On some variants of Jensen’s inequality

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Abstract.
Some variants of Jensen’s discrete inequality are derived. These include interpolations of the basic relation for subadditive maps and of the generalised triangle inequality.

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1 Introduction

Let $X$ be a real linear space and $C \subseteq X$ a convex set in $X$, that is, a set such that

$$x, y \in C \quad \text{and} \quad \lambda \in [0, 1] \quad \text{imply} \quad \lambda x + (1 - \lambda) y \in C.$$
If \( f : C \to \mathbb{R} \) is convex, \( f \) satisfies

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

for all \( x, y \in C \) and \( \lambda \in [0, 1] \). If \( p_i \geq 0 \) \((i = 1, \ldots, n)\) with \( P_n := \sum_{i=1}^{n} p_i > 0 \) and \( y_i \in C \) \((i = 1, \ldots, n)\), we have the Jensen inequality

\[
f \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i y_i \right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(y_i)
\]

(see [2] or [8, p. 6]). For some recent generalizations, refinements and applications the reader is referred to [1]–[7], [9] and [8, p. 20].

In this paper we show that several new results flow from simple but judicious applications of Abel’s identity, which gives the following. Suppose \( X \) is a linear space, \( x_i \in X \) \((i = 1, \ldots, n)\) and \( s_n := \sum_{i=1}^{n} x_i \). If \( a_i \) is real \((i = 1, \ldots, n)\), then

\[
\sum_{i=1}^{n} a_i x_i = a_1 s_1 + \sum_{i=2}^{n} a_i (s_i - s_{i-1})
\]

\[
= \sum_{i=1}^{n-1} (a_i - a_{i+1}) s_i + a_n s_n.
\]

Consequences include an interpolation of the basic inequality for subadditive maps and of the generalised triangle inequality.

2 Results

We will start with the following theorem.

**Theorem 2.1.** Let \( X \) be a linear space and \( f : X \to \mathbb{R} \) a convex mapping, \( x_1, \ldots, x_n \in X \) and \( 0 \neq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \). Then

\[
f \left( a_1^{-1} \sum_{i=1}^{n} a_i x_i \right) \leq a_1^{-1} \left\{ a_1 f(x_1) + \sum_{i=2}^{n} a_i \left[ f \left( \sum_{j=1}^{i} x_j \right) - f \left( \sum_{j=1}^{i-1} x_j \right) \right] \right\}.
\]
Proof. Choose \( p_i := a_i - a_{i+1} \) (1 \( \leq \) \( i < n \)), \( p_n := a_n \) and \( y_i = s_i (i = 1, \ldots, n) \) in Jensen’s theorem. We derive

\[
f \left[ \frac{\sum_{i=1}^{n} (a_i - a_{i+1}) s_i}{\sum_{i=1}^{n} (a_i - a_{i+1})} \right] \leq \frac{\sum_{i=1}^{n} (a_i - a_{i+1}) f(s_i)}{\sum_{i=1}^{n} (a_i - a_{i+1})},
\]

where for notational simplicity we have introduced \( a_{n+1} := 0 \). The desired result now follows by Abel’s identity. \( \square \)

**Corollary 2.2.** Let \( g : X \to (0, \infty) \) be logarithmically concave, that is, let \( \ln g \) be concave. Under the assumptions of the theorem

\[
g \left( a_1^{-1} \sum_{i=1}^{n} a_i x_i \right) \geq \left\{ [g(x_1)]^{a_1} \prod_{i=2}^{n} \left[ g \left( \sum_{j=1}^{i} x_j \right) \right]^{a_i} \right\}^{1/a_1}.
\]

The result follows from the theorem for the convex mapping \( f = -\ln g \).

Suppose that the mapping \( \varphi : X \to \mathbb{R} \) is subadditive, that is, for \( \alpha, \beta \) nonnegative we have

\[
\varphi(\alpha x + \beta y) \leq \alpha \varphi(x) + \beta \varphi(y).
\]

By mathematical induction we have for all \( \alpha_i \geq 0 \) and \( y_i \in X \) (\( i = \ldots, n \)) that

\[
\varphi \left( \sum_{i=1}^{n} \alpha_i y_i \right) \leq \sum_{i=1}^{n} \alpha_i \varphi(y_i).
\]

This inequality may be interpolated as follows.

**Corollary 2.3.** Let \( \varphi : X \to \mathbb{R} \) be subadditive, \( y_1, \ldots, y_n \in X \) and \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0 \). Then

\[
\varphi \left( \sum_{i=1}^{n} \alpha_i y_i \right) \leq \alpha_1 \varphi(y_1) + \sum_{i=2}^{n} \alpha_i \left[ \varphi \left( \sum_{j=1}^{i} y_j \right) - \varphi \left( \sum_{j=1}^{i-1} y_j \right) \right]
\leq \sum_{i=1}^{n} \alpha_i \varphi(y_i).
\]

Proof. As \( \varphi \) is subadditive, it is convex. The first desired inequality follows from Theorem 2.1.
For the second, we observe that for \( 2 \leq i \leq n \),
\[
\varphi \left( \sum_{j=1}^{i} y_j \right) - \varphi \left( \sum_{j=1}^{i-1} y_j \right) \leq \varphi(y_i).
\]
Multiplying the \( i \)th inequality by \( \alpha_i \) and summing over \( i \) provides the desired result.

Our second main result is the following.

**Theorem 2.4.** Let \( f : X \to \mathbb{R} \) be convex and \( x_i \in X \) \((i = 1, \ldots, n)\) satisfy
\[
\sum_{j=1}^{i} m_j \geq 0 \quad (1 \leq i \leq n)
\]
and
\[
\sum_{i=1}^{n} (n + 1 - i) m_i > 0.
\]
Then
\[
f \left( \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} (n + 1 - i) m_i} \right) \leq \frac{\sum_{i=1}^{n} \sum_{j=i}^{n} f(x_j - x_{j+1})}{\sum_{i=1}^{n} (n + 1 - i) m_i},
\]
where again we put \( x_{n+1} := 0 \) for notational convenience.

**Proof.** Let \( s_i = \sum_{j=1}^{i} m_j \) \((1 \leq i \leq n)\). Then by Abel’s identity
\[
\sum_{i=1}^{n} m_i x_i = s_1 x_1 + \sum_{i=2}^{n} (s_i - s_{i-1}) x_i = \sum_{i=1}^{n} s_i (x_i - x_{i+1}).
\]
Applying Jensen’s inequality provides
\[
f \left[ \frac{\sum_{i=1}^{n} s_i (x_i - x_{i+1})}{\sum_{i=1}^{n} s_i} \right] \leq \frac{\sum_{i=1}^{n} s_i f(x_i - x_{i+1})}{\sum_{i=1}^{n} x_i}.
\]
The numerator on the right–hand side may be written as
\[
\sum_{i=1}^{n} m_i \sum_{j=1}^{n} f(x_j - x_{j+1})
\]
and we have the desired result. \hfill \Box

**Corollary 2.5.** Let $g : X \to (0, \infty)$ be logarithmically concave. With the above assumptions

$$g \left( \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} (n+1-i)m_i} \right) \geq \left\{ \prod_{i=1}^{n} \left[ \prod_{j=i}^{n} g(x_j - x_{j+1}) \right]^{m_i} \right\}^{\frac{1}{\sum_{i=1}^{n} (n+1-i)m_i}}.$$

The result follows from the theorem with the choice of convex mapping $f = -\ln g$.

### 3 Applications

We now derive some particular applications relating to homely choices of convex function.

1. Let $x_i > 0 \ (i = 1, \ldots, n)$ with $0 \neq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then

$$\sum_{i=1}^{n} a_i x_i \geq a_1 \left[ x_1^{a_1} \prod_{i=2}^{n} \left( \frac{\sum_{j=1}^{i-1} x_j}{\sum_{j=1}^{i} x_j} \right)^{a_i} \right]^{1/a_1}.$$

The result follows from Corollary 2.2 with the mapping $g : (0, \infty) \to (0, \infty)$ given by $g(x) = x$.

Suppose $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and $n_i \in \mathbb{R}$ with $m_1 \geq 0$. In the same way we have from Corollary 2.5 that

$$\sum_{i=1}^{n} m_i x_i \geq \left\{ \prod_{i=1}^{n} \left( x_{j} - x_{j+1} \right) \right\}^{\frac{1}{\sum_{i=1}^{n} (n+1-i)m_i}}.$$

2. Let $x_i > 0 \ (i = 1, \ldots, n)$ and $0 \neq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then

$$a_1^2 \leq \left( \sum_{i=1}^{n} a_i x_i \right) \left( \frac{a_1}{x_1} - \sum_{i=2}^{n} \frac{a_i x_i}{\left( \sum_{j=1}^{i-1} x_j \right) \left( \sum_{k=1}^{i} x_k \right)} \right).$$

This follows from Theorem 2.1 applied to the convex mapping $f(x) = 1/x$ on the interval $(0, \infty)$. 

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3. Let \( x_i \in \mathbb{R} \) and \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \). Then
\[
\left( \sum_{i=1}^{n} a_i x_i \right)^2 \leq a_1 \left\{ a_1 x_1^2 + \sum_{i=2}^{n} a_i x_i \left[ x_i + 2 \sum_{j=1}^{i-1} x_j \right] \right\}.
\]
This follows from Theorem 2.1 applied for the convex mapping \( f(x) = x^2 \) (\( x \in \mathbb{R} \)).

4. Consider the mapping \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = \ln(1 + e^x) \). We have \( f'(x) = e^x/(1 + e^x) \) and \( f''(x) = e^x/(1 + e^x)^2 \), which shows that \( f \) is convex on \( \mathbb{R} \).

Let \( 0 \neq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \) and \( x_1, \ldots, x_n \in \mathbb{R} \). Then by Theorem 2.1
\[
\ln \left[ 1 + \exp \left( a_1^{-1} \sum_{i=1}^{n} a_i x_i \right) \right] \\
\leq a_1 \ln[1 + e^{x_1}] + \sum_{i=2}^{n} a_i \left[ \ln \left\{ 1 + \exp \left( \sum_{j=1}^{i} x_j \right) \right\} \right] \\
- \ln \left\{ 1 + \exp \left( \sum_{j=1}^{i-1} x_j \right) \right\} \\
= \ln \left\{ (1 + e^{x_1})^{a_1} \prod_{i=2}^{n} \frac{1 + \exp \left( \sum_{j=1}^{i} x_j \right)}{1 + \exp \left( \sum_{j=1}^{i-1} x_j \right)} \right\},
\]
whence
\[
1 + \exp \left( a_1^{-1} \sum_{i=1}^{n} a_i x_i \right) \leq [1 + e^{x_1}]^{a_1} \prod_{i=2}^{n} \left[ 1 + \exp \left( \sum_{j=1}^{i} x_j \right) \right]^{a_i}.
\]

5. Let \( X \) be a real normed space and \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0 \). Then for \( x_i \in X (i = 1, \ldots, n) \) we have the refinement
\[
\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq \alpha_1 \left\| x_1 \right\| + \sum_{i=2}^{n} \alpha_i \left( \left\| \sum_{j=1}^{i} x_j \right\| - \left\| \sum_{j=1}^{i-1} x_j \right\| \right) \\
\leq \sum_{i=1}^{n} \alpha_i \left\| x_i \right\|.
\]
of the generalised triangle inequality. The result follows from Corollary 2.3.

References


