p_k - 1} \cdot x_k \right) = f \left( \frac{1}{\sum_{j=1}^{n} p_j j \neq k} \cdot \sum_{j=1}^{n} p_j x_j \right) \leq \frac{\sum_{j=1}^{n} p_j f(x_j)}{\sum_{j=1}^{n} p_j},$$

from which it is obvious that,

$$p_k f(x_k) + (1 - p_k) f \left( \frac{p_k}{p_k - 1} \cdot x_k \right) \leq \sum_{j=1}^{n} p_j f(x_j)$$

(2.6)

for each $k \in \{1, \ldots, n\}$.

Taking the maximum in (2.6) over $k \in \{1, \ldots, n\}$, we deduce the first part of (2.2).

The following result can be useful for applications.

**Corollary 1.** Let $f : C \to \mathbb{R}$ be a convex function on the convex set $C$ and $q_j \in (0, 1)$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} q_j = 1$. If $v_i \in X$, $i \in \{1, \ldots, n\}$ are such that,

$$v_k = \sum_{l=1}^{n} q_l v_l, \quad \frac{q_k}{1 - q_k} \left( \sum_{l=1}^{n} q_l v_l - v_k \right) \in C \quad \text{for each } k \in \{1, \ldots, n\},$$

(2.7)

then,

$$\sum_{j=1}^{n} q_j f \left( v_j - \sum_{l=1}^{n} q_l v_l \right) \geq \max_{k \in \{1, \ldots, n\}} \left\{ q_k f \left( v_k - \sum_{l=1}^{n} q_l v_l \right) + (1 - q_k) f \left[ \frac{q_k}{1 - q_k} \left( \sum_{l=1}^{n} q_l v_l - v_k \right) \right] \right\} \geq f(0).$$

(2.8)
In particular, if,

\[ \frac{1}{n} \sum_{l=1}^{n} v_l \geq \frac{1}{n} \left( \frac{1}{n} \sum_{l=1}^{n} v_l - v_k \right) \in C \quad \text{for each } k \in \{1, \ldots, n\}, \]

then,

\[ \begin{align*}
\frac{1}{n} \sum_{j=1}^{n} f \left( v_j - \frac{1}{n} \sum_{l=1}^{n} v_l \right) & \geq \frac{1}{n} \max_{k \in \{1, \ldots, n\}} \left\{ f \left( v_k - \frac{1}{n} \sum_{l=1}^{n} v_l \right) + (n-1) f \left[ \frac{1}{n} \left( \frac{1}{n} \sum_{l=1}^{n} v_l - v_k \right) \right] \right\} \\
& \geq f(0).
\end{align*} \]

The proof follows by Theorem 1 on choosing \( x_j = v_j - \sum_{l=1}^{n} q_l v_l \) and \( p_j = q_j, \ j \in \{1, \ldots, n\} \).

**Corollary 2.** Let \( f : C \to \mathbb{R} \) be a convex function on the convex set \( C \) and \( x_i \in \{1, \ldots, n\} \) such that, for \( y_1 := x_1 - x_n, y_2 := x_2 - x_1, \ldots, y_{n-1} := x_{n-1} - x_{n-2}, y_n := x_n - x_{n-1}, \) we have \( y_k, \frac{1}{n} y_k \in C \) for each \( k \in \{1, \ldots, n\} \). It follows that,

\[ \begin{align*}
\frac{1}{n} \left[ f(x_1 - x_n) + f(x_2 - x_1) + \cdots + f(x_{n-1} - x_{n-2}) + f(x_n - x_{n-1}) \right] & \geq \frac{1}{n} \max \left\{ f(x_1 - x_n) + (n-1) f \left( \frac{1}{n} (x_1 - x_n) \right), \ldots, f(x_n - x_{n-1}) + (n-1) f \left( \frac{1}{n} (x_n - x_{n-1}) \right) \right\} \\
& \geq f(0).
\end{align*} \]

The proof is obvious by the second part of Theorem 1.

A different result is incorporated in the following.

**Theorem 2.** Let \( C \) be a convex set in the linear space \( X \) and \( f : C \to \mathbb{R} \) be a convex function on \( C \). If \( x_j \in C, p_j \in (0, 1), \ j \in \{1, \ldots, n\} \) are such that \( \sum_{j=1}^{n} p_j = 1 \) and

\[ \begin{align*}
-x_k, \quad & \frac{p_k x_k}{2 - p_k} \in C \quad \text{for each } k \in \{1, \ldots, n\}, \\
& \sum_{j=1}^{n} p_j \left[ \frac{f(x_j) + f(-x_j)}{2} \right] \\
& \geq \max_{k \in \{1, \ldots, n\}} \left[ \frac{p_k}{2} f(-x_k) + \left( 1 - \frac{p_k}{2} \right) f \left( \frac{p_k x_k}{2 - p_k} \right) \right] \\
& \geq f(0).
\end{align*} \]

In particular, if,

\[ \begin{align*}
-x_k, \quad & \frac{n}{2n - 1} x_k \in C \quad \text{for each } k \in \{1, \ldots, n\},
\end{align*} \]
then,

\[(2.13) \quad \frac{1}{n} \sum_{j=1}^{n} \frac{f(x_j) + f(-x_j)}{2} \geq \frac{1}{n} \max_{k \in \{1, \ldots, n\}} \left\{ \frac{1}{2} f(-x_k) + (2n - 1) f \left( \frac{2nx_k}{2n - 1} \right) \right\} \]

\[\geq f(0).\]

**Proof.** For any \(k \in \{1, \ldots, n\}\) we have,

\[\sum_{i=1}^{n} p_i x_i = p_k x_k + \sum_{j \neq k}^{n} p_j x_j,\]

which gives,

\[p_k x_k = \sum_{i=1}^{n} p_i x_i + \sum_{j \neq k}^{n} p_j (-x_j)\]

\[= \frac{\sum_{i=1}^{n} p_i x_i + \sum_{j \neq k}^{n} p_j (-x_j)}{\sum_{i=1}^{n} p_i + \sum_{j \neq k}^{n} p_j} \cdot \left( \sum_{i=1}^{n} p_i + \sum_{j \neq k}^{n} p_j \right)\]

\[= \sum_{i=1}^{n} p_i x_i + \sum_{j \neq k}^{n} p_j (-x_j)\]

\[= \frac{\sum_{i=1}^{n} p_i x_i + \sum_{j \neq k}^{n} p_j (-x_j)}{\sum_{i=1}^{n} p_i + \sum_{j \neq k}^{n} p_j} \cdot (1 + 1 - p_k).
\]

This obviously implies,

\[(2.14) \quad \frac{p_k x_k}{2 - p_k} = \frac{\sum_{i=1}^{n} p_i x_i + \sum_{j \neq k}^{n} p_j (-x_j)}{\sum_{i=1}^{n} p_i + \sum_{j \neq k}^{n} p_j}\]

for each \(k \in \{1, \ldots, n\}\).

Applying Jensen’s inequality, we have from (2.14) that,

\[f \left( \frac{p_k x_k}{2 - p_k} \right) = f \left( \frac{\sum_{i=1}^{n} p_i x_i + \sum_{j \neq k}^{n} p_j (-x_j)}{\sum_{i=1}^{n} p_i + \sum_{j \neq k}^{n} p_j} \right)\]

\[\leq \frac{\sum_{i=1}^{n} p_i f(x_i) + \sum_{j \neq k}^{n} p_j f(-x_j)}{1 + 1 - p_k}\]

\[= \frac{\sum_{i=1}^{n} p_i [f(x_i) + f(-x_i)] - p_k f(x_k)}{2 - p_k},\]
which is clearly equivalent to,
\begin{equation}
\frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right) \leq \sum_{i=1}^{n} p_i \frac{f(x_i) + f(-x_i)}{2},
\end{equation}
for each \( k \in \{1, \ldots, n\} \).

Taking the supremum over \( k \in \{1, \ldots, n\} \) in (2.15) produces the first inequality in (2.12).

By the convexity of \( f \) we also have:
\[
\frac{p_k}{2} f(-x_k) + \left(1 - \frac{p_k}{2}\right) f\left(\frac{p_k x_k}{2 - p_k}\right) \geq f\left[\frac{p_k}{2} (-x_k) + \left(1 - \frac{p_k}{2}\right) \frac{p_k x_k}{2 - p_k}\right] = f(0),
\]
and the last part of (2.12) is also established.

3. Applications for Normed Spaces

Let \((X, \|\cdot\|)\) be a normed space over the real or complex number field \(K\).

For the probability sequence \(\mathbf{p} = (p_1, \ldots, p_n), i \in \{1, \ldots, n\}\), the sequence of vectors \(\mathbf{x} = (x_1, \ldots, x_n) \in X^n\) and a real number \(p \geq 1\), we define the \(p\)-mean absolute deviation of \(\mathbf{x}\) with probability \(\mathbf{p}\) by:
\begin{equation}
K_p (\mathbf{p}, \mathbf{x}) := \sum_{j=1}^{n} p_j \left\|x_j - \sum_{l=1}^{n} p_l x_l\right\|^p.
\end{equation}

For the uniform probability \(\mathbf{u} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\) we have \(K_p (\mathbf{u}, \mathbf{x}) = K_p (\mathbf{x})\), where,
\begin{equation}
K_p (\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \left\|x_j - \frac{1}{n} \sum_{l=1}^{n} x_l\right\|.
\end{equation}

The following result concerning upper and lower bounds for the \(p\)-mean absolute deviation can be stated:

Proposition 1. With the above, we have,
\begin{equation}
\max_{k \in \{1, \ldots, n\}} \left\|x_k - \frac{1}{n} \sum_{l=1}^{n} p_l x_l\right\|^p \geq K_p (\mathbf{p}, \mathbf{x})
\end{equation}
\[
\geq \max_{k \in \{1, \ldots, n\}} \left\{ p_k + p_k^p (1 - p_k)^{1-p} \right\} \left\|x_k - \frac{1}{n} \sum_{l=1}^{n} p_l x_l\right\|^p,
\]
for any \(\mathbf{x} \in X^n, p \geq 1\) and \(\mathbf{p}\) a probability sequence.

In particular,
\begin{equation}
\max_{k \in \{1, \ldots, n\}} \left\|x_k - \frac{1}{n} \sum_{l=1}^{n} x_l\right\|^p \geq K_p (\mathbf{x})
\end{equation}
\[
\geq \frac{1}{n} \left[1 + (n - 1)^{-p}\right] \max_{k \in \{1, \ldots, n\}} \left\|x_k - \frac{1}{n} \sum_{l=1}^{n} x_l\right\|^p
\]
for all \(\mathbf{x} \in X^n\) and \(p \geq 1\).

Proof. The first inequality in (3.3) is obvious, the second follows by Corollary 1 applied for the convex function \(f : X \to \mathbb{R}, f(x) = \|x\|^p\). The details are omitted.
Remark 1. The case \( p = 1 \) produces the inequalities,

\[
\max_{k \in \{1, \ldots, n\}} \left\| x_k - \sum_{l=1}^{n} p_l x_l \right\| \geq K(p, x) \geq 2 \max_{k \in \{1, \ldots, n\}} \left\{ p_k \left\| x_k - \sum_{l=1}^{n} p_l x_l \right\| \right\}
\]

and

\[
\max_{k \in \{1, \ldots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^{n} x_l \right\| \geq K(x) \geq 2 \max_{k \in \{1, \ldots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^{n} x_l \right\|
\]

where \( K(p, x) = K_1(p, x) \) and \( K(x) = K_1(x) \).

If \( \sigma^2(p, x) = K_2(p, x) \), where \( \sigma^2(p, x) \) denotes the variance of \( x \) with the probability \( p \), then we have,

\[
\max_{k \in \{1, \ldots, n\}} \left\| x_k - \frac{n}{n} \sum_{l=1}^{n} p_l x_l \right\|^2 \geq \sigma^2(p, x) \geq \max_{k \in \{1, \ldots, n\}} \left\{ \frac{p_k}{p_k - 1} \left\| x_k - \sum_{l=1}^{n} p_l x_l \right\|^2 \right\},
\]

for any \( x \in X^n \) and \( p \) a probability density.

Also, if \( \sigma^2(x) = K_2(x) \),

\[
\max_{k \in \{1, \ldots, n\}} \left\| x_k - \frac{1}{n} \sum_{l=1}^{n} x_l \right\| \geq \sigma^2(x) \geq \frac{n}{n-1} \max_{k \in \{1, \ldots, n\}} \left\| x_k - \frac{1}{n} x_l \right\|^2.
\]

We notice that if \( X = H \), \( H \) an inner product space, then \( \sigma(p, x) \) and \( \sigma(x) \) can be represented as,

\[
\sigma(p, x) = \left( \sum_{j=1}^{n} p_j \|x_j\|^2 - \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \right)^{\frac{1}{2}},
\]

while

\[
\sigma(x) = \left( \frac{1}{n} \sum_{j=1}^{n} \|x_j\|^2 - \left\| \frac{1}{n} \sum_{j=1}^{n} x_j \right\|^2 \right)^{\frac{1}{2}}.
\]

Since the lower bound for \( K_p(p, x) \) may be difficult to use in applications, we provide the following coarse but perhaps more useful bound.
Corollary 3. If \( p_m := \min_{k \in \{1, \ldots, n\}} p_k, p_m \in (0, 1) \), then

\begin{equation}
K_p (p, x) \geq \left[ p_m + p_m^{p_k} (1 - p_m)^{1-p} \right] \max_{k \in \{1, \ldots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\|^p
\end{equation}

for all \( x \in X^n \).

Proof. For \( p \geq 1 \), consider the function \( h_p : [0, 1) \to \mathbb{R}, h_p (t) := t + t^p (1-t)^{1-p} \). The function \( h_p \) is differentiable on \( [0, 1) \) and

\[ h_p' (t) = 1 + pt^{p-1} (1-t)^{1-p} + (p-1) t^p (1-t)^{-p} > 0, \]

for any \( t \in [0, 1) \), showing that \( h_p \) is strictly increasing on \([0, 1)\). It follows that,

\[ \min_{k \in \{1, \ldots, n\}} \left[ p_k + p_k^p (1 - p_k)^{1-p} \right] = p_m + p_m^p (1 - p_m)^{1-p}, \]

which together with (3.3) provides the desired bound (3.9).

Remark 2. In particular, we have,

\begin{equation}
K (p, x) \geq 2p_m \max_{k \in \{1, \ldots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\|
\end{equation}

and

\begin{equation}
\sigma^2 (p, x) \geq \frac{p_m}{1-p_m} \max_{k \in \{1, \ldots, n\}} \left\| x_k - \sum_{l=1}^n p_l x_l \right\|^2.
\end{equation}

From a different perspective, we can state the following inequalities as well.

Proposition 2. Let \((X, \|\cdot\|)\) be a normed linear space, \( x = (x_1, \ldots, x_n) \in X^n \), \( p \geq 1 \) and \( p_i \in (0, 1) \) with \( \sum_{i=1}^n p_i = 1 \), then,

\begin{equation}
\sum_{i=1}^n p_i \|x_i\|^p \geq \frac{1}{2} \max_{k \in \{1, \ldots, n\}} \left[ p_k + p_k^p (2 - p_k)^{1-p} \right] \|x_k\|^p,
\end{equation}

for any \( x, p \) and \( p \) as above.

In particular,

\begin{equation}
\frac{1}{n} \sum_{i=1}^n \|x_i\|^p \geq \frac{1}{2n} \left[ 1 + (2n - 1)^{1-p} \right] \max_{k \in \{1, \ldots, n\}} \|x_k\|^p
\end{equation}

for any \( x \in X^n \).

The proof is obvious by Theorem 2 applied for the convex function \( f : X \to \mathbb{R}_+ \), \( f(x) = \|x\|^p \). The details are omitted.

Remark 3. The case \( p = 2 \) gives the simple inequalities:

\begin{equation}
\sum_{i=1}^n p_i \|x_i\|^2 \geq \max_{k \in \{1, \ldots, n\}} \left[ \frac{p_k}{2 - p_k} \|x_k\|^2 \right]
\end{equation}

and

\begin{equation}
\frac{1}{n} \sum_{i=1}^n \|x_i\|^2 \geq \frac{1}{2n - 1} \max_{k \in \{1, \ldots, n\}} \|x_k\|^2.
\end{equation}
As pointed out before, for applications, the lower bound for the quantity $\sum_{i=1}^{n} p_i \|x_i\|^2$ may not be as useful as one where the $p_k$’s and $x_k$’s are separate. This can be achieved, however, by the following coarser result:

**Corollary 4.** If $p_m := \min_{k \in \{1, \ldots, n\}} p_k$, $p_m \in (0, 1)$, then,

\[
\sum_{i=1}^{n} p_i \|x_i\|^p \geq \frac{1}{2} \left[ p_m + p_m^p (2 - p_m)^{-1-p} \right] \max_{k \in \{1, \ldots, n\}} \|x_k\|^p,
\]

for any $x \in X^n$.

**Proof.** Consider the function $g_p : [0, 1) \rightarrow \mathbb{R}$, $g_p(t) = t + t^p (2 - t)^{1-p}$ which is differentiable on $[0, 1)$ and

\[
g_p'(t) = 1 + pt^{p-1} (2 - t)^{1-p} + (p - 1) t^p (2 - t)^{-p} > 0
\]

for any $t \in [0, 1)$, showing that $g_p$ is strictly increasing on $[0, 1)$. Therefore,

\[
\min_{k \in \{1, \ldots, n\}} \left[ p_k + p_k^p (2 - p_k)^{1-p} \right] = p_m + p_m^p (2 - p_m)^{1-p},
\]

which, together with (3.12), provides the desired result (3.16). $\blacksquare$

**Remark 4.** In particular,

\[
\sum_{i=1}^{n} p_i \|x_i\|^2 \geq \frac{p_m}{2 - p_m} \max_{k \in \{1, \ldots, n\}} \|x_k\|^2,
\]

for any $x \in X^n$.

**References**


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