ON SOME INEQUALITIES OF CAUCHY-BUNYAKOVSKY-SCHWARZ TYPE AND APPLICATIONS

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ABSTRACT. Some discrete inequalities of Cauchy-Bunyakovsky-Schwarz type for complex numbers with applications for the maximal deviation of a sequence from its weighted mean are given.

1. Introduction

The following result for complex numbers \( a_k, b_k, k \in \{1, \ldots, n\} \) is well known in the literature as the Cauchy-Bunyakovsky-Schwarz (CBS) inequality:

\[
\left| \sum_{k=1}^{n} a_k b_k \right|^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2, \tag{1.1}
\]

with equality if and only if there is a complex number \( c \in \mathbb{C} \) such that \( a_k = c b_k \) for each \( k \in \{1, \ldots, n\} \), and \( \overline{b_k} \) is the complex conjugate of \( b_k \).

A simple proof of this statement can be achieved by utilising the following Lagrange identity for complex numbers (see [2, p. 3])

\[
\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left| \sum_{k=1}^{n} a_k b_k \right|^2 = \frac{1}{2} \sum_{k,l=1}^{n} |a_k b_l - \overline{a_l} b_k|^2. \tag{1.2}
\]

If \( p_k, k \in \{1, \ldots, n\} \) are positive weights, then the weighted version of (1.1) can be stated as

\[
\left( \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \right)^{\frac{1}{2}} \leq \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2. \tag{1.3}
\]

In [4], the following result connecting the unweighted version of the (CBS) inequality with the weighted one has been established (see also [2, p. 67 – 69]):

\[
\left( \sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 \right)^{\frac{1}{2}} \left| \sum_{k=1}^{n} x_k y_k \right| = \sup_{\mathbf{p} \in S_n(1)} \left\{ \sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2 - \left| \sum_{k=1}^{n} p_k x_k y_k \right| \right\}, \tag{1.4}
\]

where \( S_n(1) = \{ \mathbf{p} = (p_1, \ldots, p_n) | 0 \leq p_k \leq 1 \text{ for each } k \in \{1, \ldots, n\} \} \).

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In the same paper the authors also established the following result concerning the length of summation in the CBS inequality:

\[
(1.4) \left( \sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^{n} p_k x_k y_k \right| = \sup_{I \subseteq \{1, \ldots, n\}} \left[ \left( \sum_{k \in I} p_k |x_k|^2 \sum_{k \in I} p_k |y_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^{n} p_k x_k y_k \right| \right]
\]

and

\[
(1.5) \left( \sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^{n} p_k x_k y_k \right| \geq \max_{1 \leq k < l \leq n} \left\{ \left( p_k |x_k|^2 + p_l |x_l|^2 \right)^{\frac{1}{2}} \left( p_k |y_k|^2 + p_l |y_l|^2 \right)^{\frac{1}{2}} - |p_k x_k y_k + p_l x_l y_l| \right\},
\]

for any \( x_k, y_k \in \mathbb{C}, \ k \in \{1, \ldots, n\} \).

For some historical facts on CBS inequality, see [9] and [2]. Refinements of this inequality are provided in [1], [6], [8] and in the Chapter 2 of [2]. Other results related to CBS inequality may be found in [5] and [7].

The aim of the present paper is to establish some inequalities of CBS type under the supplementary assumption that either \( \sum_{k=1}^{n} x_k y_k = 0 \) or \( \sum_{k=1}^{n} p_k x_k y_k = 0 \), when the weighted version is considered. Applications that provide upper bounds for the maximal deviation of a sequence \( x_k \) from the weighted mean \( \sum_{j=1}^{n} p_j x_j \), namely, for the quantity

\[
(1.6) \max_{k \in \{1, \ldots, n\}} \left| x_k - \sum_{j=1}^{n} p_j x_j \right|,
\]

where \( x_k \in \mathbb{C}, \ p_k \geq 0, \ k \in \{1, \ldots, n\}, \ \sum_{k=1}^{n} p_k = 1 \), are also given.

2. The Results

The following result holds:

**Theorem 1.** Let \( a_k, b_k \in \mathbb{C}, \ k \in \{1, \ldots, n\}, \ n \geq 2 \) with the property that

\[
(2.1) \sum_{k=1}^{n} a_k b_k = 0.
\]

Then

\[
(2.2) \max_{i \in \{1, \ldots, n\}} \{|a_i b_i|\} \leq \frac{1}{2} \left( \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} |b_k|^2 \right)^{\frac{1}{2}}.
\]

The constant \( \frac{1}{2} \) in (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof.** For any \( i \in \{1, \ldots, n\} \), we have

\[
(2.3) a_i b_i = - \sum_{k=1 \atop k \neq i}^{n} a_k b_k.
\]
Taking the modulus in (2.3) we have

\[ |a_i b_i| = \left| \sum_{k=1 \atop k \neq i}^n a_k b_k \right| \leq \left( \sum_{k=1 \atop k \neq i}^n |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1 \atop k \neq i}^n |b_k|^2 \right)^{\frac{1}{2}} \]

\[ = \left( \sum_{k=1}^n |a_k|^2 - |a_i|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n |b_k|^2 - |b_i|^2 \right)^{\frac{1}{2}}, \]

for any \( i \in \{1, \ldots, n\} \), where we used the Cauchy-Bunyakovsky-Schwarz inequality to state the required inequality in (2.4).

Utilising the elementary inequality for real numbers

\[ (\alpha^2 - \beta^2)^{\frac{1}{2}} (\gamma^2 - \delta^2)^{\frac{1}{2}} \leq \alpha \gamma - \beta \delta, \]

provided \( \alpha, \beta, \gamma, \delta > 0 \) and \( \alpha \geq \beta, \gamma \geq \delta \), we have

\[ \left( \sum_{k=1}^n |a_k|^2 - |a_i|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n |b_k|^2 - |b_i|^2 \right)^{\frac{1}{2}} \]

\[ \leq \left( \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}} - |a_i b_i|, \]

for each \( i \in \{1, \ldots, n\} \).

Now, on making use of (2.4) and (2.5) we get the desired inequality (2.2).

To prove the sharpness of the constant, we assume that the inequality (2.2) holds true for a constant \( C > 0 \), i.e.,

\[ \max_{i \in \{1, \ldots, n\}} |a_i b_i| \leq C \left( \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}}, \]

provided \( a_k, b_k, k \in \{1, \ldots, n\} \) (\( n \geq 2 \)) are complex numbers such that \( \sum_{k=1}^n a_k b_k = 0 \).

Now, for \( n = 2 \), choose \( a_1 = a, a_2 = -b, b_1 = b, b_2 = -a \) with \( a, b > 0 \). Then \( a_1 b_1 + a_2 b_2 = 0, |a_1 b_1| = |a_2 b_2| = ab \) and by (2.6) we get

\[ ab \leq C (a^2 + b^2) \quad \text{for} \ a, b > 0. \]

Choosing in (2.7) \( a = b = 1 \), we deduce \( C \geq \frac{1}{2} \) and the proof is complete. \( \blacksquare \)

The following corollary is of interest.

**Corollary 1.** Let \( x_k \in \mathbb{C}, \ k \in \{1, \ldots, n\} \) and \( p_k, k \in \{1, \ldots, n\} \) be a probability sequence, i.e., \( p_k \geq 0, k \in \{1, \ldots, n\} \) and \( \sum_{k=1}^n p_k = 1 \). Then we have the
inequality:

\[
\begin{align*}
\max_{i \in \{1, \ldots, n\}} \left\{ p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right\} & \leq \frac{1}{2} \left( \sum_{k=1}^{n} p_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \left| x_k - \sum_{j=1}^{n} p_j x_j \right|^2 \right)^{\frac{1}{2}} \\
& = \frac{1}{2} \left( \sum_{k=1}^{n} p_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} |x_k|^2 + n \sum_{j=1}^{n} p_j x_j \right)^{\frac{1}{2}} \\
& \quad - 2 \Re \left[ \sum_{k=1}^{n} x_k \left( \sum_{j=1}^{n} p_j x_j \right) \right] \right) \right) \right) .
\end{align*}
\]  

(2.8)

**Proof.** If we choose \( a_k = p_k \), \( b_k := x_k - \sum_{j=1}^{n} p_j x_j \), then

\[
\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} p_k \left( x_k - \sum_{j=1}^{n} p_j x_j \right) = 0
\]

and the condition (2.1) is satisfied.

Applying the inequality (2.2), we obtain

\[
\max_{i \in \{1, \ldots, n\}} \left\{ p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right\}
\]

\[
\leq \frac{1}{2} \left( \sum_{k=1}^{n} p_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} |x_k|^2 + n \sum_{j=1}^{n} p_j x_j \right)^{\frac{1}{2}} \\
- 2 \Re \left[ \sum_{k=1}^{n} x_k \left( \sum_{j=1}^{n} p_j x_j \right) \right] \right) \right) .
\]

and the inequality (2.8) is obtained. \( \blacksquare \)

**Remark 1.** If \( \min_{i \in \{1, \ldots, n\}} p_i = p_m > 0 \), then from (2.8) we can obtain a coarser and perhaps more useful inequality, providing some upper bounds for the maximal deviation of \( x_k \) from the weighted mean \( \sum_{j=1}^{n} p_j x_j \), namely,

\[
\max_{k \in \{1, \ldots, n\}} \left| x_k - \sum_{j=1}^{n} p_j x_j \right| \leq \frac{1}{2p_m} \left( \sum_{k=1}^{n} p_k^2 \right)^{\frac{1}{2}} \left( |x_k|^2 + n \sum_{j=1}^{n} p_j x_j \right)^{\frac{1}{2}} \\
\]

(2.9)

The following weighted version of Theorem 1 may be stated as well:

**Theorem 2.** Let \( x_k, y_k \in \mathbb{C}, k \in \{1, \ldots, n\} \) and \( p_k, k \in \{1, \ldots, n\} \) be a probability sequence with the property that

\[
\sum_{k=1}^{n} p_k x_k y_k = 0.
\]

(2.10)
Then

\[
(2.11) \quad \max_{i \in \{1, \ldots, n\}} \{ p_i |x_i y_i| \} \leq \frac{1}{2} \left( \sum_{k=1}^{n} p_k |x_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} p_k |y_k|^2 \right)^{\frac{1}{2}}.
\]

The constant \( \frac{1}{2} \) in (2.11) is best possible in (2.11).

**Proof.** It follows from Theorem 1 on choosing \( a_k = \sqrt{p_k} x_k, \ b_k = \sqrt{p_k} y_k \).

**Remark 2.** One should notice that Theorem 1 and Theorem 2 are equivalent in the sense that one implies the other.

The above result provides the opportunity to obtain a different bound for the maximal deviation of \( x_k \) from the weighted mean.

**Corollary 2.** With the assumptions in Corollary 1, we have the inequality:

\[
(2.12) \quad \max_{i \in \{1, \ldots, n\}} \left\{ p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right\} \leq \frac{1}{2} \left( \sum_{k=1}^{n} p_k \left| x_k - \sum_{j=1}^{n} p_j x_j \right|^2 \right)^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \left[ \sum_{k=1}^{n} p_k |x_k|^2 - \left| \sum_{j=1}^{n} p_j x_j \right|^2 \right]^{\frac{1}{2}}.
\]

**Proof.** Follows by Theorem 2 on choosing \( y_k = 1, k \in \{1, \ldots, n\} \).

**Remark 3.** If \( \min_{i \in \{1, \ldots, n\}} p_i = p_m > 0 \), then

\[
(2.13) \quad \max_{i \in \{1, \ldots, n\}} \left| x_k - \sum_{j=1}^{n} p_j x_j \right| \leq \frac{1}{2p_m} \left( \sum_{k=1}^{n} p_k \left| x_k - \sum_{j=1}^{n} p_j x_j \right|^2 \right)^{\frac{1}{2}}.
\]

**Remark 4.** It is natural to ask which of the bounds for the maximal deviation

\[
\max_{i \in \{1, \ldots, n\}} \left\{ p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right\}
\]

provided by (2.8) and (2.12) are better and when, respectively?

For \( n = 2 \), let \( p_1 = p, \ p_2 = 1 - p, \ p \in [0, 1], \ x_1 = x, \ x_2 = y \), then we have the specific case of

\[
B_1 (p, x, y) := \frac{1}{2} \left[ p^2 + (1 - p)^2 \right]^{\frac{1}{2}} \left[ (x - px - (1 - p)y)^2 + (y - px - (1 - p)y)^2 \right]^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \left[ p^2 + (1 - p)^2 \right]^{\frac{1}{2}} \left[ (1 - p)^2 (x - y)^2 + p^2 (x - y)^2 \right]^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \cdot [p^2 + (1 - p)^2] |x - y|
\]
and

\[ B_2 (p, x, y) := \frac{1}{2} \left[ p (x - px - (1 - p) y)^2 + (1 - p) (y - px - (1 - p) y)^2 \right]^{\frac{1}{2}} \]

\[ = \frac{1}{2} \left[ p (1 - p)^2 (x - y)^2 + (1 - p)^2 (x - y)^2 \right]^{\frac{1}{2}} \]

\[ = \frac{1}{2} \sqrt{p (1 - p)} |x - y|. \]

Since \( p^2 + (1 - p)^2 \geq \sqrt{p (1 - p)} \) for \( p \in [0, 1] \), we have that the bound (2.12) is always better than (2.8) for \( n = 2 \).

**Remark 5.** For \( n = 3, p_1 = p, p_2 = q, p_3 = r, x_1 = x, x_2 = y, x_3 = z \), we should compare the bounds

\[ B_1 (p, q, r, x, y, z) = \frac{1}{2} \left( p^2 + q^2 + r^2 \right)^{\frac{1}{2}} \times \left[ p (x - px - qy - rz)^2 + \right. \]

\[ \left. + q (y - px - qy - rz)^2 + r (z - px - qy - rz)^2 \right]^{\frac{1}{2}} \]

and

\[ B_1 (p, q, r, x, y, z) = \frac{1}{2} \left[ p (x - px - qy - rz)^2 + \right. \]

\[ \left. + q (y - px - qy - rz)^2 + r (z - px - qy - rz)^2 \right]^{\frac{1}{2}}. \]

The plot of the function

\[ \Delta (0.1, 0.5, 0.4, x, y, -4) = B_1 (0.1, 0.5, 0.4, x, y, -4) - B_2 (0.1, 0.5, 0.4, x, y, -4) \]

on the box \([0, 6] \times [8, 10]\) shows that one bound is not always better the other (see Figure 1):

**Remark 6.** In the case of uniform distribution, i.e., when \( p_i = \frac{1}{n}, i \in \{1, \ldots, n\} \), we obtain from both inequalities (2.8) and (2.12) the same result:

\[ \max_{k \in \{1, \ldots, n\}} \left| x_k - \frac{1}{n} \sum_{j=1}^{n} x_j \right| \leq \frac{1}{2} \sqrt{n} \sum_{k=1}^{n} \left| x_k - \frac{1}{n} \sum_{j=1}^{n} x_j \right|^2 \]

\[ = \frac{1}{2} \left[ n \sum_{k=1}^{n} |x_k|^2 - \left( \sum_{k=1}^{n} x_k \right)^2 \right]^{\frac{1}{2}}. \]

3. Related Results

The following result may be stated as well.

**Theorem 3.** Let \( a_k, b_k \in \mathbb{C} \setminus \{0\} \), \( k \in \{1, \ldots, n\} \) so that \( \sum_{k=1}^{n} a_k b_k = 0 \). Then for any probability sequence \( p_k, k \in \{1, \ldots, n\} \), we have:

\[ \frac{\sum_{j=1}^{n} p_j |a_j|^2}{\sum_{k=1}^{n} |a_k|^2} + \frac{\sum_{j=1}^{n} p_j |b_j|^2}{\sum_{k=1}^{n} |b_k|^2} \leq 1. \]
Proof. We know, from the proof of Theorem 1, that
\[
|a_i b_i|^2 \leq \left( \sum_{k=1}^{n} |a_k|^2 - |a_i|^2 \right) \left( \sum_{k=1}^{n} |b_k|^2 - |b_i|^2 \right) 
\]
\[
= \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - |a_i|^2 \sum_{k=1}^{n} |b_k|^2 - |b_i|^2 \sum_{k=1}^{n} |a_k|^2,
\]
which is clearly equivalent with
\[
(3.2) \quad |a_i|^2 \sum_{k=1}^{n} |b_k|^2 + |b_i|^2 \sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2
\]
for each \( i \in \{1, \ldots, n\} \).

Now, if we multiply (3.2) by \( p_i \geq 0 \) and sum over \( i \in \{1, \ldots, n\} \), we deduce:
\[
(3.3) \quad \sum_{i=1}^{n} p_i |a_i|^2 \sum_{k=1}^{n} |b_k|^2 + \sum_{i=1}^{n} p_i |b_i|^2 \sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 
\]
which is clearly equivalent with (3.1).

Corollary 3. With the assumptions of the above theorem, we have:
\[
(3.4) \quad \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 \leq \frac{1}{4} \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2.
\]
The constant \( \frac{1}{4} \) is best possible in (3.4).
Proof. On utilising the inequality $\alpha^2 + \beta^2 \geq 2\alpha\beta$, $\alpha, \beta \in \mathbb{R}^+$, we have

\[
\sum_{j=1}^{n} p_j |a_j|^2 \sum_{k=1}^{n} |b_k|^2 + \sum_{j=1}^{n} p_j |b_j|^2 \sum_{k=1}^{n} |a_k|^2 \geq 2 \left( \sum_{j=1}^{n} p_j |a_j|^2 \sum_{j=1}^{n} p_j |b_j|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \right)^{\frac{1}{2}}.
\]

Now, by (3.3) and (3.5) we deduce the desired inequality (3.4).

To prove the sharpness of the constant, we assume that (3.4) holds true with a $D > 0$, i.e.,

\[
\sum_{j=1}^{n} p_j |a_j|^2 \sum_{k=1}^{n} p_j |b_j|^2 \leq D \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2,
\]

provided $\sum_{k=1}^{n} a_k b_k = 0$, $n \geq 2$.

For $n = 2$, we choose $a_1 = a, a_2 = -b, b_1 = b, b_2 = -a$ and $p_1 = p, p_2 = 1 - p$ to get:

\[
[p a^2 + (1 - p) b^2] [p b^2 + (1 - p) a^2] \leq D [a^2 + b^2]^2.
\]

If in (3.6) we choose $p = \frac{1}{2}$, then we get

\[
\frac{1}{4} (a^2 + b^2)^2 \leq D (a^2 + b^2)^2,
\]

which shows that $D \geq \frac{1}{4}$. \(\blacksquare\)

Corollary 4. Let $x_k \in \mathbb{C}, k \in \{1, \ldots, n\}$ and $p_k, k \in \{1, \ldots, n\}$ be a probability sequence. Then:

\[
\sum_{k=1}^{n} p_k |x_k|^2 - \left| \sum_{k=1}^{n} p_k x_k \right|^2 = \sum_{j=1}^{n} p_j \left| x_j - \sum_{l=1}^{n} p_l x_l \right|^2 \leq \frac{1}{4} \sum_{k=1}^{n} p_k^2 \sum_{k=1}^{n} \left| x_k - \sum_{l=1}^{n} p_l x_l \right|^2.
\]

Proof. It is obvious by (3.4) on choosing $a_k = p_k$ and $b_k = x_k - \sum_{l=1}^{n} p_l x_l, k \in \{1, \ldots, n\}$. \(\blacksquare\)

The following result that provides a refinement of Theorem 2 should be noted.

Theorem 4. Let $x_k, y_k \in \mathbb{C}, k \in \{1, \ldots, n\}$ and $p_k, k \in \{1, \ldots, n\}$ be a probability sequence with the property that

\[
\sum_{k=1}^{n} p_k x_k y_k = 0.
\]
Then

\[
(3.8) \quad \max_{i \in \{1, \ldots, n\}} \{p_i \ |x_i y_i|\}
\leq \frac{1}{2} \max_{i \in \{1, \ldots, n\}} \left[ p_i \ |x_i|^2 \sum_{k=1}^{n} p_k |y_k|^2 + p_i \ |y_i|^2 \sum_{k=1}^{n} p_k |x_k|^2 \right] \left( \sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2 \right)^{\frac{1}{2}}
\leq \frac{1}{2} \left( \sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2 \right)^{\frac{1}{2}}.
\]

**Proof.** As in the proof of Theorem 1, we have

\[ p_i \ |x_i y_i| \leq \left( \sum_{k=1}^{n} p_k |x_k|^2 - p_i \ |x_i|^2 \right) \left( \sum_{k=1}^{n} p_k |y_k|^2 - p_i \ |y_i|^2 \right)^{\frac{1}{2}}, \]

which gives

\[
p_i^2 \ |x_i y_i|^2 \leq \left( \sum_{k=1}^{n} p_k |x_k|^2 - p_i \ |x_i|^2 \right) \left( \sum_{k=1}^{n} p_k |y_k|^2 - p_i \ |y_i|^2 \right) = \sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2 + p_i^2 \ |x_i|^2 |y_i|^2 - p_i \ |y_i|^2 \sum_{k=1}^{n} p_k |x_k|^2 - p_i \ |x_i|^2 \sum_{k=1}^{n} p_k |y_k|^2 
\]

i.e.,

\[
(3.9) \quad p_i \ |x_i|^2 \sum_{k=1}^{n} p_k |y_k|^2 + p_i \ |y_i|^2 \sum_{k=1}^{n} p_k |x_k|^2 \leq \sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2
\]

for each \( i \in \{1, \ldots, n\} \).

Taking the maximum in (3.9) over \( i \in \{1, \ldots, n\} \), we get the second inequality in (3.8).

The first inequality follows by the elementary fact that

\[
p_i \ |x_i|^2 \sum_{k=1}^{n} p_k |y_k|^2 + p_i \ |y_i|^2 \sum_{k=1}^{n} p_k |x_k|^2 \geq 2p_i \ |x_i| \ |y_i| \left( \sum_{k=1}^{n} p_k |x_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} p_k |y_k|^2 \right)^{\frac{1}{2}},
\]

for each \( i \in \{1, \ldots, n\} \).

**Remark 7.** The inequality (3.8) is obviously a refinement of the inequality (2.11) in Theorem 2. However, the inequality (3.8) is not apparently useful in deriving upper bounds for the maximal deviation of \( x_k \) from its weighted mean \( \sum_{j=1}^{n} p_j x_j \), as the inequality (2.11).
References


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