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GENERALIZATIONS OF THE HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE s -CONVEX

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ABSTRACT. Some new results related to the right-hand side of the Hermite-Hadamard type inequality for the class of functions whose derivatives at certain powers are s -convex functions in the second sense are obtained.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the *Hermite-Hadamard inequality* [10]:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

For recent results, refinements, counterparts, generalizations of the Hermite-Hadamard inequality see [7] – [12] and [14] – [19].

Dragomir and Agarwal [8] established the following result connected with the right-hand side of (1.1).

Theorem 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$(1.2) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|].$$

In [13], Hudzik and Maligranda considered among others the class of functions which are s -convex in the second sense. This class is defined in the following way: a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be *s -convex in the second sense* if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of s -convex functions in the second sense is usually denoted by K_s^2 .

It can be easily seen that for $s = 1$, s -convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

For recent results and generalizations concerning s -convex functions see [1] – [6] and [14].

In [5], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s -convex functions in the second sense.

Key words and phrases. Convex function, s -Convex function, Hermite-Hadamard's inequality.

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Theorem 2. Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[0, 1]$, then the following inequalities hold:

$$(1.3) \quad 2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}.$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3). The above inequalities are sharp.

New inequalities of Hermite-Hadamard type for differentiable functions based on concavity and s -convexity established by U.S. Kirmaci et al. [14], are presented below:

Theorem 3. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:

$$(1.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left(\frac{1}{2} \right)^{\frac{q-1}{q}} \left[\frac{s + \left(\frac{1}{2}\right)^s}{(s+1)(s+2)} \right]^{\frac{1}{q}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.$$

Theorem 4. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then the following inequality holds:

$$(1.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left[\frac{q-1}{2(2q-1)} \right]^{\frac{q-1}{q}} \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \times \left[\left(|f'(a)|^q + \left| f' \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left(\left| f' \left(\frac{a+b}{2} \right) \right|^q + |f'(a)|^q \right)^{\frac{1}{q}} \right] \leq \frac{b-a}{2} \left[\left(|f'(a)|^q + \left| f' \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left(\left| f' \left(\frac{a+b}{2} \right) \right|^q + |f'(a)|^q \right)^{\frac{1}{q}} \right].$$

Theorem 5. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then the following inequality holds:

$$(1.6) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left[\frac{q-1}{2(2q-1)} \right]^{\frac{q-1}{q}} 2^{\frac{s-1}{q}} \left(\left| f' \left(\frac{a+3b}{2} \right) \right| + \left| f' \left(\frac{3a+b}{2} \right) \right| \right) \leq \frac{b-a}{2} \left(\left| f' \left(\frac{a+3b}{2} \right) \right| + \left| f' \left(\frac{3a+b}{2} \right) \right| \right).$$

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose derivatives at certain powers are s -convex functions in the second sense.

2. INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE s -CONVEX

In order to prove our main results we consider the following lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$(2.1) \quad \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{b-a}{r+1} \int_0^1 [(r+1)t-1] f'(tb + (1-t)a) dt$$

for some fixed $r \in [0, 1]$.

Proof. We note that

$$\begin{aligned} I &= \int_0^1 [(r+1)t-1] f'(tb + (1-t)a) dt \\ &= [(r+1)t-1] \frac{f(tb + (1-t)a)}{b-a} \Big|_0^1 - \frac{r+1}{b-a} \int_0^1 f(tb + (1-t)a) dt \\ &= \frac{rf(b) + f(a)}{b-a} - \frac{r+1}{b-a} \int_0^1 f(tb + (1-t)a) dt. \end{aligned}$$

Setting $x = tb + (1-t)a$, and $dx = (b-a)dt$ gives

$$I = \frac{f(a) + rf(b)}{b-a} - \frac{r+1}{(b-a)^2} \int_a^b f(x) dx.$$

Therefore,

$$\left(\frac{b-a}{r+1} \right) I = \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx$$

which gives the desired representation (2.1). ■

The next theorem gives a new refinement of the upper Hermite-Hadamard inequality for s -convex functions.

Theorem 6. *Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. If $|f'|$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$(2.2) \quad \left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{(r+1)(s+1)(s+2)} \left[\left(r(s+1) + 2 \left(\frac{1}{r+1} \right)^{s+1} - 1 \right) |f'(b)| + \left(s - r + 2(r+1) \left(\frac{r}{r+1} \right)^{s+2} + 1 \right) |f'(a)| \right],$$

for some fixed $r \in [0, 1]$.

Proof. From Lemma 1, we have

$$\begin{aligned}
& \left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{r+1} \int_0^1 |(r+1)t-1| |f'(tb+(1-t)a)| dt \\
& = \frac{b-a}{r+1} \int_0^{\frac{1}{r+1}} (1-(r+1)t) |f'(tb+(1-t)a)| dt \\
& \quad + \frac{b-a}{r+1} \int_{\frac{1}{r+1}}^1 ((r+1)t-1) |f'(tb+(1-t)a)| dt \\
& \leq \frac{b-a}{r+1} \int_0^{\frac{1}{r+1}} (1-(r+1)t) [t^s |f'(b)| + (1-t)^s |f'(a)|] dt \\
& \quad + \frac{b-a}{r+1} \int_{\frac{1}{r+1}}^1 ((r+1)t-1) [t^s |f'(b)| + (1-t)^s |f'(a)|] dt \\
& = \frac{b-a}{r+1} \left[\frac{\left(\frac{1}{r+1}\right)^{s+1}}{(s+1)(s+2)} |f'(b)| + \frac{s+2+(r+1)\left[\left(\frac{r}{r+1}\right)^{s+2} - 1\right]}{(s+1)(s+2)} |f'(a)| \right] \\
& \quad + \frac{b-a}{r+1} \left[\frac{r(s+1) + \left(\frac{1}{r+1}\right)^{s+1} - 1}{(s+1)(s+2)} |f'(b)| + \frac{(r+1)\left(\frac{r}{r+1}\right)^{s+2}}{(s+1)(s+2)} |f'(a)| \right] \\
& = \frac{(b-a)}{(r+1)(s+1)(s+2)} \left[\left(r(s+1) + 2\left(\frac{1}{r+1}\right)^{s+1} - 1 \right) |f'(b)| \right. \\
& \quad \left. + \left(s-r+2(r+1)\left(\frac{r}{r+1}\right)^{s+2} + 1 \right) |f'(a)| \right],
\end{aligned}$$

which completes the proof. ■

Therefore, we can deduce the following results.

Corollary 1. *Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. Assume that $|f'|$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$.*

(1) *If $r = 1$ in (2.2), then the following inequality holds:*

$$(2.3) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(s+2^{-s})(b-a)}{2(s+1)(s+2)} [|f'(b)| + |f'(a)|].$$

(2) *If $r = 0$ in (2.2), then the following inequality holds:*

$$(2.4) \quad \left| f(a) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{(s+1)(s+2)} [|f'(b)| + (s+1)|f'(a)|].$$

Proof. It is obvious from Theorem 6. ■

Remark 1. We note that inequality (2.3) with $s = 1$ gives an improvement for the inequality (1.2).

A similar result is embodied in the following theorem:

Theorem 7. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$, then the following inequality holds:

$$(2.5) \quad \left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [(r+1)(1+p)]^{\frac{1}{p}}} \left[\left(|f'(a)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q \right)^{\frac{1}{q}} + r^{(p+1)/p} \left(|f'(b)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q \right)^{\frac{1}{q}} \right]$$

for some fixed $r \in [0, 1]$, where $q = p/(p-1)$.

Proof. Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{r+1} \int_0^{\frac{1}{r+1}} (1 - (r+1)t) |f'(tb + (1-t)a)| dt \\ & \quad + \frac{b-a}{r+1} \int_{\frac{1}{r+1}}^1 ((r+1)t - 1) |f'(tb + (1-t)a)| dt \\ & \leq \frac{b-a}{r+1} \left(\int_0^{\frac{1}{r+1}} (1 - (r+1)t)^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{r+1}} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{b-a}{r+1} \left(\int_{\frac{1}{r+1}}^1 ((r+1)t - 1)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{r+1}}^1 |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is convex, we have

$$\int_0^{\frac{1}{r+1}} |f'(tb + (1-t)a)|^q dt \leq \frac{|f'(a)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q}{s+1}$$

and

$$\int_{\frac{1}{r+1}}^1 |f'(tb + (1-t)a)|^q dt \leq \frac{|f'(b)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q}{s+1}.$$

Therefore, we have

$$\begin{aligned} & \left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [(r+1)(1+p)]^{\frac{1}{p}}} \left[\left(|f'(a)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + r^{(p+1)/p} \left(|f'(b)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which is required. ■

Corollary 2. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. Assume that $|f'|^{p/(p-1)}$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$.

(1) If $r = 1$ in (2.5), then the following inequality holds:

$$\begin{aligned} (2.6) \quad & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[\left(|f'(a)|^q + \left| f' \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(|f'(b)|^q + \left| f' \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

(2) If $r = 0$ in (2.5), then the following inequality holds:

$$(2.7) \quad \left| f(a) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} (1+p)^{\frac{1}{p}}} (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}},$$

where $q = \frac{p}{p-1}$.

Proof. It follows directly from Theorem 7. ■

Remark 2. We observe that the inequality (2.6) is better than the inequality (1.5).

Our next result gives a new refinement for the upper Hermite-Hadamard inequality:

Theorem 8. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$, then the following inequality holds:

$$\begin{aligned} (2.8) \quad & \left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} (r+1)^{\frac{1}{p}+\frac{s}{q}} (p+1)^{1+p}} \left[\left([(r+1)^s + 1] |f'(a)|^q + r^s |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(r^s |f'(a)|^q + [(r+1)^s + 1] |f'(b)|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

for some fixed $r \in [0, 1]$, where $q = \frac{p}{p-1}$.

Proof. We consider the inequality (2.5), that is

$$\begin{aligned} & \left| \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [(r+1)(1+p)]^{\frac{1}{p}}} \left[\left(|f'(a)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + r^{(p+1)/p} \left(|f'(b)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Since $|f'|^{p/(p-1)}$ is s -convex on $[a, b]$, then we have

$$\left| f' \left(\frac{a+rb}{r+1} \right) \right|^q \leq \left(\frac{1}{r+1} \right)^s |f'(a)|^q + \left(\frac{r}{r+1} \right)^s |f'(b)|^q,$$

which gives

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [(r+1)(1+p)]^{\frac{1}{p}}} \left[\left(|f'(a)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + r^{(p+1)/p} \left(|f'(b)|^q + \left| f' \left(\frac{b+ra}{r+1} \right) \right|^q \right)^{\frac{1}{q}} \right] \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} (r+1)^{\frac{1}{p}+\frac{s}{q}} (p+1)^{1+p}} \left[[(r+1)^s + 1] |f'(a)|^q + r^s |f'(b)|^q \right)^{\frac{1}{q}} \\ & \quad + (r^s |f'(a)|^q + [(r+1)^s + 1] |f'(b)|^q)^{\frac{1}{q}} \Big], \end{aligned}$$

which completes the proof. ■

Corollary 3. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$, then the following inequality holds:

$$\begin{aligned} (2.9) \quad & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[((1+2^{-s}) |f'(a)|^q + 2^{-s} |f'(b)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + (2^{-s} |f'(a)|^q + (1+2^{-s}) |f'(b)|^q)^{\frac{1}{q}} \right], \end{aligned}$$

where $q = \frac{p}{p-1}$.

Proof. We consider the inequality (2.6), that is

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[\left(|f'(a)|^q + \left| f' \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(|f'(b)|^q + \left| f' \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Since $|f'|^{p/(p-1)}$ is s -convex on $[a, b]$, then $|f'(\frac{a+b}{2})|^q \leq \frac{|f'(a)|^q + |f'(b)|^q}{2^s}$, which gives

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[\left(|f'(a)|^q + \left| f' \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(|f'(b)|^q + \left| f' \left(\frac{a+b}{2} \right) \right|^q \right)^{\frac{1}{q}} \right] \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[((1+2^{-s}) |f'(a)|^q + 2^{-s} |f'(b)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + (2^{-s} |f'(a)|^q + (1+2^{-s}) |f'(b)|^q)^{\frac{1}{q}} \right], \end{aligned}$$

where $q = \frac{p}{p-1}$, which completes the proof. ■

Corollary 4. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$, then the following inequality holds:

$$\begin{aligned} (2.10) \quad & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(1+2^{1-s})^{\frac{1}{q}} (b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} [|f'(a)| + |f'(b)|], \end{aligned}$$

where $q = \frac{p}{p-1}$.

Proof. We consider the inequality (2.9), i.e.,

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[((1+2^{-s}) |f'(a)|^q + 2^{-s} |f'(b)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + (2^{-s} |f'(a)|^q + (1+2^{-s}) |f'(b)|^q)^{\frac{1}{q}} \right]. \end{aligned}$$

Now, let $a_1 = (1+2^{-s}) |f'(a)|^q$, $b_1 = 2^{-s} |f'(b)|^q$, $a_2 = 2^{-s} |f'(a)|^q$ and $b_2 = (1+2^{-s}) |f'(b)|^q$.

Here, $0 < \frac{1}{q} < 1$, for $q \geq 1$. Using the fact that $\sum_{i=1}^n (a_i + b_i)^k \leq \sum_{i=1}^n a_i^k + \sum_{i=1}^n b_i^k$, for $0 < k < 1$, $a_1, a_2, \dots, a_n \geq 0$ and $b_1, b_2, \dots, b_n \geq 0$, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} \left[((1+2^{-s}) |f'(a)|^q + 2^{-s} |f'(b)|^q)^{\frac{1}{q}} \right. \\ & \quad \left. + (2^{-s} |f'(a)|^q + (1+2^{-s}) |f'(b)|^q)^{\frac{1}{q}} \right] \\ & \leq \frac{(1+2^{1-s})^{\frac{1}{q}} (b-a)}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{\frac{1}{p}}} [|f'(a)| + |f'(b)|], \end{aligned}$$

where $q = \frac{p}{p-1}$, which is required. ■

Remark 3. 1. Using the technique in Corollary 4, one can obtain in a similar manner another result by considering the inequality (2.8). However, the details are left to the interested reader.

2. All of the above inequalities obviously hold for convex functions. Simply choose $s = 1$ in each of those results to get the desired results.

3. Interchanging a and b in Lemma 1, we obtain the following equality

$$\frac{1}{b-a} \int_a^b f(x) dx - \frac{rf(a) + f(b)}{r+1} = \frac{b-a}{r+1} \int_0^1 [(r+1)t - 1] f'((1-t)b + ta) dt.$$

For this reason, if we interchanging a and b in all above results, we can write new results using the above equality.

3. APPLICATIONS TO SPECIAL MEANS

We consider the means for arbitrary real numbers α, β ($\alpha \neq \beta$) as follows:

(1) *Arithmetic mean:*

$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}.$$

(2) *Generalized log-mean:*

$$L_s(\alpha, \beta) = \left[\frac{\beta^{s+1} - \alpha^{s+1}}{(s+1)(\beta - \alpha)} \right]^{\frac{1}{s}}, \quad s \in \mathbb{R} \setminus \{-1, 0\}, \quad \alpha, \beta \in \mathbb{R}, \quad \alpha \neq \beta.$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

In [13], the following example is given:

Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define a function $f : [0, \infty) \rightarrow \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0 \\ bt^s + c, & t > 0. \end{cases}$$

If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$. Hence, for $a = c = 0$, $b = 1$, we have $f : [0, 1] \rightarrow [0, 1]$, $f(t) = t^s$, $f \in K_s^2$.

Proposition 1. *Let $a, b \in I^\circ$, $a < b$ and $0 < s < 1$. Then, we have*

$$(3.1) \quad |L_s^s(a, b) - A(a^s, b^s)| \leq s(b-a) \frac{s+2^{-s}}{2(s+1)(s+2)} \left(|a|^{s-1} + |b|^{s-1} \right)$$

and

$$(3.2) \quad |L_s^s(a, b) - |a|^s| \leq \frac{s(b-a)}{(s+1)(s+2)} \left((s+1) |a|^{s-1} + |b|^{s-1} \right).$$

Proof. The assertion follows from Corollary 1 applied to the s -convex mapping $f : [0, 1] \rightarrow [0, 1]$, $f(x) = x^s$. ■

Proposition 2. *Let $a, b \in I^\circ$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have*

$$(3.3) \quad |L_s^s(a, b) - A(a^s, b^s)| \leq s \frac{b-a}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{1/p}} \left[\left(|a|^{q(s-1)} + \left| \frac{a+b}{2} \right|^{q(s-1)} \right)^{1/q} + \left(|b|^{(s-1)q} + \left| \frac{a+b}{2} \right|^{q(s-1)} \right)^{1/q} \right]$$

and

$$(3.4) \quad |L_s^s(a, b) - |a|^s| \leq \frac{s(b-a)}{(s+1)^{1+\frac{1}{q}} (1+p)^{1/p}} \left(|a|^{(s-1)q} + |b|^{(s-1)q} \right)^{1/q}.$$

Proof. The assertion follows from Corollary 2 applied to the s -convex mapping $f : [0, 1] \rightarrow [0, 1]$, $f(x) = x^s$. ■

Proposition 3. *Let $a, b \in I^\circ$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have*

$$(3.5) \quad |L_s^s(a, b) - A(a^s, b^s)| \leq s(b-a) \frac{(1+2^{1-s})^{1/q}}{(s+1)^{1+\frac{1}{q}} [2(1+p)]^{1/p}} \left(|a|^{s-1} + |b|^{s-1} \right).$$

Proof. The assertion follows from Corollary 4 applied to the s -convex mapping $f : [0, 1] \rightarrow [0, 1]$, $f(x) = x^s$. ■

REFERENCES

- [1] M. Alomari and M. Darus, Hadamard-type inequalities for s -convex functions, *Inter. Math. Forum*, **3** (40) (2008) 1965–1975.
- [2] M. Alomari and M. Darus, On co-ordinated s -convex functions, *Inter. Math. Forum*, **3** (40) (2008) 1977–1989.
- [3] M. Alomari and M. Darus, Co-ordinates s -convex function in the first sense with some Hadamard-type inequalities, *Int. J. Contemp. Math. Sci.*, **3** (32) (2008) 1557–1567.
- [4] M. Alomari and M. Darus, Refinements of s -Orlicz convex functions in normed linear spaces, *Int. J. Contemp. Math. Sci.*, **3** (32) (2008) 1969–1994.
- [5] S.S. Dragomir and S. Fitzpatrick, The Hadamard's inequality for s -convex functions in the second sense, *Demonstratio Math.*, **32** (4) (1999), 687–696.
- [6] S.S. Dragomir, s -Orlicz convex functions in linear spaces and Jensen's discrete inequality, *J. Math. Anal. Appl.*, **210** (1997), 419–439.
- [7] S.S. Dragomir, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.*, **167** (1992) 49–56.

- [8] S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, **11** (1998) 91–95.
- [9] S.S. Dragomir, Y.J. Cho and S.S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, *J. Math. Anal. Appl.*, **245** (2000), 489–501.
- [10] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. Online: [<http://www.staff.vu.edu.au/RGMIA/monographs/hermite-hadamard.html>].
- [11] S.S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1 norm and applications to some special means and to some numerical quadrature rule, *Tamkang J. Math.*, **28** (1997) 239–244.
- [12] S.S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rule, *Appl. Math. Lett.*, **11** (1998) 105–109.
- [13] H. Hudzik and L. Maligranda, Some remarks on s -convex functions, *Aequationes Math.*, **48** (1994), 100–111.
- [14] U.S. Kirmaci et al., Hadamard-type inequalities for s -convex functions, *Appl. Math. Comp.*, **193** (2007), 26–35.
- [15] U.S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, *Appl. Math. Comp.*, **147** (2004), 137–146.
- [16] U.S. Kirmaci and M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comp.*, **153** (2004), 361–368.
- [17] M.E. Özdemir, A theorem on mappings with bounded derivatives with applications to quadrature rules and means, *Appl. Math. Comp.*, **138** (2003), 425–434.
- [18] C.E.M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formula, *Appl. Math. Lett.*, **13** (2000) 51–55.
- [19] G.S. Yang, D.Y. Hwang and K.L. Tseng, Some inequalities for differentiable convex and concave mappings, *Comp. Math. Appl.*, **47** (2004), 207–216.

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