POWER INEQUALITIES FOR THE NUMERICAL RADIUS OF A
PRODUCT OF TWO OPERATORS IN HILBERT SPACES

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Abstract. Some power inequalities for the numerical radius of a product of
two operators in Hilbert spaces with applications for commutators and self-
commutators are given.

1. Introduction

Let \((H; \langle \cdot, \cdot \rangle)\) be a complex Hilbert space. The numerical range of an operator
\(T\) is the subset of the complex numbers \(\mathbb{C}\) given by [11, p. 1]:
\[
W(T) = \{ \langle Tx, x \rangle ; x \in H, \|x\| = 1 \}.
\]

The numerical radius \(w(T)\) of an operator \(T\) on \(H\) is given by [11, p. 8]:
\[
w(T) = \sup \{ |\lambda| ; \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle| ; \|x\| = 1 \}.
\]

It is well known that \(w(\cdot)\) is a norm on the Banach algebra \(B(H)\) of all bounded
linear operators \(T : H \to H\). This norm is equivalent to the operator norm. In fact,
the following more precise result holds [11, p. 9]:
\[
w(T) \leq \|T\| \leq 2w(T),
\]
for any \(T \in B(H)\).

For other results on numerical radii, see [12], Chapter 11.

If \(A, B\) are two bounded linear operators on the Hilbert space \((H, \langle \cdot, \cdot \rangle)\), then
\[
w(AB) \leq 4w(A)w(B).
\]
In the case that \(AB = BA\), then
\[
w(AB) \leq 2w(A)w(B).
\]

The following results are also well known [11, p. 38]:

If \(A\) is a unitary operator that commutes with another operator \(B\), then
\[
w(AB) \leq w(B).
\]
If \(A\) is an isometry and \(AB = BA\), then (1.5) also holds true.

We say that \(A\) and \(B\) double commute if \(AB = BA\) and \(AB^* = B^*A\). If the
operators \(A\) and \(B\) double commute, then [11, p. 38]
\[
w(AB) \leq w(B)\|A\|.
\]

As a consequence of the above, we have [11, p. 39]:

Let \(A\) be a normal operator commuting with \(B\), then
\[
w(AB) \leq w(A)w(B).
\]

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2. Inequalities for a Product of Two Operators

**Theorem 1.** For any $A, B \in B(H)$ and $r \geq 1$, we have the inequality:

$$w^r(B^*A) \leq \frac{1}{2} \| (A^*A)^r + (B^*B)^r \|.$$  

The constant $\frac{1}{2}$ is best possible.

**Proof.** By the Schwarz inequality in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ we have:

$$\langle B^*Ax, x \rangle = \|Ax\| \cdot \|Bx\|$$

$$= \langle A^*Ax, x \rangle^{1/2} \cdot (B^*Bx, x)^{1/2}, \quad x \in H.$$  

Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$\langle A^*Ax, x \rangle^{1/2} \cdot (B^*Bx, x)^{1/2} \leq \frac{\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle}{2}$$

$$\leq \left( \frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right)^{\frac{1}{r}}$$

for any $x \in H$.

It is known that if $P$ is a positive operator then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [15])

$$\langle Px, x \rangle^r \leq \langle P^r x, x \rangle.$$  

Applying this property to the positive operator $A^*A$ and $B^*B$, we deduce that

$$\left( \frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right)^{\frac{1}{r}} \leq \left( \frac{\langle (A^*A)^r x, x \rangle + \langle (B^*B)^r x, x \rangle}{2} \right)^{\frac{1}{r}}$$

$$= \left( \frac{\langle (A^*A)^r + (B^*B)^r \rangle x, x \rangle}{2} \right)^{\frac{1}{r}}$$

for any $x \in H$, $\|x\| = 1$.

Now, on making use of the inequalities (2.2), (2.3) and (2.5), we get the inequality:

$$\|((B^*A)^r x, x)\|^r \leq \frac{1}{2} \langle (A^*A)^r + (B^*B)^r \rangle x, x \rangle$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.6) and since the operator $[(A^*A)^r + (B^*B)^r]$ is self-adjoint, we deduce the desired inequality (2.1).

For $r = 1$ and $B = A$, we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1).

**Corollary 1.** For any $A \in B(H)$ and $r \geq 1$ we have the inequalities:

$$w^r(A) \leq \frac{1}{2} \| (A^*A)^r + I \|$$

and

$$w^r(A^2) \leq \frac{1}{2} \| (A^*A)^r + (AA^*)^r \|,$$
respectively.

A different approach is considered in the following result:

**Theorem 2.** For any $A, B \in B(H)$ and any $\alpha \in (0, 1)$ and $r \geq 1$, we have the inequality:

\[(2.9) \quad w^{2r} (B^* A) \leq \|\alpha (A^* A)^{\frac{1}{r}} + (1 - \alpha) (B^* B)^{\frac{1}{r}} \|.
\]

**Proof.** By Schwarz’s inequality, we have:

\[(2.10) \quad |\langle (B^* A) x, x \rangle|^2 \leq \langle (A^* A) x, x \rangle \cdot \langle (B^* B) x, x \rangle
\]

\[= \left< \left( [A^* A]^{\frac{1}{r}} \right)^{\alpha} x, x \right> \cdot \left< \left( [B^* B]^{\frac{1}{r}} \right)^{1-\alpha} x, x \right>, \]

for any $x \in H$.

It is well known that (see for instance [15]) if $P$ is a positive operator and $q \in (0, 1)$ then for any $u \in H$, $\|u\| = 1$, we have

\[(2.11) \quad \langle Pu, u \rangle \leq \langle Pu, u \rangle^q.
\]

Applying this property to the positive operators $(A^* A)^{\frac{1}{2}}$ and $(B^* B)^{\frac{1}{2}}$ ($\alpha \in (0, 1)$), we have

\[(2.12) \quad \left< \left( [A^* A]^{\frac{1}{2}} \right)^{\alpha} x, x \right> \cdot \left< \left( [B^* B]^{\frac{1}{2}} \right)^{1-\alpha} x, x \right>
\]

\[\leq \left< \left( [A^* A]^{\frac{1}{2}} x, x \right)^{\alpha} \cdot \left< (B^* B)^{\frac{1}{2}} x, x \right>^{1-\alpha},
\]

for any $x \in H$, $\|x\| = 1$.

Now, using the weighted arithmetic mean - geometric mean inequality, i.e., $a^\alpha b^{1-\alpha} \leq a \alpha + (1 - \alpha) b$, $\alpha \in (0, 1)$, $a, b \geq 0$, we get

\[(2.13) \quad \left< (A^* A)^{\frac{1}{2}} x, x \right>^\alpha \cdot \left< (B^* B)^{\frac{1}{2}} x, x \right>^{1-\alpha}
\]

\[\leq \alpha \left< (A^* A)^{\frac{1}{2}} x, x \right> + (1 - \alpha) \left< (B^* B)^{\frac{1}{2}} x, x \right>
\]

for any $x \in H$, $\|x\| = 1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t) = t^r$, $r \geq 1$, namely

\[\alpha a (1 - \alpha) b \leq (\alpha a^r + (1 - \alpha) b^r)^{\frac{1}{r}}, \quad \alpha \in (0, 1), \quad a, b \geq 0,
\]

we deduce that

\[(2.14) \quad \alpha \left< (A^* A)^{\frac{1}{2}} x, x \right> + (1 - \alpha) \left< (B^* B)^{\frac{1}{2}} x, x \right>
\]

\[\leq \left[ \alpha \left< (A^* A)^{\frac{1}{2}} x, x \right> + (1 - \alpha) \left< (B^* B)^{\frac{1}{2}} x, x \right> \right]^{\frac{1}{r}}
\]

\[\leq \left[ \alpha \left< (A^* A)^{\frac{1}{2}} x, x \right> + (1 - \alpha) \left< (B^* B)^{\frac{1}{2}} x, x \right> \right]^{\frac{1}{r}},
\]

for any $x \in H$, $\|x\| = 1$, where, for the last inequality we used the inequality (2.4) for the positive operators $(A^* A)^{\frac{1}{2}}$ and $(B^* B)^{\frac{1}{2}}$.

Now, on making use of the inequalities (2.10), (2.12), (2.13) and (2.14), we get

\[(2.15) \quad |\langle (B^* A) x, x \rangle|^{2r} \leq \left< \left[ \alpha (A^* A)^{\frac{1}{2}} + (1 - \alpha) (B^* B)^{\frac{1}{2}} \right] x, x \right>
\]
for any \( x \in H, \|x\| = 1 \). Taking the supremum over \( x \in H, \|x\| = 1 \) in (2.15) produces the desired inequality (2.9).

**Remark 1.** The particular case \( \alpha = \frac{1}{2} \) produces the inequality

\[
(2.16) \quad \|w^{2r}(B^*A)\| \leq \frac{1}{2} \left\| \left( A^*A \right)^{\frac{r}{2}} + \left( B^*B \right)^{\frac{r}{2}} \right\|,
\]

for \( r \geq 1 \). Notice that \( \frac{1}{2} \) is best possible in (2.16) since for \( r = 1 \) and \( B = A \) we get in both sides of (2.16) the same quantity \( \|A\|^4 \).

**Corollary 2.** For any \( A \in B(H) \) and \( \alpha \in (0,1) \), \( r \geq 1 \), we have the inequalities

\[
(2.17) \quad \|w^{2r}(A^2)\| \leq \left\| \alpha \left( A^*A \right)^{\frac{r}{2}} + (1 - \alpha) I \right\|
\]

and

\[
(2.18) \quad \|w^{2r}(A^2)\| \leq \left\| \alpha \left( A^*A \right)^{\frac{r}{2}} + (1 - \alpha) \left( A^*A \right)^{\frac{r}{2}} \right\|,
\]

respectively.

Moreover, we have

\[
(2.19) \quad \|A\|^{4r} \leq \left\| \alpha \left( A^*A \right)^{\frac{r}{2}} + (1 - \alpha) \left( A^*A \right)^{\frac{r}{2}} \right\|.
\]

## 3. Inequalities for the Sum of Two Products

The following result may be stated:

**Theorem 3.** For any \( A, B, C, D \in B(H) \) and \( r, s \geq 1 \) we have:

\[
(3.1) \quad \|w^{2} \left( \frac{B^*A + D^*C}{2} \right)\| \leq \frac{\left\| \left( A^*A \right)^{\frac{r}{2}} + \left( C^*C \right)^{\frac{r}{2}} \right\|^{\frac{1}{2}} \cdot \left\| \left( B^*B \right)^{\frac{s}{2}} + \left( D^*D \right)^{\frac{s}{2}} \right\|^{\frac{1}{2}}}{2}.
\]

**Proof.** By the Schwarz inequality in the Hilbert space \( (H; \langle \cdot, \cdot \rangle) \) we have:

\[
(3.2) \quad \|\langle (B^*A + D^*C), x \rangle \|^2
\]

\[
= |\langle B^*Ax, x \rangle + \langle D^*Cx, x \rangle|^2
\]

\[
\leq \left\| \langle B^*Ax, x \rangle \right\|^2 + \left\| \langle D^*Cx, x \rangle \right\|^2
\]

\[
\leq \left[ \langle A^*Ax, x \rangle \frac{1}{2} \cdot \langle B^*Bx, x \rangle \frac{1}{2} + \langle C^*Cx, x \rangle \frac{1}{2} \cdot \langle D^*Dx, x \rangle \frac{1}{2} \right]^2,
\]

for any \( x \in H \).

Now, on utilising the elementary inequality:

\[
(ab + cd)^2 \leq (a^2 + c^2) (b^2 + d^2), \quad a, b, c, d \in \mathbb{R},
\]

we then conclude that:

\[
(3.3) \quad \langle A^*Ax, x \rangle \frac{1}{2} \cdot \langle B^*Bx, x \rangle \frac{1}{2} + \langle C^*Cx, x \rangle \frac{1}{2} \cdot \langle D^*Dx, x \rangle \frac{1}{2}
\]

\[
\leq (\langle A^*Ax, x \rangle + \langle C^*Cx, x \rangle) \cdot (\langle B^*Bx, x \rangle + \langle D^*Dx, x \rangle),
\]

for any \( x \in H \).
Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for \( r, s \geq 1 \) that

\[
(A^r x, x) + (C^s x, x) = (B^r x, x) + (D^s x, x)
\]

\[
\leq 4 \left[ \frac{(A^r A + C^s C)^r}{2} x, x \right]^{\frac{1}{2}} \cdot \left[ \frac{(B^r B + D^s D)^s}{2} x, x \right]^{\frac{1}{2}}
\]

for any \( x \in H, \|x\| = 1 \).

Consequently, by (3.2) – (3.4) we have:

\[
B^r A + D^s C \leq \left[ \frac{(A^r A + C^s C)^r}{2} \right] x, x \right]^{\frac{1}{2}} \cdot \left[ \frac{(B^r B + D^s D)^s}{2} x, x \right]^{\frac{1}{2}}
\]

for any \( x \in H, \|x\| = 1 \).

Taking the supremum over \( x \in H, \|x\| = 1 \) we deduce the desired inequality (3.1).

\( \square \)

**Remark 2.** If \( s = r \), then the inequality (3.1) is equivalent with:

\[
w^{2r} \left( \frac{B^r A + D^s C}{2} \right) \leq \left( \frac{(A^r A + (C^s C)^r)}{2} \right) \cdot \left( \frac{(B^r B + (D^s D)^s)}{2} \right)
\]

**Corollary 3.** For any \( A, C \in B (H) \) we have:

\[
w^{2r} \left( \frac{A + C}{2} \right) \leq \left( \frac{(A^r A + (C^s C)^r)}{2} \right),
\]

where \( r \geq 1 \). Also, we have

\[
w^2 \left( \frac{A^2 + C^2}{2} \right) \leq \left( \frac{(A^r A + (C^s C)^r)}{2} \right) \cdot \left( \frac{(AA^* + (CC^*)^s)}{2} \right)
\]

for all \( r, s \geq 1 \), and in particular

\[
w^{2r} \left( \frac{A^2 + C^2}{2} \right) \leq \left( \frac{(A^r A + (C^s C)^r)}{2} \right) \cdot \left( \frac{(AA^*)^r + (CC^*)^s}{2} \right)
\]

for \( r \geq 1 \).

The inequality (3.7) follows from (3.1) for \( B = D = I \), while the inequality (3.8) is obtained from the same inequality (3.1) for \( B = A^* \) and \( D = C^* \).

Another particular result of interest is the following one:

**Corollary 4.** For any \( A, B \in B (H) \) we have:

\[
\left( \frac{B^r A + A^r B}{2} \right)^r \leq \left( \frac{(A^r A + (B^s B)^r)}{2} \right) \cdot \left( \frac{(AA^*)^r + (B^s B)^s}{2} \right)
\]

for \( r, s \geq 1 \) and, in particular,

\[
\left( \frac{B^r A + A^r B}{2} \right)^r \leq \left( \frac{(A^r A + (B^s B)^r)}{2} \right)
\]

for any \( r \geq 1 \).
The inequality (3.9) follows from (3.1) for \( D = A \) and \( C = B \) and taking into account that the operator \( \frac{1}{2} (B^* A + A^* B) \) is self-adjoint and that

\[
w \left[ \frac{1}{2} (B^* A + A^* B) \right] = \left\| \frac{B^* A + A^* B}{2} \right\| .
\]

Another particular case that might be of interest is the following one.

**Corollary 5.** For any \( A, D \in B(H) \) we have:

\[
w^2 \left( \frac{A + D}{2} \right) \leq \left\| \frac{(A^* A)^r + I}{2} \right\|^\frac{1}{2} \cdot \left\| \frac{(DD^*)^s + I}{2} \right\|^\frac{1}{2},
\]

where \( r, s \geq 1 \). In particular

\[
w^2 (A) \leq \left\| \frac{(A^* A)^r + I}{2} \right\|^\frac{1}{2} \cdot \left\| \frac{(AA^*)^s + I}{2} \right\|^\frac{1}{2}.
\]

Moreover, for any \( r \geq 1 \) we have

\[
w^{2r} (A) \leq \left\| \frac{(A^* A)^r + I}{2} \right\| \cdot \left\| \frac{(AA^*)^r + I}{2} \right\|.
\]

The proof is obvious by the inequality (3.1) on choosing \( B = I, C = I \) and writing the inequality for \( D^* \) instead of \( D \).

**Remark 3.** If \( T \in B(H) \) and \( T = A + iC \), i.e., \( A \) and \( C \) are its Cartesian decomposition, then we get from (3.7) that

\[w^{2r} (T) \leq 2^{2r-1} \left\| A^{2r} + C^{2r} \right\|,\]

for any \( r \geq 1 \).

Also, since \( A = \text{Re} (T) = \frac{T + T^*}{2} \) and \( C = \text{Im} (T) = \frac{T - T^*}{2i} \), then from (3.7) we get the following inequalities as well:

\[\left\| \text{Re} (T) \right\|^{2r} \leq \left\| \frac{(T^* T)^r + (TT^*)^r}{2} \right\|\]

and

\[\left\| \text{Im} (T) \right\|^{2r} \leq \left\| \frac{(T^* T)^r + (TT^*)^r}{2} \right\|\]

for any \( r \geq 1 \).

In terms of the Euclidean radius of two operators \( w_\epsilon (\cdot, \cdot) \), where, as in [1],

\[w_\epsilon (T, U) := \sup_{\|x\|=1} \left( \left| \langle T x, x \rangle \right|^2 + \left| \langle U x, x \rangle \right|^2 \right)^{\frac{1}{2}},\]

we have the following result as well.

**Theorem 4.** For any \( A, B, C, D \in B(H) \) and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have the inequality:

\[
w^2_e (B^* A, D^* C) \leq \left\| (A^* A)^p \right\|^{1/p} \cdot \left\| (B^* B)^q \right\|^{1/q}.
\]
Proof. For any $x \in H$, $\|x\| = 1$ we have the inequalities

\[
\langle |B^* Ax, x| \rangle^2 + \langle |D^* Cx, x| \rangle^2 \\
\leq \langle A^* Ax, x \cdot (B^* Bx, x) + \langle C^* Cx, x \cdot (D^* Dx, x) \\
\leq \langle (A^* Ax, x) + (C^* Cx, x)^p \rangle^{1/p} \cdot \langle (B^* Bx, x)^q + (D^* Dx, x)^q \rangle^{1/q} \\
\leq ((A^* A)^p x, x) + (C^* C)^p x, x) \rangle^{1/p} \cdot ((B^* B)^q x, x) + (D^* D)^q x, x) \rangle^{1/q} \\
\leq \langle (A^* A)^p + (C^* C)^p \rangle^{1/p} \cdot ((B^* B)^q + (D^* D)^q \rangle^{1/q}.
\]

Taking the supremum over $x \in H$, $\|x\| = 1$ and noticing that the operators $(A^* A)^p + (C^* C)^p$ and $(B^* B)^q + (D^* D)^q$ are self-adjoint, we deduce the desired inequality (3.14).

\[\square\]

The following particular case is of interest.

Corollary 6. For any $A, C \in B (H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:

\[w_e^2 (A, C) \leq 2^{1/q} \|(A^* A)^p + (C^* C)^p\|^{1/p}.\]

The proof follows from (3.14) for $B = D = I$.

Corollary 7. For any $A, D \in B (H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:

\[w_e^2 (A, D) \leq \|(A^* A)^p + I\|^{1/p} \cdot \|(D^* D)^q + I\|^{1/q}.\]

4. Vector Inequalities for the Commutator

The commutator of two bounded linear operators $T$ and $U$ is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators $T$ and $U$, by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality:

\[(4.1) \quad \|TU - UT\| \leq 2 \|T\| \|U\|\]

In [10], the following result has been obtained as well

\[(4.2) \quad \|TU - UT\| \leq 2 \min \{\|T\|, \|U\|\} \min \{\|T - U\|, \|T + U\|\}.
\]

By utilising Theorem 3 we can state the following result for the numerical radius of the commutator.

Proposition 1. For any $T, U \in B (H)$ and $r, s \geq 1$ we have

\[(4.3) \quad w^2 (TU - UT) \leq 2^{2^{r} - 2} \|(T^* T)^r + (U^* U)^s\|^\frac{1}{2} \cdot \|(TT^*)^r + (UU^*)^s\|^\frac{1}{2}.
\]

Proof. Follows by Theorem 3 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. \[\square\]

Remark 4. In particular, for $r = s$ we get from (4.3) that

\[(4.4) \quad w^2 (TU - UT) \leq 2 \|T^* T + U^* U\| \cdot \|TT^* + UU^*\|
\]

and for $r = 1$ we get

\[(4.5) \quad w^2 (TU - UT) \leq \|T^* T + U^* U\| \cdot \|TT^* + UU^*\|.
\]
For a bounded linear operator $T \in B(H)$, the self-commutator is the operator $T^*T - TT^*$. Observe that the operator $V := -i(T^*T - TT^*)$ is self-adjoint and $w(V) = \|V\|$, i.e.,

$$w(T^*T - TT^*) = \|T^*T - TT^*\|.$$

Now, utilising (4.3) for $U = T^*$ we can state the following corollary.

**Corollary 8.** For any $T \in B(H)$ we have the inequality:

$$\|T^*T - TT^*\| \leq 2^{2^{-\frac{1}{2} - \frac{1}{2}}} \|(T^*T)^r + (TT^*)^r\|^{\frac{1}{2}} \cdot \|(T^*T)^s + (TT^*)^s\|^{\frac{1}{2}}.$$

In particular, we have

$$\|T^*T - TT^*\|^r \leq 2^{r-1} \|(T^*T)^r + (TT^*)^r\|,$$

for any $r \geq 1$.

Moreover, for $r = 1$ we have

$$\|T^*T - TT^*\| \leq \|T^*T + TT^*\|.$$

**References**


