THE BEESACK-DARST-POLLARD INEQUALITIES AND APPROXIMATIONS OF THE RIEMANN-STIELTJES INTEGRAL

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Abstract. Utilising the Beesack version of the Darst-Pollard inequality, some error bounds for approximating the Riemann-Stieltjes integral are given. Some applications related to the trapezoid and midpoint quadrature rules are provided.

1. Introduction

In 1970, R. Darst and H. Pollard [3] obtained the following inequality for the Riemann-Stieltjes integral:

\[ \int_a^b h(t) \, dg(t) \leq \inf_{t \in [a, b]} h(t) [g(b) - g(a)] + S(g; a, b) \int_a^b (h) \]

where \( \int_a^b (h) \) denotes the total variation of \( h \) on \([a, b]\) and

\[ S(g; a, b) := \sup_{a \leq \alpha < \beta \leq b} [g(\beta) - g(\alpha)] \]

under the assumption that \( h \) is of bounded variation and \( g \) is continuous on \([a, b]\).

As P.R. Beesack observed in [1] that, by replacing \( g \) with \((-g)\) in (1.1), we can also obtain the “dual” Darst-Pollard inequality

\[ \int_a^b h(t) \, dg(t) \geq \inf_{t \in [a, b]} h(t) [g(b) - g(a)] + s(g; a, b) \int_a^b (h) \]

where

\[ s(g; a, b) := \inf_{a \leq \alpha < \beta \leq b} [g(\beta) - g(\alpha)] . \]

Beesack also showed that the inequalities (1.1) and (1.4) remain valid even if \( g \) is not continuous on \([a, b]\), provided only that \( g \) is bounded on \([a, b]\) and \( \int_a^b h(t) \, dg(t) \) exists.

In a recent paper [6], in order to approximate the Riemann-Stieltjes integral \( \int_a^b f(t) \, du(t) \) by the quadrature rule

\[ \frac{m + M}{2} [u(b) - u(a)] \]
where \( m \leq f(t) \leq M \) for each \( t \in [a, b] \), the second author defined the error functional
\[
\Delta(f, u, m, M; a, b) := \int_a^b f(t) \, du(t) - \frac{m + M}{2} [u(b) - u(a)]
\]
and showed that
\[
|\Delta(f, u, m, M; a, b)| \leq \begin{cases} \frac{1}{2} (M - m) \vee^b_a (u) & \text{if } u \text{ is of bounded variation;} \\ \frac{1}{2} (M - m) L(b - a) & \text{if } f \text{ is } L \text{-Lipschitzian;} \\ \int_a^b |f(t) - \frac{m + M}{2} du(t) | & \text{if } u \text{ is monotonic nondecreasing.} \end{cases}
\]

The constant \( \frac{1}{2} \) is the best possible in both inequalities. The last inequality in (1.5) is also sharp.

In the same paper [6], in order to approximate the integral \( \int_a^b f(t) \, du(t) \) in terms of the generalised trapezoid rule
\[
\left[ u(b) - \frac{n + N}{2} \right] f(b) + \left[ \frac{n + N}{2} - u(a) \right] f(a),
\]
the second author introduced the error functional
\[
\nabla(f, u, n, N; a, b) := \left[ u(b) - \frac{n + N}{2} \right] f(b) + \left[ \frac{n + N}{2} - u(a) \right] f(a) - \int_a^b f(t) \, du(t),
\]
where \(-\infty < n \leq u(t) \leq N < \infty\) for \( t \in [a, b] \) and showed that
\[
|\nabla(f, u, n, N; a, b)| \leq \begin{cases} \frac{1}{2} (N - n) \vee^b_a (f) & \text{if } f \text{ is of bounded variation;} \\ \frac{1}{2} (N - n) K(b - a) & \text{if } f \text{ is } K \text{-Lipschitzian;} \\ \int_a^b |u(t) - \frac{n + N}{2} | df(t) & \text{if } f \text{ is monotonic nondecreasing.} \end{cases}
\]

The constant \( \frac{1}{2} \) is the best possible in (1.6) and the last inequality is sharp.

In this paper, by use of the Beesack-Darst-Pollard inequalities (1.1) and (1.3), we provide other error bounds for the functionals \( \Delta \) and \( \nabla \). Applications for the generalised trapezoid and Ostrowski inequalities are also given.

2. The Results

We can state the following result concerning the error bounds for the error functional \( \Delta(f, u, m, M; a, b) \).

**Theorem 1.** Let \( f : [a, b] \to \mathbb{R} \) be a function of bounded variation and assume that
\[
-\infty < m = \inf_{t \in [a, b]} f(t), \quad \sup_{t \in [a, b]} f(t) = M < \infty.
\]
If $u$ is bounded and the Riemann-Stieltjes integral $\int_a^b f(t) \, du(t)$ exists, then
\begin{equation}
|\Delta (f, u, m, M; a, b)| \leq \min \left\{ \frac{1}{2} (M - m) [u(b) - u(a)] - \int_a^b (f) \cdot s(u; a, b), \right. \\
\frac{1}{2} M [u(b) - u(a)] - \int_a^b (f) \cdot S(u; a, b) \left. \right\}
\end{equation}

The constant $\frac{1}{2}$ is the best possible and the inequalities are sharp.

Proof. If we apply the inequality (1.3) for $h(t) = f(t)$, $g(t) = u(t)$, we can write,
\begin{equation}
\int_a^b f(t) \, du(t) \geq m [u(b) - u(a)] + s(u; a, b) \int_a^b (f).
\end{equation}

If we apply the same inequality (1.3) for $h(t) = M - f(t)$ and $g(t) = u(t)$, we get
\begin{equation}
M [u(b) - u(a)] - \int_a^b f(t) \, du(t) \geq s(u; a, b) \int_a^b (f)
\end{equation}
since, obviously, $\int_a^b (M - f) = \int_a^b (f)$.

The inequalities (2.3) and (2.4) give the following double inequality that is of interest:
\begin{equation}
M [u(b) - u(a)] - s(u; a, b) \geq \int_a^b f(t) \, du(t) \geq m [u(b) - u(a)] + s(u; a, b).
\end{equation}

Now, if we subtract from all terms the same quantity
\begin{equation}
\frac{M + m}{2} [u(b) - u(a)]
\end{equation}
we get
\begin{equation}
\frac{1}{2} (M - m) [u(b) - u(a)] - s(u; a, b)
\end{equation}
\begin{equation}
\geq \int_a^b f(t) \, du(t) - \frac{M + m}{2} [u(b) - u(a)]
\end{equation}
\begin{equation}
\geq - \frac{1}{2} (M - m) [u(b) - u(a)] + s(u; a, b),
\end{equation}
which is equivalent to
\begin{equation}
|\Delta (f, u, m, M; a, b)| \leq \frac{1}{2} (M - m) [u(b) - u(a)] - s(u; a, b).
\end{equation}

On utilising (1.1) we can also prove in a similar way that
\begin{equation}
|\Delta (f, u, m, M; a, b)| \leq \int_a^b (f) \, S(u; a, b) - \frac{1}{2} (M - m) [u(b) - u(a)].
\end{equation}

These show that the first inequality in (2.2) is valid. The second part is obvious since for any $\alpha, \beta \in \mathbb{R}$, $\min (\alpha, \beta) \leq \frac{\alpha + \beta}{2}$. 

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For the sharpness of the inequality, we assume that \( u(t) = t, t \in [a, b] \). Since for this selection of \( u \) we have
\[
S(u; a, b) = b - a \quad \text{and} \quad s(u; a, b) = 0,
\]
hence the inequality (2.3) becomes
\[
|\int_a^b f(t) \, du(t) - \frac{M + m}{2} (b - a)| \\
\leq \min \left\{ (b - a) \sqrt{\left( \int_a^b f(t)^2 \, dt \right)} - \frac{1}{2} (M - m) (b - a), \frac{1}{2} (M - m) (b - a) \right\} \\
\leq \frac{1}{2} \sqrt{\left( \int_a^b f(t)^2 \, dt \right)} (b - a).
\]
If we consider the function \( f_0 : [a, b] \to \mathbb{R} \),
\[
f_0(t) = \begin{cases} 
0 & \text{if } t \in [a, b) \\
k & \text{if } t = b,
\end{cases}
\]
where \( k > 0 \), then obviously \( m = 0, M = k, \int_a^b f_0(t) \, dt = 0, \sqrt{\int_a^b f_0^2(t) \, dt} = k \) and in all parts of (2.9) we get the same quantity \( \frac{1}{2} k (b - a) \).

The following corollary that provides error bounds for the error functional \( \nabla(f, u, n, N; a, b) \) can be stated as well.

**Corollary 1.** Let \( u : [a, b] \to \mathbb{R} \) be a function of bounded variation such that there exist the constants \( n, N \) with
\[
-\infty < n = \inf_{t \in [a, b]} u(t), \sup_{t \in [a, b]} u(t) = N < \infty.
\]
If \( f \) is bounded and the Riemann-Stieltjes integral \( \int_a^b f(t) \, du(t) \) exists, then
\[
|\nabla(f, u, n, N; a, b)| \leq \min \left\{ \sqrt{\int_a^b (u(t) - n)^2 \, dt} S(f; a, b) - \frac{1}{2} (N - n) [f(b) - f(a)], \frac{1}{2} (N - n) [f(b) - f(a)] - \sqrt{\int_a^b (u(t) - N)^2 \, dt} s(f; a, b) \right\} \\
\leq \frac{1}{2} \sqrt{\int_a^b (u(t) - N)^2 \, dt} S(f; a, b) - s(f; a, b).
\]
The constant \( \frac{1}{2} \) is the best possible and the inequalities are sharp.

**Proof.** Follows by Theorem 1 on utilising the identity
\[
f(b) \left[ u(b) - \frac{n + N}{2} \right] + f(a) \left[ \frac{n + N}{2} - u(a) \right] - \int_a^b f(t) \, du(t) \\
= \int_a^b u(t) - \frac{n + N}{2} \, df(t) \\
= \int_a^b u(t) \, df(t) - \frac{n + N}{2} [f(b) - f(a)].
\]
The details are omitted.

The following particular cases of Theorem 1 may be of interest in applications.

**Corollary 2.** Assume that \( f : [a, b] \to \mathbb{R} \) is as in Theorem 1. If \( u : [a, b] \to \mathbb{R} \) is of the \( r-H \)-Hölder type, i.e.,

\[
|u(t) - u(s)| \leq H |t - s|^r \quad \text{for any} \quad t, s \in [a, b],
\]

where \( H > 0 \) and \( s \in (0, 1] \) are given, then

\[
\Delta \left( f, u, m, M; a, b \right) \leq \min \left\{ H \left( b - a \right)^r \sqrt{b} \left( f \right) - \frac{1}{2} \left( M - m \right) \left[ u \left( b \right) - u \left( a \right) \right], \right.
\]

\[
\left. \frac{1}{2} \left( M - m \right) \left[ u \left( b \right) - u \left( a \right) \right] + H \left( b - a \right)^r \sqrt{a} \left( f \right) \right\}
\]

\[
\leq H \left( b - a \right)^r \sqrt{b} \left( f \right).
\]

**Proof.** For any \( a \leq \alpha < \beta \leq b \) we have, by (2.12), that

\[
-H \left( \beta - \alpha \right)^r \leq u \left( \beta \right) - u \left( s \right) \leq H \left( \beta - \alpha \right)^r.
\]

This implies that

\[
S \left( u; a, b \right) \leq \sup_{a \leq \alpha < \beta \leq b} \left[ H \left( \beta - \alpha \right)^r \right] = H \left( b - a \right)^r
\]

and

\[
s \left( u; a, b \right) \geq \inf_{a \leq \alpha < \beta \leq b} \left[ -H \left( \beta - \alpha \right)^r \right] = - \sup_{a \leq \alpha < \beta \leq b} \left[ H \left( \beta - \alpha \right)^r \right] = -H \left( b - a \right)^r.
\]

Utilising (2.2) we deduce the desired inequality (2.13). 

**Corollary 3.** Assume that \( f \) is as in Theorem 1. If \( u : [a, b] \to \mathbb{R} \) is monotonic non-decreasing and such that the Riemann-Stieltjes integral \( \int_a^b f \left( t \right) du \left( t \right) \) exists, then

\[
\Delta \left( f, u, m, M; a, b \right) \leq \min \left\{ \left[ u \left( b \right) - u \left( a \right) \right], \right.
\]

\[
\left. \frac{1}{2} \left[ u \left( b \right) - u \left( a \right) \right] + H \left( b - a \right)^r \sqrt{b} \left( f \right) \right\}
\]

The proof is obvious by Theorem 1 on taking into account that for the monotonic nondecreasing function \( u : [a, b] \to \mathbb{R} \) we have:

\[
S \left( u; a, b \right) = u \left( b \right) - u \left( a \right)
\]

and

\[
s \left( u; a, b \right) = 0.
\]
3. Applications

The following inequality obtained in [2] is known as the trapezoid inequality for functions of bounded variation:

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{2} \sqrt{\mathcal{V}(f)},
\]

where the constant \( \frac{1}{2} \) is best possible in the sense that it cannot be replaced by a smaller constant.

The following trapezoid inequality for the larger class of Riemann integrable functions can be stated:

**Proposition 1.** Let \( f : [a, b] \to \mathbb{R} \) be Riemann integrable on \([a, b]\), then:

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(a) + f(b)}{2} \right| \leq \min \left\{ S(f; a, b) - \frac{1}{2} [f(b) - f(a)], \frac{1}{2} [f(b) - f(a)] - s(f; a, b) \right\}
\]

\[
\leq \frac{1}{2} [S(f; a, b) - s(f; a, b)].
\]

**Proof.** We use the following identity holding for the Riemann integrable function \( f : [a, b] \to \mathbb{R} \):

\[
f(b)(b-x) + f(a)(x-a) - \int_a^b f(t) \, dt = \int_a^b (t-x) \, df(t)
\]

for any \( x \in [a, b] \), see [2].

We observe that \( \sup_{t \in [a,b]} (t-x) = b-a \), \( \inf_{t \in [a,b]} (t-x) = a-x \), for \( x \in [a,b] \) and, applying Theorem 1 for the Stieltjes integral \( \int_a^b (t-x) \, df(t) \), \( x \in [a,b] \), we obtain:

\[
\left| \int_a^b (t-x) \, df(t) - \left( \frac{a+b}{2} - x \right) [f(b) - f(a)] \right|
\]

\[
\leq \min \left\{ \int_a^b (t-x) \, S(f; a, b) - \frac{1}{2} (b-a) [f(b) - f(a)], \right\}
\]

\[
\frac{1}{2} (b-a) [f(b) - f(a)] - \int_a^b (t-x) \, s(f; a, b) \right\}
\]

\[
\leq \frac{1}{2} \sqrt{\mathcal{V}(f)},
\]

On utilising the identity (3.3) and the fact that \( \sqrt{\mathcal{V}(f)} = b-a \), we deduce from (3.4) the desired result (3.2). 

In [5], S.S. Dragomir obtained the following Ostrowski type inequality for functions of bounded variation:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \sqrt{\mathcal{V}(f)},
\]

where \( \mathcal{V}(f) \) is the total variation of \( f \) on \([a, b]\).
for any \( x \in [a, b] \). The constant \( \frac{1}{2} \) is the best possible in (3.5).

The best inequality one can obtain from (3.5) is the mid-point inequality:

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f (t) \, dt \right| \leq \frac{1}{2} \sqrt{f},
\]

for which \( \frac{1}{2} \) is also the best possible constant.

In order to extend (3.5) to the larger class of Riemann integrable functions, we can state:

**Proposition 2.** Let \( f : [a, b] \to \mathbb{R} \) be Riemann integrable on \([a, b]\), then:

\[
\left| f (x) - \left( x - \frac{a + b}{2} \right) \cdot \frac{f (b) - f (a)}{b - a} - \frac{1}{b - a} \int_a^b f (t) \, dt \right| \\
\leq \min \left\{ S (f; a, b) - \frac{1}{2} [f (b) - f (a)], \frac{1}{2} [f (b) - f (a)] - s (f; a, b) \right\} \\
\leq \frac{1}{2} \left[ S (f; a, b) - s (f; a, b) \right].
\]

**Proof.** We use the Montgomery type identity [5] for the Riemann integrable function \( f : [a, b] \to \mathbb{R} \):

\[
\int_a^b p (t, x) \, df (t) = f (x) (b - a) - \int_a^b f (t) \, dt
\]

for any \( x \in [a, b] \), where the kernel \( p : [a, b]^2 \to \mathbb{R} \) is defined by

\[
p (t, x) := \\
\begin{cases} 
  t - a & \text{if } t \in [a, x], \\
  t - b & \text{if } t \in (x, b].
\end{cases}
\]

For any fixed \( x \in [a, b] \), the function \( p (\cdot, x) \) is of bounded variation, and

\[
\sqrt[\big]{\int_a^b \left[ p (\cdot, x) \right]} = \sqrt[\big]{\int_a^b \left[ p (\cdot, x) \right]} + \sqrt[\big]{\int_x^b \left[ p (\cdot, x) \right]}
\]

\[
= x - a + b - x = b - a.
\]

Also, observe that

\[
\sup_{t \in [a, b]} p (t, x) = x - a \quad \text{and} \quad \inf_{t \in [a, b]} p (t, x) = x - b
\]

for any \( x \in [a, b] \).

Now, applying Theorem 1 for the Riemann-Stieltjes integral \( \int_a^b p (t, x) \, df (t) \), we can write that

\[
\left| \int_a^b p (t, x) \, df (t) - \left( x - \frac{a + b}{2} \right) \cdot [f (b) - f (a)] \right| \\
\leq (b - a) \min \left\{ S (f; a, b) - \frac{1}{2} [f (b) - f (a)], \frac{1}{2} [f (b) - f (a)] - s (f; a, b) \right\} \\
\leq \frac{1}{2} \left[ S (f; a, b) - s (f; a, b) \right],
\]

which is clearly equivalent to (3.2). \( \square \)

The following mid-point inequality holds.
Corollary 4. Let $f$ be as in Proposition 2, then

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| 
\leq \min \left\{ S(f; a, b) - \frac{1}{2} |f(b) - f(a)|, \frac{1}{2} |f(b) - f(a)| - s(f; a, b) \right\} 
\leq \frac{1}{2} [S(f; a, b) - s(f; a, b)].
\]