OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE MODULUS OF THE DERIVATIVES ARE CONVEX AND APPLICATIONS

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Abstract. Some inequalities of the Ostrowski type for functions whose modulus of derivatives are convex and applications for special means and to the $f$ and $HH$-divergences in Information Theory are given.

1. Introduction

The following Ostrowski type inequalities for absolutely continuous functions are known (see [2], [3] and [4]).

Theorem 1. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ we have

\begin{equation}
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \begin{cases}
\frac{1}{4} \left( \frac{x-a+b}{b-a} \right)^2 (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\
\frac{1}{(p+1)^{\frac{1}{p}}} \left[ \left( \frac{x-a}{b-a} \right)^{p+1} + \left( \frac{b-x}{b-a} \right)^{p+1} \right] (b-a)^{\frac{1}{q}} \|f'\|_q & \text{if } f' \in L_q[a,b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1;
\end{cases}
\end{equation}

where $\| \cdot \|_r \ (r \in [1, \infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

\[ \|g\|_\infty := \text{ess sup}_{t \in [a,b]} |g(t)|. \]

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ are sharp in the sense that they cannot be replaced by smaller constants.

The above inequalities may also be obtained from Fink’s result in [5] on choosing $n = 1$ and performing some appropriate computations.

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2. The Results

We start with the following lemma which is of intrinsic interest (see also [1]).

**Lemma 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous function on \([a, b]\), then, for any \( x \in [a, b] \),

\[
(2.1) \quad f(x) = \frac{1}{b-a} \int_a^b f(t)dt + \frac{1}{b-a} \int_a^b (x-t) \left[ \int_0^1 f'(1 - \lambda) (x + \lambda t) d\lambda \right] dt
\]

**Proof.** For any \( x, t \in [a, b], x \neq t \), one has

\[
\frac{f(x) - f(t)}{x-t} = \frac{1}{x-t} \int_t^x f'(u)du = \int_0^1 f'((1 - \lambda) x + \lambda t) d\lambda
\]

showing that

\[
(2.2) \quad f(x) = f(t) + (x-t) \int_0^1 f'((1 - \lambda) x + \lambda t) d\lambda \text{ for any } x, t \in [a, b].
\]

Integrating (2.2) over \( t \) on \([a, b]\) and dividing the result by \((b-a)\), gives the desired identity (2.1). \( \blacksquare \)

Using the above lemma the following result can be pointed out improving Ostrowski’s inequality.

**Theorem 2.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous function on \([a, b]\) so that \(|f'|\) is convex on \((a, b)\). If \( f' \in L_\infty[a, b] \), then for any \( x \in [a, b] \),

\[
(2.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{2} \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'(x)\| + \|f''\|_\infty.
\]

The constant \( \frac{1}{2} \) is sharp in the sense that it cannot be replaced by a smaller constant.

**Proof.** Using (2.1) and taking the modulus, we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| = \frac{1}{b-a} \left| \int_a^b \int_0^1 (x-t) f'((1 - \lambda) x + \lambda t) d\lambda dt \right|
\]

\[
\leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t||f'((1 - \lambda) x + \lambda t)| d\lambda dt
\]
\[
\begin{align*}
\int_{a}^{b} \int_{0}^{1} |x - t| [(1 - \lambda) |f'(x)| + \lambda |f'(t)|] \ d\lambda dt \\
&\leq \frac{1}{b - a} \int_{a}^{b} \int_{0}^{1} |x - t| [(1 - \lambda) |f'(x)| + \lambda |f'(t)|] d\lambda dt \\
&= \frac{1}{b - a} \int_{a}^{b} \int_{0}^{1} |x - t| \left[ \frac{|f'(x)| + |f'(t)|}{2} \right] d\lambda dt := M(x)
\end{align*}
\]

(by convexity of $|f'|$)

\[
\begin{align*}
&\leq \frac{1}{2} \frac{1}{b - a} \ \text{ess. sup}_{t \in [a, b]} [(|f'(x)| + |f'(t)|)] \int_{a}^{b} |x - t| dt \\
&= \frac{1}{2} \left[ \frac{(x - a)^2 + (b - x)^2}{2(b - a)} \right] \left[ |f'(x)| + \|f'\|_{\infty} \right] \\
&= \frac{1}{2} \left[ \frac{1}{4} + \left( \frac{x - a + b}{b - a} \right)^2 \right] (b - a) \left[ |f'(x)| + \|f'\|_{\infty} \right],
\end{align*}
\]

and the inequality (2.3) is proved.

Assume that (2.3) holds with a constant $C > 0$, that is,

\[
|f(x) - \frac{1}{b - a} \int_{a}^{b} f(t) dt| \leq C \left[ \frac{1}{4} + \left( \frac{x - a + b}{b - a} \right)^2 \right] (b - a) \left[ |f'(x)| + \|f'\|_{\infty} \right]
\]

for any $x \in [a, b]$ with $f$ as in the hypothesis of the theorem.

Consider the function

\[
f_0 : [a, b] \to \mathbb{R}, f_0(t) = k \left| t - \frac{a + b}{2} \right|, \quad k > 0, t \in [a, b].
\]

Since $|f_0'(t)| = k$, for any $t \in [a, b]$ and

\[
\frac{1}{b - a} \int_{a}^{b} f_0(t) dt = \frac{k}{4} (b - a), \quad \|f_0'\|_{\infty} = k
\]

then choosing $f = f_0$ and $x = \frac{a + b}{2}$ in (2.4), we get

\[
\frac{k}{4} (b - a) \leq \frac{Ck(b - a)}{2}
\]

giving $C \geq \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$.

The following particular case is interesting.

**Corollary 1.** With the assumptions of Theorem 3, we have the inequality

\[
|f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_{a}^{b} f(t) dt| \leq \frac{1}{8} (b - a) \left[ \left| f'\left(\frac{a + b}{2}\right) \right| + \|f'\|_{\infty} \right]
\]

and the constant $\frac{1}{8}$ is the best possible.

The following result in terms of the $p$-norms also holds:
Theorem 3. Let \( f : [a, b] \to \mathbb{R} \) be as in Theorem 3. If \( f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \) then for any \( x \in [a, b], \)

\[
(2.6) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \\
\leq \frac{1}{2(q+1)^\frac{q}{q}} \left[ \left( \frac{b-x}{b-a} \right)^{q+1} + \left( \frac{x-a}{b-a} \right)^{q+1} \right] \left( b-a \right)^{\frac{1}{2}} \|f'(x)| + |f'|_p \]

The constant \( \frac{1}{2} \) is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. According to the proof of Theorem 2, we have

\[
M(x) = \frac{1}{2(b-a)} \left( \int_a^b |x-t|^q dt \right)^\frac{1}{q} \left( \int_a^b (|f'(x)| + |f'|)^p dt \right)^\frac{1}{p}
\]

and the inequality (2.6) is proved.

Reconsider the function utilised in Theorem 2,

\[
f_0 : [a, b] \to \mathbb{R}, \quad f_0(t) = k \left| t - \frac{a+b}{2} \right|, \quad k > 0, \quad t \in [a, b]
\]

which has \( |f_0'(t)| = k \) convex in \([a, b]\). If we assume that (2.6) holds with a constant \( D > 0 \) instead of \( \frac{1}{2} \), so that

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \\
\leq \frac{D}{(q+1)^\frac{q}{q}} \left[ \left( \frac{b-x}{b-a} \right)^{q+1} + \left( \frac{x-a}{b-a} \right)^{q+1} \right] \left( b-a \right)^{\frac{1}{2}} \|f'(x)| + |f'|_p \,
\]

then taking \( f = f_0 \) over \( x = \frac{a+b}{2} \), we get,

\[
\frac{k}{4} (b-a) \leq \frac{D}{(q+1)^\frac{q}{q}} \left( \frac{1}{2^q} \right)^\frac{1}{q} \left( b-a \right)^{\frac{1}{2}} k (b-a)^{\frac{1}{2}} , \quad q > 1, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1
\]

giving, on simplification,

\[
D \geq \frac{1}{2} (q+1)^{\frac{q}{2}}, \quad q > 1.
\]

Taking the limit as \( q \to \infty \) and since,

\[
\lim_{q \to \infty} (q+1)^{\frac{1}{2}} = \exp \left\{ \lim_{q \to \infty} \frac{\ln(1+q)}{q} \right\} = \exp 0 = 1,
\]

we deduce that \( D \geq \frac{1}{2} \), which proves the sharpness of the constant.
A particular case is the following mid-point inequality:

**Corollary 2.** With the assumptions of Theorem 3, we have,

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| 
\leq \frac{1}{4} \left( b-a \right)^\frac{1}{2} \left( \int_a^b \left[ \left| f'\left(\frac{a+b}{2}\right)\right| + |f'(t)| \right]^p dt \right)^\frac{1}{p} \quad (p > 1, \frac{1}{p} + \frac{1}{q} = 1)
\]

The constant \(\frac{1}{4}\) is sharp in the previous sense.

Finally, the case involving the 1-norm is embodied in the following theorem:

**Theorem 4.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be as in Theorem 2. If \( f' \in L_1[a, b] \), then, for any \( x \in [a, b] \),

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} \left[ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[ (b-a) |f'(x)| + \|f'\|_1 \right].
\]

**Proof.** We have, from the proof of Theorem 2, that

\[
M(x) \leq \sup_{t \in [a, b]} |x - t| \frac{1}{b-a} \int_a^b \left[ \left| f'(x) \right| + |f'(t)| \right] \frac{1}{2} dt
\]

\[
= \frac{1}{2(b-a)} \max (x-a, b-x) \left[ (b-a) |f'(x)| + \int_a^b |f'(t)| dt \right]
\]

\[
= \frac{1}{2} \left[ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[ (b-a) |f'(x)| + \|f'\|_1 \right]
\]

and the inequality (2.8) is proved.

In particular, we have the mid-point inequality:

**Corollary 3.** Assume that \( f \) is as in Theorem 4. Then

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{4} \left( b-a \right) \left| f'\left(\frac{a+b}{2}\right)\right| + \int_a^b |f'(t)| dt \]

Another way to estimate the difference

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right|
\]

is presented in the following theorem.
Theorem 5. Let \( f : [a, b] \to \mathbb{R} \) be an absolutely continuous function on \([a, b]\) so that \(|f'|\) is convex on \((a, b)\). Then, for any \(x \in [a, b]\),

\[
(2.10) \quad |f(x) - \frac{1}{b - a} \int_a^b f(t) dt| \leq \frac{1}{2} \left\{ \frac{\left| f'(x) \right|}{4} \sqrt{\int_a^b |f'(t)|^2 dt} + \int_a^b \frac{\left| f'(t) \right|}{b - a} dt \right\},
\]

where \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. With the notation of Theorem 2, we have,

\[
M(x) = \frac{1}{2} \left( \left| f'(x) \right| \int_a^b |x - t| dt + \int_a^b |x - t| |f'(t)| dt \right) \leq \frac{1}{2} \left| f'(x) \right| \int_a^b |x - t| dt + \int_a^b |x - t| |f'(t)| dt \leq \frac{1}{2} \left[ \left| f'(x) \right| \left( \frac{x - a}{b - a} \right)^2 + \left( \frac{x - a}{b - a} \right) \right] (b - a) + \frac{1}{b - a} \int_a^b |x - t| |f'(t)| dt.
\]

Using Hölder’s inequality,

\[
\frac{1}{b - a} \int_a^b |x - t| |f'(t)| dt \leq \frac{1}{b - a} \left( \int_a^b |x - t|^q dt \right)^{\frac{1}{q}} \left( \int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} \leq \frac{1}{b - a} \left[ (b - x)^{q+1} + (x - a)^{q+1} \right]^{\frac{1}{q+1}} \left( b - a \right)^{\frac{1}{q}} \left( b - a \right)^{\frac{1}{p}} \||f'|_p\|
\]

and the theorem is proved. 

The following particular corollary is of interest providing a bound for the midpoint.

Corollary 4. Let \( f \) be as in the previous theorem. Then one has the inequality:

\[
(2.11) \quad \left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{1}{4} \left\{ \frac{1}{2} \left| f' \left( \frac{a + b}{2} \right) \right| (b - a) + \frac{1}{(q + 1)^{\frac{1}{q}}} (b - a)^{\frac{1}{p}} \||f'|_p\| \right\}
\]
3. Applications for Special Means

In the applications below we consider the following definitions of some special means:

- Arithmetic mean,
  \[ A = A(a, b) = \frac{a + b}{2}; \quad a, b > 0. \]

- Geometric mean,
  \[ G = G(a, b) = \sqrt{ab}; \quad a, b > 0. \]

- Logarithmic mean,
  \[ L = L(a, b) = \begin{cases} \frac{b - a}{\ln b - \ln a}, & a \neq b > 0, \\ a, & a = b. \end{cases} \]

- \(p\)-Logarithmic mean,
  \[ L_p(a, b) = \begin{cases} \frac{a}{p} & \text{if } a = b, \\ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \frac{1}{p} & \text{if } a \neq b \quad \text{for} \quad p \in \mathbb{R} \setminus \{0, -1\}. \]

- Identric mean,
  \[ I = I(a, b) = \begin{cases} \frac{a}{e} & \text{if } a = b, \\ \left(\frac{b}{a^b}\right)^{\frac{1}{b-a}} & \text{if } a \neq b. \end{cases} \]

The well known fact that \(G < L < I < A\) will be used in the following.

1. Consider the function \(f\) with domain \([a, b] \subset (0, \infty)\), \(f(x) = x^p\) and \(p \in \mathbb{R}, p \geq 2\), which is absolutely continuous, and whose modulus of the first derivative is a convex function.

1.1 If we use this function in Corollary 1, we get that
\[
\left| \left(\frac{a + b}{2}\right)^p - \frac{1}{b - a} \int_a^b t^p dt \right| \leq \frac{1}{8} (b - a) \left[ p \left(\frac{a + b}{2}\right)^{p-1} \right] + p(a^{p-1})
\]
so that
\[
|A^p(a, b) - L_p^p(a, b)| \leq \frac{p}{8} (b - a)[A^{p-1}(a, b) + b^{p-1}]
\]
or equivalently
\[
0 \leq L_p^p(a, b) - A^p(a, b) \leq \frac{p}{8} (b - a)[A^{p-1}(a, b) + b^{p-1}].
\]

1.2 For the same function, we get from Corollary 3 that
\[
|A^p(a, b) - L_p^p(a, b)| \leq \frac{1}{4} (b - a) \left| p \left(\frac{a + b}{2}\right)^{p-1} \right| + \int_a^b |pt^{p-1}| dt
\]
\[
= \frac{p}{4} (b - a) A^{p-1}(a, b) + \frac{b^p - a^p}{p}
\]
\[
= \frac{p}{4} (b - a) \left[ A^{p-1}(a, b) + L_{p-1}^p(a, b) \right].
\]
That is,
\[ 0 \leq L_p^p(a, b) - A_p^p(a, b) \leq \frac{p}{4}(b - a) \left[ A_p^{p-1}(a, b) + L_{p-1}^p(a, b) \right]. \]

2. Now, consider the function \( f \) with domain \([a, b] \subset (0, \infty)\), \( f(x) = \ln(x) \). The function is absolutely continuous, and the modulus of the first derivative is convex.

2.1 From Corollary 1, we obtain,
\[ 0 \leq \left| \ln \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b \ln(t) \, dt \right| = \left| \ln \left( \frac{a + b}{2} \right) - \ln(I(a, b)) \right| \]
\[ = \ln \frac{A(a, b)}{I(a, b)} \]
\[ \leq \frac{1}{8}(b - a) \left[ \frac{2}{a + b} + \frac{1}{a} \right] \]
\[ = \frac{1}{8}(b - a) \left[ A^{-1}(a, b) + \frac{1}{a} \right] \]
and so
\[ 0 \leq \frac{A(a, b)}{I(a, b)} \leq \frac{b - a}{8} \left[ A^{-1}(a, b) + a^{-1} \right] \]
or, equivalently,
\[ 1 \leq \frac{A(a, b)}{I(a, b)} \leq \exp \left[ \frac{b - a}{8} \left[ A^{-1}(a, b) + a^{-1} \right] \right]. \]

2.2 From Corollary 3, we get that
\[ 0 \leq |\ln A(a, b) - \ln I(a, b)| = \ln \frac{A(a, b)}{I(a, b)} \]
\[ \leq \frac{1}{4} \left[ (b - a) \left| \frac{2}{a + b} \right| + \int_a^b \frac{1}{t} \, dt \right]. \]
That is,
\[ 0 \leq \ln \frac{A(a, b)}{I(a, b)} \leq \frac{b - a}{4} A^{-1}(a, b) + \frac{1}{4} \ln \frac{b}{a} \]
or, equivalently,
\[ 1 \leq \frac{A(a, b)}{I(a, b)} \leq \left( \frac{b}{a} \right)^{\frac{1}{4}} \exp \left[ \frac{b - a}{4} \left[ A^{-1}(a, b) \right] \right]. \]

2.3 Taking \( f(x) = \ln x \) in Corollary 4, gives
\[ 1 \leq \frac{A(a, b)}{I(a, b)} \leq \frac{b - a}{4} \left[ \frac{A^{-1}(a, b)}{2} + \frac{1}{(q + 1)\frac{1}{L_p^{-1}(a, b)}} \right]. \]

3. Now, consider the function \( f(x) = \frac{1}{x} \) which has domain \([a, b] \subset (0, \infty)\). This function is absolutely continuous and the modulus of the first derivative is convex.

3.1 From Corollary 1, we have,
\[ \left| \frac{2}{a + b} - L^{-1}(a, b) \right| \leq \frac{1}{8}(b - a) \left[ \left| \frac{1}{\left( \frac{a + b}{2} \right)^2} \right| + \frac{1}{a^2} \right] \]
giving
\[ |A^{-1}(a, b) - L^{-1}(a, b)| \leq \frac{1}{8} (b - a) \left[ A^{-2}(a, b) + a^{-2} \right] \]
or equivalently
\[ 0 \leq L^{-1}(a, b) - A^{-1}(a, b) \leq \frac{1}{8} (b - a) \left[ A^{-2}(a, b) + a^{-2} \right], \quad \text{(since } A(a, b) \geq L(a, b)) \]
which may further be represented as
\[ 0 \leq A(a, b) - L(a, b) \leq \frac{1}{8} (b - a) A(a, b) L(a, b) \left[ A^{-2}(a, b) + a^{-2} \right]. \]

3.2 From Corollary 3, we get,
\[ |A^{-1}(a, b) - L^{-1}(a, b)| \leq \frac{1}{4} (b - a) \left[ A^{-2}(a, b) + G^{-2}(a, b) \right] \]
or equivalently
\[ 0 \leq L^{-1}(a, b) - A^{-1}(a, b) \leq \frac{1}{4} (b - a) \left[ A^{-2}(a, b) + G^{-2}(a, b) \right] \]
or still further
\[ 0 \leq A(a, b) - L(a, b) \leq \frac{1}{4} (b - a) A(a, b) L(a, b) \left[ A^{-2}(a, b) + G^{-2}(a, b) \right]. \]

3.3 Taking \( f(x) = \frac{1}{x} \) in Corollary 4, produces
\[ |A^{-1}(a, b) - L^{-1}(a, b)| \leq \frac{b - a}{4} \left[ \frac{A^{-2}(a, b)}{2} + \frac{1}{(q + 1)^{\frac{1}{2}}} L_{-2p}^{-2}(a, b) \right] \]
or
\[ 0 \leq L^{-1}(a, b) - A^{-1}(a, b) \leq \frac{b - a}{4} \left[ \frac{A^{-2}(a, b)}{2} + \frac{1}{(q + 1)^{\frac{1}{2}}} L_{-2p}^{-2}(a, b) \right] \]
which may be further expressed as
\[ 0 \leq A(a, b) - L(a, b) \leq \frac{b - a}{4} A(a, b) L(a, b) \left[ \frac{A^{-2}(a, b)}{2} + \frac{1}{(q + 1)^{\frac{1}{2}}} L_{-2p}^{-2}(a, b) \right]. \]

4. Applications for \( f \) and \( HH \)–Divergence Measures in Information Theory

Assume that a set \( \chi \) and the \( \sigma \)–finite measure \( \mu : \chi \rightarrow \mathbb{R} \) are given. Consider the set of all probability densities on \( \mu \) to be
\[ (4.1) \quad \Omega := \left\{ p | p : \chi \rightarrow \mathbb{R}, p(x) \geq 0, \int_{\chi} p(x) \, d\mu(x) = 1 \right\}. \]
The \( f \)–divergence on \( \Omega \) is defined as follows
\[ (4.2) \quad D_{f}(p, q) := \int_{\chi} p(x) f \left[ \frac{q(x)}{p(x)} \right] \, d\mu(x), \quad p, q \in \Omega, \]
where \( f \) is convex on \((0, \infty)\). It is also assumed that \( f(u) \) is zero and strictly convex at \( u = 1 \).

By appropriately defining this convex function, various divergences such as the Kullback-Leibler divergence \( D_{KL} \), variation distance \( D_{v} \), Hellinger distance \( D_{H} \),
\( \chi^2 \)-divergence \( D_{\chi^2} \), Jeffrey's distance \( D_J \), triangular discrimination \( D_\Delta \), etc. may be obtained. They are defined as follows:

\[
D_v (p, q) := \int_{\chi} \left| p(x) - q(x) \right| d\mu(x), \quad p, q \in \Omega;
\]

\[
D_H (p, q) := \int_{\chi} \left( \sqrt{p(x)} - \sqrt{q(x)} \right)^2 d\mu(x), \quad p, q \in \Omega;
\]

\[
D_{\chi^2} (p, q) := \int_{\chi} p(x) \left[ \left( \frac{q(x)}{p(x)} \right)^2 - 1 \right] d\mu(x), \quad p, q \in \Omega;
\]

\[
D_J (p, q) := \int_{\chi} \left[ p(x) - q(x) \right] \ln \left( \frac{p(x)}{q(x)} \right) d\mu(x), \quad p, q \in \Omega;
\]

\[
D_\Delta (p, q) := \int_{\chi} \frac{\left[ p(x) - q(x) \right]^2}{p(x) + q(x)} d\mu(x), \quad p, q \in \Omega.
\]

In [6], Shioya and Da-te introduced the generalised Ling-Wong \( f \)-divergence \( D_f (p, \frac{1}{2}p + \frac{1}{2}q) \) and the Hermite-Hadamard (HH) \( f \)-divergence

\[
D_{f, HH} (p, q) := \int_{\chi} \frac{p^2(x)}{q(x) - p(x)} \left( \int_a^b f(t) dt \right) d\mu(x), \quad p, q \in \Omega.
\]

They proved, by the use of the Hermite-Hadamard inequality for convex functions,

\[
D_f \left( p, \frac{1}{2}p + \frac{1}{2}q \right) \leq D_{f, HH} (p, q) \leq \frac{1}{2} D_f (p, q),
\]

provided that \( f \) is convex and normalised, i.e., \( f(1) = 0 \).

We will illustrate the approach to developing bounds and expressions involving various divergence measures from the inequalities developed in Section 2.

We will use the inequality (2.5), namely

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{1}{8} \left| b - a \right| \left| f' \left( \frac{a + b}{2} \right) \right| + \| f' \|_{\infty},
\]

where \( a, b \in \hat{I}, \ a \neq b \) and \( f : \hat{I} \subset \mathbb{R} \to \mathbb{R} \) is a differentiable function on the interior of \( I \) with \( |f'| : \hat{I} \to \mathbb{R} \) convex on \( \hat{I} \), to prove the following result.

**Theorem 6.** Let \( r, R \) be such that \( 0 \leq r \leq 1 \leq R \leq \infty \) and \( p, q \in \Omega \) with

\[
r \leq \frac{q(x)}{p(x)} \leq R, \quad \text{for a.e.} \ x \in \chi.
\]

If \( f : [0, \infty) \to \mathbb{R} \) is differentiable on \( (0, \infty) \) and \( |f'| \) is convex on \( [r, R] \) then,

\[
\left| D_f \left( p, \frac{1}{2}p + \frac{1}{2}q \right) - D_{f, HH} (p, q) \right|
\]

\[
\leq \frac{1}{8} \left( \| f' \|_{[r, R], \infty} D_v (p, q) + D_f^* (p, q) \right),
\]

where \( f^* (x) = |x - 1| |f' \left( \frac{x + 1}{2} \right)|, \ x \in [r, R] \) and \( \| h \|_{[a, b], \infty} := \text{ess sup}_{t \in [a, b]} |h(t)|. \)
Proof. If in (4.10) we choose $a = 1$, $b = \frac{q(x)}{p(x)}$, $x \in \chi$, then

$$\left| f \left( \frac{p(x) + q(x)}{2p(x)} \right) - \frac{p(x)}{q(x) - p(x)} \left( \int_1^{q(x)} f(t) \, dt \right) \right| \leq \frac{1}{8} \left| q(x) - p(x) \right| \left\| f' \left( \frac{p(x) + q(x)}{2p(x)} \right) \right\| + \left\| f'' \right\|_{[r,R], \infty}.$$  

Multiplying (4.13) with $p(x) \geq 0$ and integrating on $\chi$, we deduce the desired inequality (4.12).

Another approach is embodied in the following theorem.

**Theorem 7.** Let $r, R$ be as in Theorem 6. If $f : [0, \infty) \to \mathbb{R}$ is twice differentiable on $(0, \infty)$ and $|f''|$ is convex on $[r,R]$, then

$$|D_f(p,q) - f(1) - D_f #(p,q)| \leq \frac{1}{8} \left[ \left\| f' \right\|_{[r,R], \infty} D \chi^2_p (p, q) + D f'(p, q) \right],$$

where $f #(x) := (x-1) f' \left( \frac{1+x}{2} \right)$, and $f^1(x) := (x-1)^2 \left| f'' \left( \frac{1+x}{2} \right) \right|$, $x \in [0, \infty)$.

**Proof.** Applying the inequality (4.10) for $a = 1$, $b = u$ and choosing instead of $f$, its derivative $f'$, one may state the inequality

$$\left| f(u) - f(1) - (u-1) f' \left( \frac{u+1}{2} \right) \right| \leq \frac{1}{8} (u-1)^2 \left[ \left\| f'' \left( \frac{u+1}{2} \right) \right\| + \left\| f' \right\|_{[r,R], \infty} \right].$$

If in this inequality we choose $u = \frac{q(x)}{p(x)}$, $x \in \chi$, then we get

$$\left| f \left( \frac{q(x)}{p(x)} \right) - f(1) - \frac{q(x)}{p(x)} f' \left( \frac{p(x) + q(x)}{2p(x)} \right) \right| \leq \frac{1}{8} \frac{\left| p(x) - q(x) \right|^2}{p^2(x)} \left[ \left\| f'' \left( \frac{p(x) + q(x)}{2p(x)} \right) \right\| + \left\| f' \right\|_{[r,R], \infty} \right].$$

Multiplying (4.15) by $p(x) \geq 0$, $x \in \chi$ and then integrating on $\chi$, we deduce the desired inequality (4.14).

**References**


