POWER INEQUALITIES FOR THE NUMERICAL RADIUS OF A PRODUCT OF TWO OPERATORS IN HILBERT SPACES

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Abstract. Some power inequalities for the numerical radius of a product of two operators in Hilbert spaces with applications for commutators and self-commutators are given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [11, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [11, p. 8]:

$$w(T)=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [11, p. 9]:

$$w(T)\le\|T\|\le 2w(T), \tag{1.2}$$

for any $T\in B(H)$

For other results on numerical radii, see [12], Chapter 11.

If A,B are two bounded linear operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then

$$w(AB)\le 4w(A)w(B). \tag{1.3}$$

In the case that $AB=BA$, then

$$w(AB)\le 2w(A)w(B). \tag{1.4}$$

The following results are also well known [11, p. 38]:

If A is a unitary operator that commutes with another operator B, then

$$w(AB)\le w(B). \tag{1.5}$$

If A is an isometry and $AB=BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB=BA$ and $AB^*=B^*A$. If the operators A and B double commute, then [11, p. 38]

$$w(AB)\le w(B)\|A\|. \tag{1.6}$$

As a consequence of the above, we have [11, p. 39]:

Let A be a normal operator commuting with B, then

$$w(AB)\le w(A)w(B). \tag{1.7}$$

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For other results and historical comments on the above see [11, p. 39–41]. For recent inequalities involving the numerical radius, see [1]-[9], [13], [14]-[16] and [17].

2. Inequalities for a Product of Two Operators

Theorem 1. *For any $A, B \in B(H)$ and $r \geq 1$, we have the inequality:*

$$w^r(B^*A) \leq \frac{1}{2}\|(A^*A)^r + (B^*B)^r\|. \tag{2.1}$$

The constant $\frac{1}{2}$ is best possible.

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle .,. \rangle)$ we have:

$$\begin{aligned} |\langle B^*Ax, x\rangle| &= |\langle Ax, Bx\rangle| \leq \|Ax\| \cdot \|Bx\| \\ &= \langle A^*Ax, x\rangle^{1/2} \cdot \langle B^*Bx, x\rangle^{1/2}, \qquad x \in H. \end{aligned} \tag{2.2}$$

Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$\begin{aligned} \langle A^*Ax, x\rangle^{1/2} \cdot \langle B^*Bx, x\rangle^{1/2} &\leq \frac{\langle A^*Ax, x\rangle + \langle B^*Bx, x\rangle}{2} \\ &\leq \left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*Bx, x\rangle^r}{2}\right)^{\frac{1}{r}} \end{aligned} \tag{2.3}$$

for any $x \in H$.

It is known that if P is a positive operator then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [15])

$$\langle Px, x\rangle^r \leq \langle P^r x, x\rangle. \tag{2.4}$$

Applying this property to the positive operator A^*A and B^*B, we deduce that

$$\begin{aligned} \left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*Bx, x\rangle^r}{2}\right)^{\frac{1}{r}} &\leq \left(\frac{\langle (A^*A)^r x, x\rangle + \langle (B^*B)^r x, x\rangle}{2}\right)^{\frac{1}{r}} \\ &= \left(\frac{\langle [(A^*A)^r + (B^*B)^r] x, x\rangle}{2}\right)^{\frac{1}{r}} \end{aligned} \tag{2.5}$$

for any $x \in H$, $\|x\| = 1$.

Now, on making use of the inequalities (2.2), (2.3) and (2.5), we get the inequality:

$$|\langle (B^*A)^r x, x\rangle|^r \leq \frac{1}{2}\langle [(A^*A)^r + (B^*B)^r] x, x\rangle \tag{2.6}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.6) and since the operator $[(A^*A)^r + (B^*B)^r]$ is self-adjoint, we deduce the desired inequality (2.1).

For $r = 1$ and $B = A$, we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1). □

Corollary 1. *For any $A \in B(H)$ and $r \geq 1$ we have the inequalities:*

$$w^r(A) \leq \frac{1}{2}\|(A^*A)^r + I\| \tag{2.7}$$

and

$$w^r(A^2) \leq \frac{1}{2}\|(A^*A)^r + (AA^*)^r\|, \tag{2.8}$$

respectively.

A different approach is considered in the following result:

Theorem 2. *For any $A, B \in B(H)$ and any $\alpha \in (0,1)$ and $r \geq 1$, we have the inequality:*

$$w^{2r}(B^*A) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(B^*B)^{\frac{r}{1-\alpha}} \right\|. \tag{2.9}$$

Proof. By Schwarz's inequality, we have:

$$\begin{aligned} |\langle (B^*A)x, x\rangle|^2 &\leq \langle (A^*A)x, x\rangle \cdot \langle (B^*B)x, x\rangle \\ &= \left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x \right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} x, x \right\rangle, \end{aligned} \tag{2.10}$$

for any $x \in H$.

It is well known that (see for instance [15]) if P is a positive operator and $q \in (0,1]$ then for any $u \in H$, $\|u\| = 1$, we have

$$\langle P^q u, u\rangle \leq \langle Pu, u\rangle^q. \tag{2.11}$$

Applying this property to the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$ $(\alpha \in (0,1))$, we have

$$\begin{aligned} &\left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x \right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} x, x \right\rangle \\ &\qquad \leq \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^{\alpha} \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha}, \end{aligned} \tag{2.12}$$

for any $x \in H$, $\|x\| = 1$.

Now, utilising the weighted arithmetic mean - geometric mean inequality, i.e., $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b$, $\alpha \in (0,1)$, $a, b \geq 0$, we get

$$\begin{aligned} &\left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^{\alpha} \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha} \\ &\qquad \leq \alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle \end{aligned} \tag{2.13}$$

for any $x \in H$, $\|x\| = 1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t) = t^r$, $r \geq 1$, namely

$$\alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}, \qquad \alpha \in (0,1), \ a, b \geq 0,$$

we deduce that

$$\begin{aligned} &\alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle \\ &\leq \left[\alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^r + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^r\right]^{\frac{1}{r}} \\ &\leq \left[\alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} x, x \right\rangle\right]^{\frac{1}{r}}, \end{aligned} \tag{2.14}$$

for any $x \in H$, $\|x\| = 1$, where, for the last inequality we used the inequality (2.4) for the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$.

Now, on making use of the inequalities (2.10), (2.12), (2.13) and (2.14), we get

$$|\langle (B^*A)x, x\rangle|^{2r} \leq \left\langle \left[\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(B^*B)^{\frac{r}{1-\alpha}}\right] x, x \right\rangle \tag{2.15}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.15) produces the desired inequality (2.9). □

Remark 1. *The particular case* $\alpha = \frac{1}{2}$ *produces the inequality*

$$w^{2r}(B^*A) \le \frac{1}{2}\left\|(A^*A)^{2r} + (B^*B)^{2r}\right\|, \tag{2.16}$$

for $r \ge 1$. *Notice that* $\frac{1}{2}$ *is best possible in (2.16) since for* $r = 1$ *and* $B = A$ *we get in both sides of (2.16) the same quantity* $\|A\|^4$.

Corollary 2. *For any* $A \in B(H)$ *and* $\alpha \in (0,1)$, $r \ge 1$, *we have the inequalities*

$$w^{2r}(A) \le \left\|\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) I\right\| \tag{2.17}$$

and

$$w^{2r}\left(A^2\right) \le \left\|\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(AA^*)^{\frac{r}{1-\alpha}}\right\|, \tag{2.18}$$

respectively.

Moreover, we have

$$\|A\|^{4r} \le \left\|\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(A^*A)^{\frac{r}{1-\alpha}}\right\|. \tag{2.19}$$

3. Inequalities for the Sum of Two Products

The following result may be stated:

Theorem 3. *For any* $A, B, C, D \in B(H)$ *and* $r, s \ge 1$ *we have:*

$$w^2\left(\frac{B^*A + D^*C}{2}\right) \le \left\|\frac{(A^*A)^r + (C^*C)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(B^*B)^s + (D^*D)^s}{2}\right\|^{\frac{1}{s}}. \tag{3.1}$$

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle .,. \rangle)$ we have:

$$\begin{aligned} &|\langle (B^*A + D^*C)x, x\rangle|^2 \\ &= |\langle B^*Ax, x\rangle + \langle D^*Cx, x\rangle|^2 \\ &\le [|\langle B^*Ax, x\rangle| + |\langle D^*Cx, x\rangle|]^2 \\ &\le \left[\langle A^*Ax, x\rangle^{\frac{1}{2}} \cdot \langle B^*Bx, x\rangle^{\frac{1}{2}} + \langle C^*Cx, x\rangle^{\frac{1}{2}} \cdot \langle D^*Dx, x\rangle^{\frac{1}{2}}\right]^2, \end{aligned} \tag{3.2}$$

for any $x \in H$.

Now, on utilising the elementary inequality:

$$(ab + cd)^2 \le \left(a^2 + c^2\right)\left(b^2 + d^2\right), \qquad a, b, c, d \in \mathbb{R},$$

we then conclude that:

$$\begin{aligned} &\langle A^*Ax, x\rangle^{\frac{1}{2}} \cdot \langle B^*Bx, x\rangle^{\frac{1}{2}} + \langle C^*Cx, x\rangle^{\frac{1}{2}} \cdot \langle D^*Dx, x\rangle^{\frac{1}{2}} \\ &\qquad \le (\langle A^*Ax, x\rangle + \langle C^*Cx, x\rangle) \cdot (\langle B^*Bx, x\rangle + \langle D^*Dx, x\rangle), \end{aligned} \tag{3.3}$$

for any $x \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for $r,s\geq 1$ that

$$\begin{aligned}(3.4)\quad & \left(\langle A^{*}Ax,x\rangle+\langle C^{*}Cx,x\rangle\right)\cdot\left(\langle B^{*}Bx,x\rangle+\langle D^{*}Dx,x\rangle\right)\\ &\leq 4\left\langle\left[\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right]x,x\right\rangle^{\frac{1}{r}}\cdot\left\langle\left[\frac{(B^{*}B)^{s}+(D^{*}D)^{s}}{2}\right]x,x\right\rangle^{\frac{1}{s}}\end{aligned}$$

for any $x\in H$, $\|x\|=1$.

Consequently, by (3.2) – (3.4) we have:

$$\begin{aligned}(3.5)\quad & \left|\left\langle\left[\frac{B^{*}A+D^{*}C}{2}\right]x,x\right\rangle\right|^{2}\\ &\leq\left\langle\left[\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right]x,x\right\rangle^{\frac{1}{r}}\cdot\left\langle\left[\frac{(B^{*}B)^{s}+(D^{*}D)^{s}}{2}\right]x,x\right\rangle^{\frac{1}{s}}\end{aligned}$$

for any $x\in H$, $\|x\|=1$.

Taking the supremum over $x\in H$, $\|x\|=1$ we deduce the desired inequality (3.1). □

Remark 2. *If $s=r$, then the inequality (3.1) is equivalent with:*

$$w^{2r}\left(\frac{B^{*}A+D^{*}C}{2}\right)\leq\left\|\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right\|\cdot\left\|\frac{(B^{*}B)^{r}+(D^{*}D)^{r}}{2}\right\|.\tag{3.6}$$

Corollary 3. *For any $A,C\in B(H)$ we have:*

$$w^{2r}\left(\frac{A+C}{2}\right)\leq\left\|\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right\|,\tag{3.7}$$

where $r\geq 1$. Also, we have

$$w^{2}\left(\frac{A^{2}+C^{2}}{2}\right)\leq\left\|\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(AA^{*})^{s}+(CC^{*})^{s}}{2}\right\|^{\frac{1}{s}}\tag{3.8}$$

for all $r,s\geq 1$, and in particular

$$w^{2r}\left(\frac{A^{2}+C^{2}}{2}\right)\leq\left\|\frac{(A^{*}A)^{r}+(C^{*}C)^{r}}{2}\right\|\cdot\left\|\frac{(AA^{*})^{r}+(CC^{*})^{r}}{2}\right\|\tag{3.9}$$

for $r\geq 1$.

The inequality (3.7) follows from (3.1) for $B=D=I$, while the inequality (3.8) is obtained from the same inequality (3.1) for $B=A^{*}$ and $D=C^{*}$.

Another particular result of interest is the following one:

Corollary 4. *For any $A,B\in B(H)$ we have:*

$$\left\|\frac{B^{*}A+A^{*}B}{2}\right\|^{2}\leq\left\|\frac{(A^{*}A)^{r}+(B^{*}B)^{r}}{2}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(A^{*}A)^{s}+(B^{*}B)^{s}}{2}\right\|^{\frac{1}{s}}\tag{3.10}$$

for $r,s\geq 1$ and, in particular,

$$\left\|\frac{B^{*}A+A^{*}B}{2}\right\|^{r}\leq\left\|\frac{(A^{*}A)^{r}+(B^{*}B)^{r}}{2}\right\|\tag{3.11}$$

for any $r\geq 1$.

The inequality (3.9) follows from (3.1) for $D = A$ and $C = B$ and taking into account that the operator $\frac{1}{2}(B^*A + A^*B)$ is self-adjoint and that

$$w\left[\frac{1}{2}(B^*A + A^*B)\right] = \left\|\frac{B^*A + A^*B}{2}\right\|.$$

Another particular case that might be of interest is the following one.

Corollary 5. *For any $A, D \in B(H)$ we have:*

$$w^2\left(\frac{A+D}{2}\right) \le \left\|\frac{(A^*A)^r + I}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(DD^*)^s + I}{2}\right\|^{\frac{1}{s}}, \tag{3.12}$$

where $r, s \ge 1$. In particular

$$w^2(A) \le \left\|\frac{(A^*A)^r + I}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(AA^*)^s + I}{2}\right\|^{\frac{1}{s}}. \tag{3.13}$$

Moreover, for any $r \ge 1$ we have

$$w^{2r}(A) \le \left\|\frac{(A^*A)^r + I}{2}\right\| \cdot \left\|\frac{(AA^*)^r + I}{2}\right\|.$$

The proof is obvious by the inequality (3.1) on choosing $B = I$, $C = I$ and writing the inequality for D^* instead of D.

Remark 3. *If $T \in B(H)$ and $T = A + iC$, i.e., A and C are its Cartesian decomposition, then we get from (3.7) that*

$$w^{2r}(T) \le 2^{2r-1}\left\|A^{2r} + C^{2r}\right\|,$$

for any $r \ge 1$.

Also, since $A = \operatorname{Re}(T) = \frac{T+T^*}{2}$ and $C = \operatorname{Im}(T) = \frac{T-T^*}{2i}$, then from (3.7) we get the following inequalities as well:

$$\|\operatorname{Re}(T)\|^{2r} \le \left\|\frac{(T^*T)^r + (TT^*)^r}{2}\right\|$$

and

$$\|\operatorname{Im}(T)\|^{2r} \le \left\|\frac{(T^*T)^r + (TT^*)^r}{2}\right\|$$

for any $r \ge 1$.

In terms of the *Euclidean radius* of two operators $w_e(\cdot,\cdot)$, where, as in [1],

$$w_e(T, U) := \sup_{\|x\|=1}\left(|\langle Tx, x\rangle|^2 + |\langle Ux, x\rangle|^2\right)^{\frac{1}{2}},$$

we have the following result as well.

Theorem 4. *For any $A, B, C, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the inequality:*

$$w_e^2(B^*A, D^*C) \le \|(A^*A)^p + (C^*C)^p\|^{1/p} \cdot \|(B^*B)^q + (D^*D)^q\|^{1/q}. \tag{3.14}$$

Proof. For any $x \in H$, $\|x\| = 1$ we have the inequalities

$$\begin{aligned}
&|\langle B^*Ax, x\rangle|^2 + |\langle D^*Cx, x\rangle|^2 \\
&\le \langle A^*Ax, x\rangle \cdot \langle B^*Bx, x\rangle + \langle C^*Cx, x\rangle \cdot \langle D^*Dx, x\rangle \\
&\le (\langle A^*Ax, x\rangle^p + \langle C^*Cx, x\rangle^p)^{1/p} \cdot (\langle B^*Bx, x\rangle^q + \langle D^*Dx, x\rangle^q)^{1/q} \\
&\le (\langle (A^*A)^p x, x\rangle + \langle (C^*C)^p x, x\rangle)^{1/p} \cdot (\langle (B^*B)^q x, x\rangle + \langle (D^*D)^q x, x\rangle)^{1/q} \\
&\le \langle [(A^*A)^p + (C^*C)^p]\, x, x\rangle^{1/p} \cdot \langle [(B^*B)^q + (D^*D)^q]\, x, x\rangle^{1/q}.
\end{aligned}$$

Taking the supremum over $x \in H$, $\|x\| = 1$ and noticing that the operators $(A^*A)^p + (C^*C)^p$ and $(B^*B)^q + (D^*D)^q$ are self-adjoint, we deduce the desired inequality (3.14). □

The following particular case is of interest.

Corollary 6. *For any $A, C \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$w_e^2(A, C) \le 2^{1/q} \|(A^*A)^p + (C^*C)^p\|^{1/p}.$$

The proof follows from (3.14) for $B = D = I$.

Corollary 7. *For any $A, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$w_e^2(A, D) \le \|(A^*A)^p + I\|^{1/p} \cdot \|(D^*D)^q + I\|^{1/q}.$$

4. Vector Inequalities for the Commutator

The commutator of two bounded linear operators T and U is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators T and U, by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality:

$$\|TU - UT\| \le 2\|T\|\|U\|. \tag{4.1}$$

In [10], the following result has been obtained as well

$$\|TU - UT\| \le 2 \min\{\|T\|, \|U\|\} \min\{\|T - U\|, \|T + U\|\}. \tag{4.2}$$

By utilising Theorem 3 we can state the following result for the numerical radius of the commutator.

Proposition 1. *For any $T, U \in B(H)$ and $r, s \ge 1$ we have*

$$w^2(TU - UT) \le 2^{2-\frac{1}{r}-\frac{1}{s}} \|(T^*T)^r + (U^*U)^r\|^{\frac{1}{r}} \cdot \|(TT^*)^s + (UU^*)^s\|^{\frac{1}{s}}. \tag{4.3}$$

Proof. Follows by Theorem 3 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. □

Remark 4. *In particular, for $r = s$ we get from (4.3) that*

$$w^{2r}(TU - UT) \le 2^{2r-2} \|(T^*T)^r + (U^*U)^r\| \cdot \|(TT^*)^r + (UU^*)^r\| \tag{4.4}$$

and for $r = 1$ we get

$$w^2(TU - UT) \le \|T^*T + U^*U\| \cdot \|TT^* + UU^*\|. \tag{4.5}$$

For a bounded linear operator $T \in B(H)$, the self-commutator is the operator $T^{*}T - TT^{*}$. Observe that the operator $V := -i(T^{*}T - TT^{*})$ is self-adjoint and $w(V) = \|V\|$, i.e.,

$$w(T^{*}T - TT^{*}) = \|T^{*}T - TT^{*}\|.$$

Now, utilising (4.3) for $U = T^{*}$ we can state the following corollary.

Corollary 8. *For any $T \in B(H)$ we have the inequality:*

$$\|T^{*}T - TT^{*}\|^{2} \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \left\|(T^{*}T)^{r} + (TT^{*})^{r}\right\|^{\frac{1}{r}} \cdot \left\|(T^{*}T)^{s} + (TT^{*})^{s}\right\|^{\frac{1}{s}}. \tag{4.6}$$

In particular, we have

$$\|T^{*}T - TT^{*}\|^{r} \leq 2^{r-1} \left\|(T^{*}T)^{r} + (TT^{*})^{r}\right\|, \tag{4.7}$$

for any $r \geq 1$.

Moreover, for $r = 1$ we have

$$\|T^{*}T - TT^{*}\| \leq \|T^{*}T + TT^{*}\|. \tag{4.8}$$

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