A GENERALISED TRAPEZOID TYPE INEQUALITY FOR
CONVEX FUNCTIONS

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ABSTRACT. A generalised trapezoid inequality for convex functions and applications for quadrature rules are given. A refinement and a counterpart result for the Hermite-Hadamard inequalities are obtained and some inequalities for pdf’s and (HH) — divergence measure are also mentioned.

1. Introduction

The following integral inequality for the generalised trapezoid formula was obtained in [2] (see also [1, p. 68]):

**Theorem 1.** Let \( f : [a, b] \to \mathbb{R} \) be a function of bounded variation. We have the inequality

\[
\left| \int_a^b f(t) \, dt - [(x-a)f(a) + (b-x)f(b)] \right| \leq \left\lfloor \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right\rfloor \nabla_a^b(f),
\]

holding for all \( x \in [a, b] \), where \( \nabla_a^b(f) \) denotes the total variation of \( f \) on the interval \([a,b]\).

The constant \( \frac{1}{2} \) is the best possible one.

This result may be improved if one assumes the monotonicity of \( f \) as follows (see [1, p. 76]):

**Theorem 2.** Let \( f : [a, b] \to \mathbb{R} \) be a monotonic nondecreasing function on \([a, b]\). Then we have the inequality:

\[
\left| \int_a^b f(t) \, dt - [(x-a)f(a) + (b-x)f(b)] \right| \\
\leq (b-x)f(b) - (x-a)f(a) + \int_a^b \text{sgn}(x-t)f(t) \, dt \\
\leq (x-a)(f(x) - f(a)) + (b-x)(f(b) - f(x)) \\
\leq \left\lfloor \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right\rfloor |f(b) - f(a)|
\]

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for all $x \in [a, b]$.

The above inequalities are sharp.

If the mapping is Lipschitzian, then the following result holds as well [3] (see also [1, p. 82]).

**Theorem 3.** Let $f : [a, b] \to \mathbb{R}$ be an $L$–Lipschitzian function on $[a, b]$, i.e., $f$ satisfies the condition:

$$(L) \quad |f(s) - f(t)| \leq L |s - t| \quad \text{for any } s, t \in [a, b] \quad (L > 0 \text{ is given}).$$

Then we have the inequality:

$$\left| \int_a^b f(t) \, dt - [(x - a)f(a) + (b - x)f(b)] \right| \leq \left[ \frac{1}{4} (b - a)^2 + \left( x - \frac{a + b}{2} \right)^2 \right] L$$

for any $x \in [a, b]$.

The constant $\frac{1}{4}$ is best in (1.3).

If we would assume absolute continuity for the function $f$, then the following estimates in terms of the Lebesgue norms of the derivative $f'$ hold [1, p. 93].

**Theorem 4.** Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then for any $x \in [a, b]$, we have

$$\left| \int_a^b f(t) \, dt - [(x - a)f(a) + (b - x)f(b)] \right| 
\leq \begin{cases} 
\left[ \frac{1}{4} (b - a)^2 + \left( x - \frac{a + b}{2} \right)^2 \right] \|f'\|_\infty & \text{if } f' \in L_\infty [a, b]; \\
\frac{1}{(q + 1)^{\frac{1}{q}}} \left[ (x - a)^{q+1} + (b - x)^{q+1} \right]^{\frac{1}{q}} \|f'\|_p & \text{if } f' \in L_p [a, b], \\
\left[ \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right] \|f'\|_1, & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1;
\end{cases}$$

where $\|\cdot\|_p (p \in [1, \infty])$ are the Lebesgue norms, i.e.,

$$\|f'\|_\infty = \text{ess sup}_{s \in [a, b]} |f'(s)|$$

and

$$\|f'\|_p := \left( \int_a^b |f'(s)|^p \, ds \right)^{\frac{1}{p}}, \quad p \geq 1.$$
2. The Results

The following theorem providing a lower bound for the difference

\[(x - a) f(a) + (b - x) f(b) - \int_a^b f(t) \, dt\]

holds.

**Theorem 5.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\). Then for any \( x \in (a, b) \) we have the inequality

\[
\frac{1}{2} \left[ (b - x)^2 f_+'(x) - (x - a)^2 f_-'(x) \right] \leq (x - a) f(a) + (b - x) f(b) - \int_a^b f(t) \, dt.
\]

The constant \( \frac{1}{2} \) in the left hand side of (2.1) is sharp in the sense that it cannot be replaced by a larger constant.

**Proof.** It is easy to see that for any locally absolutely continuous function \( f : (a, b) \to \mathbb{R} \), we have the identity

\[(x - a) f(a) + (b - x) f(b) - \int_a^b f(t) \, dt = \int_a^b (t - x) f'(t) \, dt\]

for any \( x \in (a, b) \), where \( f' \) is the derivative of \( f \) which exists a.e. on \([a, b]\).

Since \( f \) is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any \( x \in (a, b) \), we have the inequalities:

\[(2.3) \quad f'(t) \leq f_-'(x) \quad \text{for a.e.} \quad t \in [a, x]\]

and

\[(2.4) \quad f'(t) \geq f_+'(x) \quad \text{for a.e.} \quad t \in [x, b].\]

If we multiply (2.3) by \( x - t \geq 0, \ t \in [a, x] \) and integrate on \([a, x]\), we get

\[(2.5) \quad \int_a^x (x - t) f'(t) \, dt \leq \frac{1}{2} (x - a)^2 f_-'(x)\]

and if we multiply (2.4) by \( t - x \geq 0, \ t \in [x, b] \) and integrate on \([x, b]\), we also have

\[(2.6) \quad \int_x^b (t - x) f'(t) \, dt \geq \frac{1}{2} (b - x)^2 f_+'(x).\]

Finally, if we subtract (2.5) from (2.6) and use the representation (2.2), we deduce the desired inequality (2.1).

Now, assume that (2.1) holds with a constant \( C > 0 \) instead of \( \frac{1}{2} \), i.e.,

\[(2.7) \quad C \left[ (b - x)^2 f_+'(x) - (x - a)^2 f_-'(x) \right] \leq (x - a) f(a) + (b - x) f(b) - \int_a^b f(t) \, dt.\]
Consider the convex function $f_0(t) := k \left| t - \frac{a+b}{2} \right|$, $k > 0$, $t \in [a,b]$. Then
\[
f'_0 \left( \frac{a+b}{2} \right) = k, \quad f'_0 \left( \frac{a+b}{2} \right) = -k,
\]
\[
f_0(a) = \frac{k(b-a)}{2} = f_0(b), \quad \int_a^b f_0(t) \, dt = \frac{1}{4} k(b-a)^2.
\]
If in (2.7) we choose $f_0$ as above and $x = \frac{a+b}{2}$, then we get
\[
C \left[ \frac{1}{4} (b-a)^2 k + \frac{1}{4} (b-a)^2 k \right] \leq \frac{1}{4} k(b-a)^2
\]
giving $C \leq \frac{1}{2}$, and the sharpness of the constant is proved. \( \blacksquare \)

Now, recall that the following inequality which is well known in the literature as the *Hermite-Hadamard inequality* for convex functions holds
\[
\text{(H-H)} \quad f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2}.
\]
The following corollary gives a sharp lower bound for the difference
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt.
\]
**Corollary 1.** Let $f : [a,b] \rightarrow \mathbb{R}$ be a convex function on $[a,b]$. Then
\[
(2.8) \quad 0 \leq \frac{1}{8} \left[ f'_+ \left( \frac{a+b}{2} \right) - f'_- \left( \frac{a+b}{2} \right) \right] (b-a)
\]
\[
\leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt.
\]
The constant $\frac{1}{8}$ is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for $f_0(t) = k \left| t - \frac{a+b}{2} \right|$, $t \in [a,b]$, $k > 0$.

When $x$ is a point of differentiability, we may state the following corollary as well.

**Corollary 2.** Let $f$ be as in Theorem 5. If $x \in (a,b)$ is a point of differentiability for $f$, then
\[
(2.9) \quad (b-a) \left( \frac{a+b}{2} - x \right) f'(x) \leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t) \, dt.
\]

**Remark 1.** If $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is convex on $I$ and if we choose $x \in \bar{I}$ ($\bar{I}$ is the interior of $I$), $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a, b \in I$, then from (2.1) we may write
\[
(2.10) \quad 0 \leq \frac{1}{8} h^2 \left[ f'_+ (x) - f'_- (x) \right] \leq \frac{f(a) + f(b)}{2} \cdot h - \int_{x - \frac{h}{2}}^{x + \frac{h}{2}} f(t) \, dt
\]
and the constant $\frac{1}{8}$ is sharp in (2.10).

The following result providing an upper bound for the difference
\[
(x-a) f(a) + (b-x) f(b) - \int_a^b f(t) \, dt
\]
also holds.
Theorem 6. Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\). Then for any \( x \in [a, b] \), we have the inequality:

\[
(x - a) f (a) + (b - x) f (b) - \int_{a}^{b} f (t) \, dt \leq \frac{1}{2} \left[ (b - x)^2 f_-'(b) - (x - a)^2 f_+'(a) \right].
\]

The constant \( \frac{1}{2} \) is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. If either \( f_+'(a) = -\infty \) or \( f_-'(b) = +\infty \), then the inequality (2.11) evidently holds true.

Assume that \( f_+'(a) \) and \( f_-'(b) \) are finite.

Since \( f \) is convex on \([a, b]\), we have

\[
f' (t) \geq f_+'(a) \quad \text{for a.e. } t \in [a, x]
\]

and

\[
f' (t) \leq f_-'(b) \quad \text{for a.e. } t \in [x, b].
\]

If we multiply (2.12) by \((x - t) \geq 0, t \in [a, x]\) and integrate on \([a, x]\), then we deduce

\[
\int_{a}^{x} (x - t) f' (t) \, dt \geq \frac{1}{2} (x - a)^2 f_+'(a)
\]

and if we multiply (2.13) by \( t - x \geq 0, t \in [x, b]\) and integrate on \([x, b]\), then we also have

\[
\int_{x}^{b} (t - x) f' (t) \, dt \leq \frac{1}{2} (b - x)^2 f_-'(b).
\]

Finally, if we subtract (2.14) from (2.15) and use the representation (2.2), we deduce the desired inequality (2.11).

Now, assume that (2.11) holds with a constant \( D > 0 \) instead of \( \frac{1}{2} \), i.e.,

\[
(x - a) f (a) + (b - x) f (b) - \int_{a}^{b} f (t) \, dt \leq D \left[ (b - x)^2 f_-'(b) - (x - a)^2 f_+'(a) \right].
\]

If we consider the convex function \( f_0 : [a, b] \to \mathbb{R}, f_0 (t) = k \left| t - \frac{a + b}{2} \right| \), then we have \( f_-'(b) = k, f_+'(a) = -k \) and by (2.16) we deduce for \( x = \frac{a + b}{2} \) that

\[
\frac{1}{4} k (b - a)^2 \leq D \left[ \frac{1}{4} k (b - a)^2 + \frac{1}{4} k (b - a)^2 \right]
\]

giving \( D \geq \frac{1}{2} \), and the sharpness of the constant is proved. \( \blacksquare \)

The following corollary related to the Hermite-Hadamard inequality is interesting as well.

Corollary 3. Let \( f : [a, b] \to \mathbb{R} \) be convex on \([a, b]\). Then

\[
0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f (t) \, dt \leq \frac{1}{8} \left[ f_-'(b) - f_+'(a) \right] (b - a)
\]

and the constant \( \frac{1}{8} \) is sharp.
Remark 2. Denote $B := f' (b)$, $A := f'_+ (a)$ and assume that $B \neq A$, i.e., $f$ is not constant on $(a, b)$. Then

$$(b - x)^2 B - (x - a)^2 A = (B - A) \left[ x - \left( \frac{bB - aA}{B - A} \right) \right]^2 - \frac{AB}{B - A} (b - a)^2$$

and by (2.11) we get

$$(x - a) f (a) + (b - x) f (b) - \int_a^b f (t) dt \leq (B - A) \left[ x - \left( \frac{bB - aA}{B - A} \right) \right]^2 - \frac{AB}{(B - A)^2} (b - a)^2$$

for any $x \in [a, b]$.

If $A \geq 0$, then $x_0 = \frac{bB - aA}{B - A} \in [a, b]$, and by (2.18) for $x = \frac{bB - aA}{B - A}$ we get that

$$(2.19) \quad 0 \leq \frac{1}{2} \frac{AB}{B - A} (b - a) \leq \frac{bf (a) - Af (b)}{B - A} - \frac{1}{b - a} \int_a^b f (t) dt$$

which is an interesting inequality in itself as well.

3. The Composite Case

Consider the division $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and denote $h_i := x_{i+1} - x_i$ ($i = 0, n - 1$). If $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, n - 1$) are intermediate points, then we will denote by

$$G_n (f; I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - x_i) f (x_i) + (x_{i+1} - \xi_i) f (x_{i+1})]$$

the generalised trapezoid rule associated to $f$, $I_n$ and $\xi$.

The following theorem providing upper and lower bounds for the remainder in approximating the integral $\int_a^b f (t) dt$ of a convex function $f$ in terms of the generalised trapezoid rule holds.

Theorem 7. Let $f : [a, b] \to \mathbb{R}$ be a convex function and $I_n$ and $\xi$ be as above. Then we have:

$$(3.2) \quad \int_a^b f (t) dt = G_n (f; I_n, \xi) - S_n (f; I_n, \xi),$$

where $G_n (f; I_n, \xi)$ is the generalised Trapezoid Rule defined by (3.1) and the remainder $S_n (f; I_n, \xi)$ satisfies the estimate:

$$(3.3) \quad \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_+ (\xi_i) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+ (\xi_i) \leq S_n (f; I_n, \xi) \leq \frac{1}{2} \left[ (b - \xi_{n-1})^2 f'_+ (b) + \sum_{i=1}^{n-1} [(x_i - \xi_{i-1})^2 f'_+ (x_i) - (\xi_i - x_i)^2 f'_+ (x_i)] \right. - (\xi_0 - a)^2 f'_+ (a) \right] .$$
Proof. If we write the inequalities (2.1) and (2.11) on the interval \([x_i, x_{i+1}]\) and for the intermediate points \(\xi_i \in [x_i, x_{i+1}]\), then we have
\[
\frac{1}{2} \left[ (x_{i+1} - \xi_i)^2 f'_+(x_i) - (\xi_i - x_i)^2 f'_-(\xi_i) \right]
\]
\[
\leq (\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) - \int_{x_i}^{x_{i+1}} f(t) \, dt
\]
\[
\leq \frac{1}{2} \left[ (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) - (\xi_i - x_i)^2 f'_+(x_i) \right] .
\]
Summing the above inequalities over \(i\) from 0 to \(n - 1\), we deduce
\[
(3.4) \quad \frac{1}{2} \sum_{i=0}^{n-1} \left[ (x_{i+1} - \xi_i)^2 f'_+(\xi_i) - (\xi_i - x_i)^2 f'_-(\xi_i) \right]
\leq G_n(f; I_n; \xi) - \int_a^b f(t) \, dt
\leq \frac{1}{2} \left[ \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) \right] .
\]
However,
\[
\sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) = (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=0}^{n-2} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1})
\]
\[
= (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=1}^{n-1} (x_i - \xi_{i-1})^2 f'_-(x_i)
\]
and
\[
\sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) = \sum_{i=1}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) + (\xi_0 - a)^2 f'_+(a)
\]
and then, by (3.4), we deduce the desired estimate (3.3).  

The following corollary may be useful in practical applications.

**Corollary 4.** Let \(f : [a, b] \rightarrow \mathbb{R}\) be a differentiable convex function on \([a, b]\). Then we have the representation (3.2) and \(S_n(f; I_n, \xi)\) satisfies the estimate:
\[
(3.5) \quad \sum_{i=0}^{n-1} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right) h_i f'(\xi_i)
\leq S_n(f; I_n, \xi)
\leq \frac{1}{2} \left[ (b - \xi_{n-1})^2 f'_-(b) - (\xi_0 - a)^2 f'_+(a)
\right.
\left. + \sum_{i=1}^{n-1} \left( x_i - \frac{\xi_i + \xi_{i-1}}{2} \right) (\xi_i - \xi_{i-1}) f'(x_i) \right] .
\]

We may also consider the trapezoid quadrature rule:
\[
(3.6) \quad T_n(f; I_n) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i .
\]
Using the above results, we may state the following corollary.
Corollary 5. Assume that $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$ and $I_n$ is a division as above. Then we have the representation

$$
\int_a^b f(t) \, dt = T_n(f; I_n) - Q_n(f; I_n)
$$

where $T_n(f; I_n)$ is the mid-point quadrature formula given in (3.6) and the remainder $Q_n(f; I_n)$ satisfies the estimates

$$
0 \leq \frac{1}{8} \sum_{i=0}^{n-1} \left[ f'_+(\frac{x_i + x_{i+1}}{2}) - f'_-(\frac{x_i + x_{i+1}}{2}) \right] h_i^2 \leq Q_n(f; I_n) \leq \frac{1}{8} \sum_{i=0}^{n-1} \left[ f'_+(x_{i+1}) - f'_-(x_i) \right] h_i^2.
$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

4. Applications for P.D.F.s

Let $X$ be a random variable with the probability density function $f : [a, b] \subset \mathbb{R} \to [0, \infty)$ and with cumulative distribution function $F(x) = \Pr(X \leq x)$.

The following theorem holds.

Theorem 8. If $f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ is monotonically increasing on $[a, b]$, then we have the inequality:

$$
\frac{1}{2} \left[ (b - x)^2 f_+(x) - (x - a)^2 f_-(x) \right] + x \leq E(X) \leq \frac{1}{2} \left[ (b - x)^2 f_+(b) - (x - a)^2 f_-(a) \right] + x
$$

for any $x \in (a, b)$, where $f_\pm(\alpha)$ represent respectively the right and left limits of $f$ in $\alpha$ and $E(X)$ is the expectation of $X$.

The constant $\frac{1}{2}$ is sharp in both inequalities.

The second inequality also holds for $x = a$ or $x = b$.

Proof. Follows by Theorem 5 and 6 applied for the convex cdf function $F(x) = \int_a^x f(t) \, dt$, $x \in [a, b]$ and taking into account that

$$
\int_a^b F(x) \, dx = b - E(X).
$$

Finally, we may state the following corollary in estimating the expectation of $X$.

Corollary 6. With the above assumptions, we have

$$
\frac{1}{8} \left[ f_+(\frac{a+b}{2}) - f_-(\frac{a+b}{2}) \right] (b-a)^2 + \frac{a+b}{2} \leq E(X) \leq \frac{1}{8} \left[ f_+(b) - f_-(a) \right] (b-a)^2 + \frac{a+b}{2}.
$$
5. **Applications for $HH$–Divergence**

Assume that a set $\chi$ and the $\sigma$–finite measure $\mu$ are given. Consider the set of all probability densities on $\mu$ to be

\begin{equation}
\Omega := \left\{ p \mid p : \Omega \to \mathbb{R}, \ p(x) \geq 0, \ \int_\chi p(x) \, d\mu(x) = 1 \right\}.
\end{equation}

Csiszár’s $f$–divergence is defined as follows [4]

\begin{equation}
D_f(p, q) := \int_\chi p(x) f \left( \frac{q(x)}{p(x)} \right) \, d\mu(x), \ p, q \in \Omega,
\end{equation}

where $f$ is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived.

In [5], Shioya and Da-te introduced the generalised Lin-Wong $f$–divergence $D_f(p, \frac{1}{2}p + \frac{1}{2}q)$ and the Hermite-Hadamard ($HH$) divergence

\begin{equation}
D^f_{HH}(p, q) := \int_\chi \frac{p^2(x)}{q(x) - p(x)} \left( \int_1^{\frac{q(x)}{p(x)}} f(t) \, dt \right) \, d\mu(x), \ p, q \in \Omega,
\end{equation}

and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

\begin{equation}
D_f(p, \frac{1}{2}p + \frac{1}{2}q) \leq D^f_{HH}(p, q) \leq \frac{1}{2} D_f(p, q),
\end{equation}

provided that $f$ is convex and normalised, i.e., $f(1) = 0$.

The following result in estimating the difference

\begin{equation}
\frac{1}{2} D_f(p, q) - D^f_{HH}(p, q)
\end{equation}

holds.

**Theorem 9.** Let $f : [0, \infty) \to \mathbb{R}$ be a normalised convex function and $p, q \in \Omega$. Then we have the inequality:

\begin{equation}
0 \leq \frac{1}{8} \left[ f'_{+} \left( \frac{a + b}{2} \right) - f'_{-} \left( \frac{a + b}{2} \right) \right] |b - a|
\end{equation}

Proof. Using the double inequality

\begin{align*}
0 & \leq \frac{1}{8} \left[ f'_{+} \left( \frac{a + b}{2} \right) - f'_{-} \left( \frac{a + b}{2} \right) \right] |b - a| \\
& \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) \, dt \\
& \leq \frac{1}{8} \left[ f_{-}(b) - f'_{+}(a) \right] (b - a)
\end{align*}
for the choices $a = 1$, $b = \frac{q(x)}{p(x)}$, $x \in \chi$, multiplying with $p(x) \geq 0$ and integrating over $x$ on $\chi$ we get

$$0 \leq \frac{1}{8} \int_{\chi} \left[ f'_+ \left( \frac{p(x) + q(x)}{2p(x)} \right) - f'_- \left( \frac{p(x) + q(x)}{2p(x)} \right) \right] |q(x) - p(x)| d\mu(x)$$

$$\leq \frac{1}{2} D_f(p, q) - D_{f_{HH}}(p, q)$$

$$\leq \frac{1}{8} \int_{\chi} \left[ f'_+ \frac{q(x)}{p(x)} - f'_+(1) \right] (q(x) - p(x)) d\mu(x),$$

which is clearly equivalent to (5.5).

**Corollary 7.** With the above assumptions and if $f$ is differentiable on $(0, \infty)$, then

$$0 \leq \frac{1}{2} D_f(p, q) - D_{f_{HH}}(p, q) \leq \frac{1}{8} D_{f^\prime(-1)}(p, q).$$

### References


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