Saturation Assumption and Finite Element Method for a One–Dimensional Model

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Abstract

In this paper we refer to the hierarchical finite element method and stabilization techniques for convection–diffusion equations. In particular, the aim is to outline an application of saturation assumption to a posteriori error estimates for such problems. We consider here a simple one–dimensional model; the inequality is proved from an analitical point of view for the stabilized finite element solutions in two cases: artificial diffusion and SUPG stabilization techniques.

Key words: hierarchical methods, convection–diffusion equations, approximation methods for differential equations, applications of mathematical inequalities.

1. Introduction

One of the more studied and recently discussed subjects in the field of numerical approximation of PDEs solution is the adaptive method [6], [8]. It is known that some phenomenons are described by functions with singular points, in which the approximation results compromise; consider for example the convection–diffusion problem [7]. We know that the standard Galerkin discretization gives rise to unstable oscillations if the exact solution is not regular and if the discretization parameter is not sufficiently small. The remedies to the oscillations are in general two: the mesh refinement and the use of stabilization techniques. Moreover, adaptive methods used in last years are generally based on a posteriori error indicators; among them indicators based on the hierarchical decomposition of the solution have been studied. The advantages in the use of hierarchical bases are principally joined to the possibility of obtaining a posteriori error indicators from the analysis of the solution components of high level [1], [4], [9]. A posteriori error estimates often need a hypothesis named *saturation assumption* [1], [2], [3], [5]. In particular, in [2] and [5] the control of the discretization error, relative to a certain choice of the approximation subspace, is made by the discrete solution component of high level found in a larger subspace. More precisely, we suppose to solve a problem for a fixed value of the discretization parameter h and then on a mesh $\mathcal{T}_h$ corresponding to a space $\mathcal{V}_h$. If we refine the mesh, that is if we consider a $\bar{h} < h$, we obtain a new grid $\mathcal{T}_{\bar{h}}$ and so we enrich the space adding appropriate hierarchical basis functions to the set already used for $\mathcal{V}_h$. The new space will be indicated with $\mathcal{V}_{\bar{h}}$. The used error indicator is simply the solution component of $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ in the complementary space (space of the details)

$$e_h = u_{\bar{h}} - u_h$$

we obtain

$$C_1 |||u - u_h||| \leq |||e_h||| \leq C_2 |||u - u_h||| \tag{1.1}$$

where u is the exact solution, $|||\cdot|||$ represents an appropriate norm (in general the energy norm), C_1, C_2 are positive constants. The estimates were proved for various cases under two hypotheses, the strengthened Cauchy–Schwarz inequality and the saturation assumption. This last one states that in the energy norm the solution $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ constitues a better approximation to the exact solution u than $u_h \in \mathcal{V}_h$, more precisely, that exists $\beta < 1$ independent on the discretization parameter h such that

$$|||u - u_{\bar{h}}||| \leq \beta |||u - u_h|||.$$

The saturation assumption is difficult to obtain, in practice. In this paper a proof is presented in the case of a finite element approximation of unidimensional convection–diffusion equation stabilized with artificial diffusion or SUPG techniques. It is a very simple case but, we think, representative of the behaviour in higher dimensions of this important inequality. More precisely, the contents of the paper are the following. In Section 2 we present the problem, given by a second order differential equation with boundary conditions, which we modify by an artificial diffusion term. In Section 3 we prove the saturation condition for the modified equation. In Section 4 the same result is obtained for h–norms.

2. Finite element method for a one–dimensional model.

Let us consider the unidimensional convection–diffusion equation

$$-\nu D^2 u + \alpha D u = 0, \quad 0 < x < L, \tag{2.1}$$

where $\nu > 0$, with Dirichlet conditions

$$u(0) = 1, \; u(L) = 0.$$

The exact solution of the problem (2.1) is the function

$$u(x) = (1 - e^{-\frac{\alpha L}{\nu}(1 - \frac{x}{L})})(1 - e^{-\frac{\alpha L}{\nu}})^{-1}.$$

We know that it presents a boundary layer of size $O(\nu)$ near $x = 1$ if ν is small. We begin approximating (2.1) by the Galerkin method with linear finite elements on a uniform grid. More precisely, we choose a positive integer N, and we set $h = 1/N$, $x_j = j \cdot h$, for $j = 0, 1, \ldots, N$. Then we approximate the Dirichlet problem (2.1) in the space

$$\mathcal{V}_h = \{v \in \mathcal{C}_0^0([0,1])/v|_{[x_j,x_{j+1}]} \in P_1,\ j = 0, 1, \ldots, N-1\}.$$

Thus the solution will be of the form

$$u_h(x) = \sum_{j=0}^{N-1} u_j w_j(x)$$

where $w_j \in \mathcal{V}_h$ are Lagrange basis functions. But it is well known that, if the Péclet number $Pe = \alpha h/2\nu$ is less than 1, the solution u_h has an oscillatory behaviour. Then let us consider a modification of the problem with the adding of an artificial diffusion term

$$-(\nu + \tau_h)\, D^2\bar{u} + D\bar{u} = 0, \quad 0 < x < L,$$

with $\bar{u}(0) = 1$, $\bar{u}(L) = 0$. Let us denote $\bar{u}_h$ the corresponding approximated solution in $\mathcal{V}_h$. Among effective values that τ_h could assume, it is possible to choose the one that determines an *optimal upwind*, that is the τ_h such that

$$\bar{u}_j = \bar{u}_h(x_j) \equiv u(x_j) = \left(1 - e^{-\dfrac{\alpha L}{\nu}(1 - \frac{x_j}{L})}\right)\left(1 - e^{-\dfrac{\beta L}{\nu}}\right)^{-1}, \quad j = 0, 1, \ldots, N.$$

Easy computations, which we leave to the reader, show that such τ_h is given by

$$\tau_h = \frac{\alpha h}{2 \tanh Pe} - \nu. \tag{2.2}$$

To be definite, it results that the problem to analize is the following one

$$-\frac{\alpha h}{2\tanh \dfrac{\alpha h}{2\nu}} D^2\bar{u}_h + D\bar{u}_h = 0, \quad 0 < x < 1, \tag{2.3}$$

with $\bar{u}_h(0) = 1$, $\bar{u}_h(L) = 0$.

3. Saturation assumption for the artificial diffusion.

For the model presented in section 2, since we know the exact solution of the problem, it is possible to verify the saturation assumption. For simplicity we assume $\alpha = L = 1$.

Theorem 3.1. Let $\bar{u}_h \in \mathcal{V}_h$, $\bar{u}_{\frac{h}{2}} \in \mathcal{V}_{\frac{h}{2}}$, be finite element solutions of the problem (2.3). Then in the energy norm the solution $\bar{u}_{\frac{h}{2}}$ is a better approximation than $\bar{u}_h$ of the exact solution u of (2.1), namely: $\exists \beta < 1$ independent of h such that

$$|||u - \bar{u}_{\frac{h}{2}}||| \le \beta\, |||u - \bar{u}_h|||. \tag{3.1}$$

Proof. To prove (3.1) we evaluate at first the error in the generic interval $[x_j, x_{j+1}]$, that is the integral

$$I_j(h) = \int_{x_j}^{x_{j+1}} |u'(x) - \bar{u}'_h |_{[x_j,x_{j+1}]} |^2 dx, \quad j = 0, 1, \ldots, N-1. \tag{3.2}$$

We have

$$u'(x) = -K\frac{1}{\nu}e^{-\frac{1}{\nu}(1-x)}, \quad \text{where } K = \left(1 - e^{-\frac{1}{\nu}}\right)^{-1},$$

and

$$\bar{u}'_h |_{[x_j,x_{j+1}]} = \frac{\bar{u}_{j+1} - \bar{u}_j}{h} = \frac{K}{h}e^{-\frac{1}{\nu}}e^{\frac{jh}{\nu}}\left(1 - e^{\frac{h}{\nu}}\right).$$

Thus

$$\begin{aligned} I_j(h) &= K\left[\frac{1}{2\nu}e^{-\frac{2}{\nu}}e^{\frac{2jh}{\nu}}\left(e^{\frac{2h}{\nu}} - 1\right) - \frac{1}{h}e^{-\frac{2}{\nu}}e^{\frac{2jh}{\nu}}\left(e^{\frac{2h}{\nu}} - 1\right)^2\right] \quad (3.3) \\ &= K\left[\frac{1}{2\nu}\left(1 - e^{-\frac{2h}{\nu}}\right) - \frac{1}{h}\left(1 - e^{-\frac{h}{\nu}}\right)^2\right]\exp\left(-\frac{2}{\nu}\left(1 - (j+1)h\right)\right). \end{aligned}$$

Now we consider

$$b(h) = b(h,\nu)\frac{\sum_{j=0}^{\frac{2}{h}-1} I_j\left(\frac{h}{2}\right)}{\sum_{j=0}^{\frac{1}{h}-1} I_j(h)}.$$

If we prove that: $\exists\beta$ independent on h such that $b(h) \le \beta^2 < 1$ then (3.1) is verified. Thus we compute $b(h)$. Since

$$\begin{aligned} \sum_{j=0}^{\frac{1}{h}-1} e^{-\frac{2}{\nu}(1-(j+1)h)} &= e^{-\frac{2}{\nu}}e^{\frac{2h}{\nu}}\sum_{j=0}^{\frac{1}{h}-1}\left(e^{\frac{2h}{\nu}}\right)^j \\ &= e^{-\frac{2}{\nu}}e^{\frac{2h}{\nu}}\frac{1 - \left(e^{\frac{2h}{\nu}}\right)^{\frac{1}{h}}}{1 - e^{\frac{2h}{\nu}}} \\ &= e^{-\frac{2}{\nu}}e^{\frac{2h}{\nu}}\frac{1 - e^{\frac{2}{\nu}}}{1 - e^{\frac{2h}{\nu}}}, \end{aligned}$$

we have

$$b(h,\nu) = B\left(\frac{h}{\nu}\right),$$

where, for $x = h/\nu \in \mathbf{R}^+$,

$$B(x) = \frac{(1-e^{-x}) - \frac{4}{x}(1-e^{-\frac{x}{2}})^2}{(1-e^{-2x}) - \frac{2}{x}(1-e^{-x})^2} \frac{1-e^{-2x}}{1-e^{-x}}. \tag{3.4}$$

It is simple to prove that

$$\lim_{x \to 0^+} B(x) = \frac{1}{4},$$

c. f. the end of this paragraph, and

$$\lim_{x \to +\infty} B(x) = 1.$$

Simple considerations give also to us that $B(x)$ is a strictly increasing function. Returning to consider $b(h, \nu)$, we observe that it is homogeneous of degree zero with respect to the two variables h, ν. Taking then advantage from the bound $h \leq 1$, we deduce

$$b(h, \nu) \leq \beta = B\left(\frac{1}{\nu}\right) < 1.$$

We have therefore proved (3.1).

For $x = h/\nu$ small we have a sharper result, namely $\beta^2 = 1/4 + \epsilon$, with ϵ as small as we want. In fact substituing in (3.4) the expansion

$$e^{-x} \simeq 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + o(x^3) \quad \text{for} \quad x \to 0$$

we have easily $B(x) = 1/4 + o(1)$ and therefore

$$b(h, \nu) = B\left(\frac{h}{\nu}\right) \leq \frac{1}{4} + \epsilon < 1$$

for $h/\nu < x_0$, with x_0 depending on ϵ.

△

4. Saturation assumption for SUPG stabilization..

We consider now the unidimensional convection–diffusion problem stabilized with the SUPG method. Let us τ_h given by (2.2), that is we choose an optimal upwind. In practice, the method consists in using the *h–norm*

$$||| \cdot |||_h^2 = (\nu + \tau_h) ||| \cdot |||^2.$$

Theorem 4.1. Let us $\bar{u}_h \in \mathcal{V}_h$, $\bar{u}_{\frac{h}{2}} \in \mathcal{V}_{\frac{h}{2}}$, the finite element solutions of the problem (2.3). Then in h–norm the solution $\bar{u}_{\frac{h}{2}}$ is a better approximation than $\bar{u}_h$ of the exact solution u of (2.1), that is: $\exists \beta < 1$ independent of h such that

$$|||u - \bar{u}_{\frac{h}{2}}|||_h \leq \beta |||u - \bar{u}_h|||_h. \tag{4.1}$$

Proof. It is sufficient to prove that: $\exists\beta$ independent of h such that $b_\tau(h) \le \beta < 1$ where

$$b_\tau(h) = \frac{\nu + \tau_{\frac{h}{2}}}{\nu + \tau_h}\, b(h) \tag{4.2}$$

with $b(h)$ as before. By the proof of the theorem 3.1 we know that the second factor in (4.2) is less than $\beta^2 < 1$. Then it remains to prove that the first factor is less or equal than 1. We have

$$\frac{\nu + \tau_{\frac{h}{2}}}{\nu + \tau_h} = \frac{\tanh \dfrac{h}{2\nu}}{2\tanh \dfrac{h}{4\nu}} \tag{4.3}$$

and, considered the function $y(x) = \tanh 2x - 2\tanh x$, we have $y(x) < 0$ for $x > 0$, and thus the quantity in (3.3) is less than 1. This concludes the proof. $\triangle$

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