

**SUPERADDITIVITY OF SOME FUNCTIONALS ASSOCIATED
TO JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS ON
LINEAR SPACES WITH APPLICATIONS**

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ABSTRACT. Some new results related to Jensen's celebrated inequality for convex functions defined on convex sets in linear spaces are given. Applications for norm inequalities in normed linear spaces and f -divergences in Information Theory are provided as well.

1. INTRODUCTION

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the generalised triangle inequality, the arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C . If I denotes a finite subset of the set \mathbb{N} of natural numbers, $x_i \in C, p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i \in I} p_i > 0$, then

$$(1.1) \quad f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \leq \frac{1}{P_I} \sum_{i \in I} p_i f(x_i),$$

is well known in the literature as *Jensen's inequality*.

We introduce the following notations (see also [16]):

$$\begin{aligned} F(C, \mathbb{R}) & : = \text{the linear space of all real functions on } C, \\ F^+(C, \mathbb{R}) & : = \{f \in F(C, \mathbb{R}) : f(x) > 0 \text{ for all } x \in C\}, \\ P_f(\mathbb{N}) & : = \{I \subset \mathbb{N} : I \text{ is finite}\}, \\ J(\mathbb{R}) & : = \{p = \{p_i\}_{i \in \mathbb{N}}, p_i \in \mathbb{R} \text{ are such that } P_I \neq 0 \text{ for all } I \in P_f(\mathbb{N})\}, \end{aligned}$$

and

$$\begin{aligned} J^+(\mathbb{R}) & : = \{p \in J(\mathbb{R}) : p_i \geq 0 \text{ for all } i \in \mathbb{N}\}, \\ J_*(C) & : = \{x = \{x_i\}_{i \in \mathbb{N}} : x_i \in C \text{ for all } i \in \mathbb{N}\} \end{aligned}$$

and

$$\text{Conv}(C, \mathbb{R}) := \text{the cone of all convex functions defined on } C,$$

respectively.

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In [16] the authors considered the following functional associated with the Jensen inequality:

$$(1.2) \quad J(f, I, p, x) := \sum_{i \in I} p_i f(x_i) - P_I f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right)$$

where $f \in F(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^+(\mathbb{R})$, $x \in J_*(C)$. They established some quasi-linearity and monotonicity properties and applied the obtained results for norm and means inequalities.

The following result concerning the properties of the functional $J(f, I, \cdot, x)$ as a *function of weights* holds (see [16, Theorem 2.4]):

Theorem 1. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$.*

(i) *If $p, q \in J^+(\mathbb{R})$ then*

$$(1.3) \quad J(f, I, p + q, x) \geq J(f, I, p, x) + J(f, I, q, x) (\geq 0)$$

i.e., $J(f, I, \cdot, x)$ is superadditive on $J^+(\mathbb{R})$;

(ii) *If $p, q \in J^+(\mathbb{R})$ with $p \geq q$, meaning that $p_i \geq q_i$ for each $i \in \mathbb{N}$, then*

$$(1.4) \quad J(f, I, p, x) \geq J(f, I, q, x) (\geq 0)$$

i.e., $J(f, I, \cdot, x)$ is monotonic nondecreasing on $J^+(\mathbb{R})$.

The behavior of this functional as an *index set function* is incorporated in the following (see [16, Theorem 2.1]):

Theorem 2. *Let $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$.*

(i) *If $I, H \in P_f(\mathbb{N})$ with $I \cap H = \emptyset$, then*

$$(1.5) \quad J(f, I \cup H, p, x) \geq J(f, I, p, x) + J(f, H, p, x) (\geq 0)$$

i.e., $J(f, \cdot, p, x)$ is superadditive as an index set function on $P_f(\mathbb{N})$;

(ii) *If $I, H \in P_f(\mathbb{N})$ with $H \subset I$, then*

$$(1.6) \quad J(f, I, p, x) \geq J(f, H, p, x) (\geq 0)$$

i.e., $J(f, \cdot, p, x)$ is monotonic nondecreasing as an index set function on $P_f(\mathbb{N})$.

As pointed out in [16], the above Theorem 2 is a generalisation of the Vasić-Mijalković result for convex functions of a real variable obtained in [26] and therefore creates the possibility to obtain vectorial inequalities as well.

For applications of the above results to logarithmic convex functions, to norm inequalities, in relation with the arithmetic mean-geometric mean inequality and with other classical results, see [16].

Motivated by the above results, we introduce in the present paper a more general functional, establish its main properties and use it for some particular cases that provide inequalities of interest. Applications for norm inequalities in normed linear spaces and f -divergences in Information Theory are provided as well.

2. SOME SUPERADDITIVITY PROPERTIES FOR THE WEIGHTS

We consider the more general functional

$$(2.1) \quad D(f, I, p, x; \Phi) := P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right],$$

where $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^+(\mathbb{R})$, $x \in J_*(C)$ and $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is a function whose properties will determine the behavior of the functional D as follows. Obviously, for $\Phi(t) = t$ we recapture from D the functional J considered in [16].

First of all we observe that, by Jensen's inequality, the functional D is well defined and *positive homogeneous* in the third variable, i.e.,

$$D(f, I, \alpha p, x; \Phi) = \alpha D(f, I, p, x; \Phi),$$

for any $\alpha > 0$ and $p \in J^+(\mathbb{R})$.

The following result concerning the superadditivity and monotonicity of the functional D as a function of weights holds:

Theorem 3. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is monotonic nondecreasing and concave where is defined.*

(i) *If $p, q \in J^+(\mathbb{R})$ then*

$$(2.2) \quad D(f, I, p + q, x; \Phi) \geq D(f, I, p, x; \Phi) + D(f, I, q, x; \Phi)$$

i.e., D is superadditive as a function of weights;

(ii) *If $p, q \in J^+(\mathbb{R})$ with $p \geq q$, meaning that $p_i \geq q_i$ for each $i \in \mathbb{N}$ and $\Phi : [0, \infty) \rightarrow [0, \infty)$ then*

$$(2.3) \quad D(f, I, p, x; \Phi) \geq D(f, I, q, x; \Phi) (\geq 0)$$

i.e., D is monotonic nondecreasing as a function of weights.

Proof. (i). Let $p, q \in J^+(\mathbb{R})$. By the convexity of the function f on C we have

$$(2.4) \quad \begin{aligned} & \frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) f(x_i) - f \left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right) \\ &= \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) \right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) \right)}{P_I + Q_I} \\ & - f \left(\frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right)}{P_I + Q_I} \right) \\ & \geq \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) \right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) \right)}{P_I + Q_I} \\ & - \frac{P_I f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) + Q_I f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right)}{P_I + Q_I} \\ &= \frac{P_I \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]}{P_I + Q_I} \\ & + \frac{Q_I \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]}{P_I + Q_I}. \end{aligned}$$

Since Φ is monotonic nondecreasing and concave, then by (2.4) we have

$$\begin{aligned} & \Phi \left[\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) f(x_i) - f \left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right) \right] \\ & \geq \frac{P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]}{P_I + Q_I} \\ & \quad + \frac{Q_I \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]}{P_I + Q_I}, \end{aligned}$$

which, by multiplication with $P_I + Q_I > 0$ produces the desired result (2.2).

(ii). If $p \geq q$, then by (i) we have

$$\begin{aligned} D(f, I, p, x; \Phi) &= D(f, I, (p - q) + q, x; \Phi) \\ &\geq D(f, I, p - q, x; \Phi) + D(f, I, q, x; \Phi) \\ &\geq D(f, I, p, x; \Phi) \end{aligned}$$

since $D(f, I, p - q, x; \Phi) \geq 0$. \square

Corollary 1. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave where is defined.*

If there exists the numbers $M \geq m \geq 0$ such that $Mq \geq p \geq mq$, then we have

$$\begin{aligned} (2.5) \quad & MQ_I \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \\ & \geq P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] \\ & \geq mQ_I \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]. \end{aligned}$$

In particular

$$\begin{aligned} (2.6) \quad & \frac{M}{m} \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \\ & \geq \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] \\ & \geq \frac{m}{M} \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]. \end{aligned}$$

Now, if we denote by

$$S(\mathbf{1}) := \{p \in J^+(\mathbb{R}) : p_i \leq 1 \text{ for all } i \in \mathbb{N}\},$$

then we can state the following result as well:

Corollary 2. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave where is defined.*

Then we have the bound

$$(2.7) \quad \sup_{p \in S(\mathbf{1})} \left\{ P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] \right\} \\ = \text{card}(I) \Phi \left[\frac{1}{\text{card}(I)} \sum_{i \in I} f(x_i) - f \left(\frac{1}{\text{card}(I)} \sum_{i \in I} x_i \right) \right],$$

where $\text{card}(I)$ denotes the cardinal of the finite set I .

Remark 1. If we consider the concave and monotonic increasing function $\Phi(t) = \ln t$ and assume that $f \in \text{Conv}(C, \mathbb{R})$ and $x \in J_*(C)$ are selected such that $\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) > f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right)$ for any $I \in P_f(\mathbb{N})$ with $\text{card}(I) \geq 2$ and $p \in J^+(\mathbb{R})$ (notice that is enough to assume that f is strictly convex and x is not constant) then by the superadditivity of the functional

$$D(f, I, p, x; \ln) \quad : \quad = P_I \ln \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] \\ = \ln K(f, I, p, x)$$

where

$$(2.8) \quad K(f, I, p, x) := \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^{P_I}$$

we deduce that $K(f, I, \cdot, x)$ is supermultiplicative, i.e., it satisfies the property

$$(2.9) \quad K(f, I, p + q, x) \geq K(f, I, p, x) K(f, I, q, x)$$

for any $p, q \in J^+(\mathbb{R})$.

The proof is obvious by the monotonicity and the positive homogeneity of the functional $D(f, I, \cdot, x; \ln)$.

Notice that the inequality (2.9) has been obtain in a different way by Agarwal & Dragomir in [1].

Another important example of concave and monotonic increasing function is $\Phi(t) = t^s$ with $s \in (0, 1]$. In this situation the functional

$$(2.10) \quad D_s(f, I, p, x) := \left[P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^s \geq 0$$

is superadditive and monotonic nondecreasing as a function of the weights p .

It might be useful for applications to observe that the superadditivity property is translated into the following version of the Jensen's inequality

$$(2.11) \quad \left[(P_I + Q_I)^{s-1} \sum_{i \in I} (p_i + q_i) f(x_i) - (P_I + Q_I)^s f \left(\frac{\sum_{i \in I} (p_i + q_i) x_i}{P_I + Q_I} \right) \right]^s \\ \geq \left[P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^s \\ + \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]^s (\geq 0),$$

where $p, q \in J^+(\mathbb{R})$.

The property of monotonicity provides the following double inequality for $p, q \in J^+(\mathbb{R})$ such that $Mq \geq p \geq mq$ and $M \geq m \geq 0$:

$$(2.12) \quad \begin{aligned} & M \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]^s \\ & \geq \left[P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^s \\ & \geq m \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]^s. \end{aligned}$$

This inequality has the following equivalent form

$$(2.13) \quad \begin{aligned} & M^{1/s} \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \\ & \geq P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \\ & \geq m^{1/s} \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]. \end{aligned}$$

Finally, from the Corollary 2 we also have the bound

$$(2.14) \quad \begin{aligned} & \sup_{p \in S(\mathbf{1})} \left\{ P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right\} \\ & = [\text{card}(I)]^{s-1} \sum_{i \in I} f(x_i) - [\text{card}(I)]^s f \left(\frac{1}{\text{card}(I)} \sum_{i \in I} x_i \right). \end{aligned}$$

For a function $\Psi : (0, \infty) \rightarrow (0, \infty)$ we consider now the functional

$$(2.15) \quad \begin{aligned} & D(f, I, p, x; \Phi, \Psi) \\ & := \sum_{i \in I} \Psi(p_i) \Phi \left[\frac{1}{\sum_{i \in I} \Psi(p_i)} \sum_{i \in I} \Psi(p_i) f(x_i) - f \left(\frac{1}{\sum_{i \in I} \Psi(p_i)} \sum_{i \in I} \Psi(p_i) x_i \right) \right] \end{aligned}$$

where $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^+(\mathbb{R})$, $x \in J_*(C)$. Now, if we denote by $\Psi(p)$ the sequence $\{\Psi(p_i)\}_{i \in \mathbb{N}}$, then we observe that

$$D(f, I, p, x; \Phi, \Psi) = D(f, I, \Psi(p), x; \Phi).$$

The following result may be stated:

Corollary 3. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave. If $\Psi : (0, \infty) \rightarrow (0, \infty)$ is concave, then $D(f, I, \cdot, x; \Phi, \Psi)$ is also concave on $J^+(\mathbb{R})$.*

Proof. Utilising the properties of monotonicity, superadditivity and positive homogeneity of the functional $D(f, I, \cdot, x; \Phi)$ we have successively

$$\begin{aligned}
D(f, I, tp + (1-t)q, x; \Phi, \Psi) &= D(f, I, \Psi(tp + (1-t)q), x; \Phi) \\
&\geq D(f, I, t\Psi(p) + (1-t)\Psi(q), x; \Phi) \\
&\geq D(f, I, t\Psi(p), x; \Phi) + D(f, I, (1-t)\Psi(q), x; \Phi) \\
&= tD(f, I, \Psi(p), x; \Phi) + (1-t)D(f, I, \Psi(q), x; \Phi) \\
&= tD(f, I, p, x; \Phi, \Psi) + (1-t)D(f, I, q, x; \Phi, \Psi)
\end{aligned}$$

for any $p, q \in J^+(\mathbb{R})$ and $t \in [0, 1]$, which proves the statement. \square

3. SOME SUPERADDITIVITY PROPERTIES FOR THE INDEX

The following result concerning the superadditivity and monotonicity of the functional D as an index set function holds:

Theorem 4. *Let $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is monotonic nondecreasing and concave where is defined.*

(i) *If $I, H \in P_f(\mathbb{N})$ with $I \cap H = \emptyset$, then*

$$(3.1) \quad D(f, I \cup H, p, x; \Phi) \geq D(f, I, p, x; \Phi) + D(f, H, p, x; \Phi)$$

i.e., $D(f, \cdot, p, x; \Phi)$ is superadditive as an index set function on $P_f(\mathbb{N})$;

(ii) *If $I, H \in P_f(\mathbb{N})$ with $H \subset I$ and $\Phi : [0, \infty) \rightarrow [0, \infty)$, then*

$$(3.2) \quad D(f, I, p, x; \Phi) \geq D(f, H, p, x; \Phi) (\geq 0)$$

i.e., $D(f, \cdot, p, x; \Phi)$ is monotonic nondecreasing as an index set function on $P_f(\mathbb{N})$.

Proof. (i). Let $I, H \in P_f(\mathbb{N})$ with $I \cap H = \emptyset$. By the convexity of the function f on C we have

$$\begin{aligned}
(3.3) \quad & \frac{1}{P_{I \cup H}} \sum_{k \in I \cup H} p_k f(x_k) - f\left(\frac{1}{P_{I \cup H}} \sum_{k \in I \cup H} p_k x_k\right) \\
&= \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i)\right) + P_H \left(\frac{1}{P_H} \sum_{j \in H} p_j f(x_j)\right)}{P_I + P_H} \\
&\quad - f\left(\frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + P_H \left(\frac{1}{P_H} \sum_{j \in H} p_j x_j\right)}{P_I + P_H}\right) \\
&\geq \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i)\right) + P_H \left(\frac{1}{P_H} \sum_{j \in H} p_j f(x_j)\right)}{P_I + P_H} \\
&\quad - \frac{P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + P_H f\left(\frac{1}{P_H} \sum_{j \in H} p_j x_j\right)}{P_I + P_H} \\
&= \frac{P_I \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)\right]}{P_I + P_H} \\
&\quad + \frac{P_H \left[\frac{1}{P_H} \sum_{j \in H} p_j f(x_j) - f\left(\frac{1}{P_H} \sum_{j \in H} p_j x_j\right)\right]}{P_I + P_H}.
\end{aligned}$$

Since Φ is monotonic nondecreasing and concave, then by (3.3) we have

$$\begin{aligned} & \Phi \left[\frac{1}{P_{I \cup H}} \sum_{k \in I \cup H} p_k f(x_k) - f \left(\frac{1}{P_{I \cup H}} \sum_{k \in I \cup H} p_k x_k \right) \right] \\ & \geq \frac{P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]}{P_I + P_H} \\ & \quad + \frac{P_H \Phi \left[\frac{1}{P_H} \sum_{j \in H} p_j f(x_j) - f \left(\frac{1}{P_H} \sum_{j \in H} p_j x_j \right) \right]}{P_I + P_H}, \end{aligned}$$

which, by multiplication with $P_I + P_H > 0$ produces the desired result (3.2).

(ii). If $I, H \in P_f(\mathbb{N})$ with $H \subset I$, then

$$\begin{aligned} D(f, I, p, x; \Phi) &= D(f, (I \setminus H) \cup H, p, x; \Phi) \\ &\geq D(f, I \setminus H, p, x; \Phi) + D(f, H, p, x; \Phi) \geq D(f, H, p, x; \Phi) (\geq 0) \end{aligned}$$

since $D(f, I \setminus H, p, x; \Phi) \geq 0$. \square

For the special case $I = I_n := \{1, \dots, n\}$ we write $D_n(f, p, x; \Phi)$ instead of $D(f, I_n, p, x; \Phi)$, i.e.,

$$(3.4) \quad D_n(f, p, x; \Phi) = P_n \Phi \left[\frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) - f \left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i \right) \right]$$

where $P_n = P_{I_n} = \sum_{i=1}^n p_i > 0$.

The following particular case is of interest:

Corollary 4. *Let $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave where is defined. Then*

$$(3.5) \quad \max_{I \subseteq I_n} D(f, I, p, x; \Phi) = D_n(f, p, x; \Phi) \geq 0,$$

$$(3.6) \quad \begin{aligned} & D_n(f, p, x; \Phi) \\ & \geq \max_{1 \leq i < j \leq n} \left\{ (p_i + p_j) \Phi \left[\frac{p_i f(x_i) + p_j f(x_j)}{p_i + p_j} - f \left(\frac{p_i x_i + p_j x_j}{p_i + p_j} \right) \right] \right\} \geq 0 \end{aligned}$$

and

$$(3.7) \quad D_n(f, p, x; \Phi) \geq D_{n-1}(f, p, x; \Phi) \geq \dots \geq D_2(f, p, x; \Phi) \geq 0.$$

The proof is obvious by the monotonicity property of the functional $D(f, \cdot, p, x; \Phi)$ as an index set function.

If we use the superadditivity property, then we can state the following result as well:

Corollary 5. *Let $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is monotonic nondecreasing and concave where is defined. Then*

$$(3.8) \quad P_{2n}\Phi \left[\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i f(x_i) - f \left(\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i x_i \right) \right] \\ \geq \sum_{i=1}^n p_{2i} \Phi \left[\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} f(x_{2i}) - f \left(\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} x_{2i} \right) \right] \\ + \sum_{i=1}^n p_{2i-1} \Phi \left[\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} f(x_{2i-1}) - f \left(\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} x_{2i-1} \right) \right]$$

and

$$(3.9) \quad P_{2n+1}\Phi \left[\frac{1}{P_{2n+1}} \sum_{i=1}^{2n+1} p_i f(x_i) - f \left(\frac{1}{P_{2n+1}} \sum_{i=1}^{2n+1} p_i x_i \right) \right] \\ \geq \sum_{i=1}^n p_{2i} \Phi \left[\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} f(x_{2i}) - f \left(\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} x_{2i} \right) \right] \\ + \sum_{i=1}^n p_{2i+1} \Phi \left[\frac{1}{\sum_{i=1}^n p_{2i+1}} \sum_{i=1}^n p_{2i+1} f(x_{2i+1}) - f \left(\frac{1}{\sum_{i=1}^n p_{2i+1}} \sum_{i=1}^n p_{2i+1} x_{2i+1} \right) \right].$$

Remark 2. *If we consider the functional defined in (2.7), namely*

$$K(f, I, p, x) := \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^{P_I}$$

then by Theorem 4 we have that

$$(3.10) \quad K(f, I \cup H, p, x) \geq K(f, I, p, x) \cdot K(f, H, p, x)$$

for any $I, H \in P_f(\mathbb{N})$ with $I \cap H = \emptyset$ meaning that the functional $K(f, \cdot, p, x)$ is supermultiplicative as an index set mapping.

This fact obviously imply the following multiplicative inequalities of interest:

$$(3.11) \quad \left[\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i f(x_i) - f \left(\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i x_i \right) \right]^{P_{2n}} \\ \geq \left[\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} f(x_{2i}) - f \left(\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} x_{2i} \right) \right]^{\sum_{i=1}^n p_{2i}} \\ \times \left[\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} f(x_{2i-1}) - f \left(\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} x_{2i-1} \right) \right]^{\sum_{i=1}^n p_{2i-1}}$$

and where $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$.

Moreover, if we consider the functional defined in (2.10) by

$$D_s(f, I, p, x) := \left[P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^s \geq 0$$

where $s \in (0, 1]$ and introduce the associated functional

$$(3.12) \quad F_s(f, I, p, x) := P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right),$$

then by denoting

$$(3.13) \quad F_{s,n}(f, p, x) := F_s(f, I_n, p, x) = P_n^{s-1} \sum_{i=1}^n p_i f(x_i) - P_n^s f \left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i \right)$$

where $I_n = \{1, \dots, n\}$, we have that the sequence $\{F_{s,n}(f, p, x)\}_{n \geq 2}$ is nondecreasing and the following bounds are valid

$$(3.14) \quad \max_{I \subseteq I_n} F_s(f, I, p, x) = F_{s,n}(f, p, x)$$

and

$$(3.15) \quad F_{s,n}(f, p, x) \geq \max_{1 \leq i < j \leq n} \left\{ \frac{p_i f(x_i) + p_j f(x_j)}{(p_i + p_j)^{1-s}} - (p_i + p_j)^s f \left(\frac{p_i x_i + p_j x_j}{p_i + p_j} \right) \right\} \geq 0.$$

4. APPLICATIONS FOR NORM INEQUALITIES

Let $(X, \|\cdot\|)$ be a real or complex normed linear space. It is well known that the function $f_p : X \rightarrow \mathbb{R}$, $f_p(x) = \|x\|^p$, $p \geq 1$ is convex on X . Assume that $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ are probability distributions with all q_j nonzero. In [11] we obtained the following refinements of the generalised triangle inequality:

$$(4.1) \quad \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\|^p - \left\| \sum_{j=1}^n q_j x_j \right\|^p \right] \geq \sum_{j=1}^n p_j \|x_j\|^p - \left\| \sum_{j=1}^n p_j x_j \right\|^p \\ \geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\|^p - \left\| \sum_{j=1}^n q_j x_j \right\|^p \right] \quad (\geq 0)$$

and

$$(4.2) \quad \max_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^n x_j \right\|^p \right] \geq \sum_{j=1}^n p_j \|x_j\|^p - \left\| \sum_{j=1}^n p_j x_j \right\|^p \\ \geq \min_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^n x_j \right\|^p \right] \quad (\geq 0)$$

for all $p \geq 1$.

We remark that, for $p = 1$ one may get out of the previous results the following inequalities that are intimately related with the generalised triangle inequality in normed spaces:

$$(4.3) \quad \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\| - \left\| \sum_{j=1}^n q_j x_j \right\| \right] \geq \sum_{j=1}^n p_j \|x_j\| - \left\| \sum_{j=1}^n p_j x_j \right\|$$

$$\geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\| - \left\| \sum_{j=1}^n q_j x_j \right\| \right] \quad (\geq 0),$$

$$(4.4) \quad \max_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right] \geq \sum_{j=1}^n p_j \|x_j\| - \left\| \sum_{j=1}^n p_j x_j \right\|$$

$$\geq \min_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right] \quad (\geq 0).$$

If in (4.4) we take

$$p_j := \frac{1}{\|x_j\|} / \sum_{k=1}^n \frac{1}{\|x_k\|} \text{ with } x_j \neq 0 \text{ for all } j \in \{1, \dots, n\},$$

then, by rearranging the inequality, we get the result:

$$(4.5) \quad \max_{1 \leq j \leq n} \{\|x_j\|\} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right] \geq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\|$$

$$\geq \min_{1 \leq j \leq n} \{\|x_j\|\} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right].$$

We note that the inequality (4.5) has been obtained in a different way by M. Kato, K.-S. Saito & T. Tamura in [17] where an analysis of the equality case for strictly convex spaces has been performed as well.

We can state the following result that provides a generalization of the inequality (4.1) above:

Proposition 1. *Let $(X, \|\cdot\|)$ be a normed linear space, $x = (x_1, \dots, x_n)$ an n -tuple of vectors in X , $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ are probability distributions with all q_j nonzero. If $t \geq 1$ and $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave where is defined, then we have*

$$(4.6) \quad \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \Phi \left[\sum_{i=1}^n q_i \|x_i\|^t - \left\| \sum_{i=1}^n q_i x_i \right\|^t \right]$$

$$\geq \Phi \left[\sum_{i=1}^n p_i \|x_i\|^t - \left\| \sum_{i=1}^n p_i x_i \right\|^t \right]$$

$$\geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \Phi \left[\sum_{i=1}^n q_i \|x_i\|^t - \left\| \sum_{i=1}^n q_i x_i \right\|^t \right]$$

and, in particular,

$$\begin{aligned}
(4.7) \quad n \max_{1 \leq i \leq n} \{p_i\} \Phi & \left[n^{-1} \sum_{i=1}^n \|x_i\|^t - n^{-t} \left\| \sum_{i=1}^n x_i \right\|^t \right] \\
& \geq \Phi \left[\sum_{i=1}^n p_i \|x_i\|^t - \left\| \sum_{i=1}^n q_i x_i \right\|^t \right] \\
& \geq n \min_{1 \leq i \leq n} \{p_i\} \Phi \left[n^{-1} \sum_{i=1}^n \|x_i\|^t - n^{-t} \left\| \sum_{i=1}^n x_i \right\|^t \right].
\end{aligned}$$

The proof follows from Corollary 1 and the details are omitted.

Now, if $p = (p_1, \dots, p_n)$ are positive weights with $P_n = \sum_{i=1}^n p_i > 0$ and $x = (x_1, \dots, x_n)$ is an n -tuple of vectors in X , then by defining the functional

$$(4.8) \quad D_n(t, \|\cdot\|, p, x; \Phi) = P_n \Phi \left[P_n^{-1} \sum_{i=1}^n p_i \|x_i\|^t - P_n^{-t} \left\| \sum_{i=1}^n p_i x_i \right\|^t \right]$$

we can state the following result as well:

Proposition 2. *If $t \geq 1$ and $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave where is defined, then we have*

$$\begin{aligned}
(4.9) \quad D_n(t, \|\cdot\|, p, x; \Phi) \\
& \geq \max_{1 \leq i < j \leq n} \left\{ (p_i + p_j) \Phi \left[\frac{p_i \|x_i\|^t + p_j \|x_j\|^t}{p_i + p_j} - \left\| \frac{p_i x_i + p_j x_j}{p_i + p_j} \right\|^t \right] \right\} \geq 0
\end{aligned}$$

and

$$(4.10) \quad D_n(t, \|\cdot\|, p, x; \Phi) \geq D_{n-1}(t, \|\cdot\|, p, x; \Phi) \geq \dots \geq D_2(t, \|\cdot\|, p, x; \Phi) \geq 0.$$

The proof follows from Corollary 4 and the details are omitted.

5. APPLICATIONS FOR f -DIVERGENCES

Given a convex function $f : [0, \infty) \rightarrow \mathbb{R}$, the f -divergence functional

$$(5.1) \quad I_f(p, q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right),$$

was introduced by Csiszár [3]-[4] as a generalized measure of information, a “distance function” on the set of probability distribution \mathbb{P}^n . The restriction here to discrete distributions is only for convenience, similar results hold for general distributions. As in Csiszár [3]-[4], we interpret undefined expressions by

$$\begin{aligned}
f(0) &= \lim_{t \rightarrow 0^+} f(t), \quad 0 f\left(\frac{0}{0}\right) = 0, \\
0 f\left(\frac{a}{0}\right) &= \lim_{\varepsilon \rightarrow 0^+} \varepsilon f\left(\frac{a}{\varepsilon}\right) = a \lim_{t \rightarrow \infty} \frac{f(t)}{t}, \quad a > 0.
\end{aligned}$$

The following results were essentially given by Csiszár and Körner [5].

Proposition 3. *(Joint Convexity) If $f : [0, \infty) \rightarrow \mathbb{R}$ is convex, then $I_f(p, q)$ is jointly convex in p and q .*

Proposition 4. (*Jensen's inequality*) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex. Then for any $p, q \in [0, \infty)^n$ with $P_n := \sum_{i=1}^n p_i > 0$, $Q_n := \sum_{i=1}^n q_i > 0$, we have the inequality

$$(5.2) \quad I_f(p, q) \geq Q_n f\left(\frac{P_n}{Q_n}\right).$$

If f is strictly convex, equality holds in (5.2) iff

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \dots = \frac{p_n}{q_n}.$$

It is natural to consider the following corollary.

Corollary 6. (*Nonnegativity*) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex and normalized, i.e.,

$$(5.3) \quad f(1) = 0.$$

Then for any $p, q \in [0, \infty)^n$ with $P_n = Q_n$, we have the inequality

$$(5.4) \quad I_f(p, q) \geq 0.$$

If f is strictly convex, equality holds in (5.4) iff

$$p_i = q_i \text{ for all } i \in \{1, \dots, n\}.$$

In particular, if p, q are probability vectors, then Corollary 6 shows that, for strictly convex and normalized $f : [0, \infty) \rightarrow \mathbb{R}$ that

$$(5.5) \quad I_f(p, q) \geq 0 \text{ and } I_f(p, q) = 0 \text{ iff } p = q.$$

We now give some examples of divergence measures in Information Theory which are particular cases of f -divergences.

Kullback-Leibler distance ([20]). The *Kullback-Leibler distance* $D(\cdot, \cdot)$ is defined by

$$D(p, q) := \sum_{i=1}^n p_i \log\left(\frac{p_i}{q_i}\right).$$

If we choose $f(t) = t \ln t$, $t > 0$, then obviously

$$I_f(p, q) = D(p, q).$$

Variational distance (l_1 -distance). The *variational distance* $V(\cdot, \cdot)$ is defined by

$$V(p, q) := \sum_{i=1}^n |p_i - q_i|.$$

If we choose $f(t) = |t - 1|$, $t \in [0, \infty)$, then we have

$$I_f(p, q) = V(p, q).$$

Hellinger discrimination ([2]). The *Hellinger discrimination* is defined by $\sqrt{2}h^2(\cdot, \cdot)$, where $h^2(\cdot, \cdot)$ is given by

$$h^2(p, q) := \frac{1}{2} \sum_{i=1}^n (\sqrt{p_i} - \sqrt{q_i})^2.$$

It is obvious that if $f(t) = \frac{1}{2}(\sqrt{t} - 1)^2$, then

$$I_f(p, q) = h^2(p, q).$$

Triangular discrimination ([24]). We define *triangular discrimination* between p and q by

$$\Delta(p, q) = \sum_{i=1}^n \frac{|p_i - q_i|^2}{p_i + q_i}.$$

It is obvious that if $f(t) = \frac{(t-1)^2}{t+1}$, $t \in (0, \infty)$, then

$$I_f(p, q) = \Delta(p, q).$$

Note that $\sqrt{\Delta(p, q)}$ is known in the literature as the Le Cam distance.

χ^2 -distance. We define the χ^2 -distance (chi-square distance) by

$$D_{\chi^2}(p, q) := \sum_{i=1}^n \frac{(p_i - q_i)^2}{q_i}.$$

It is clear that if $f(t) = (t-1)^2$, $t \in [0, \infty)$, then

$$I_f(p, q) = D_{\chi^2}(p, q).$$

Rényi's divergences ([23]). For $\alpha \in \mathbb{R} \setminus \{0, 1\}$, consider

$$\rho_\alpha(p, q) := \sum_{i=1}^n p_i^\alpha q_i^{1-\alpha}.$$

It is obvious that if $f(t) = t^\alpha$ ($t \in (0, \infty)$), then

$$I_f(p, q) = \rho_\alpha(p, q).$$

Rényi's divergences $R_\alpha(p, q) := \frac{1}{\alpha(\alpha-1)} \ln[\rho_\alpha(p, q)]$ have been introduced for all real orders $\alpha \neq 0$, $\alpha \neq 1$ (and continuously extended for $\alpha = 0$ and $\alpha = 1$) in [21], where the reader may find many inequalities valid for these divergences, without, as well as with, some restrictions for p and q .

For other examples of divergence measures, see the paper [18] and the books [21] and [25], where further references are given.

For a function $f : (0, \infty) \rightarrow \mathbb{R}$ we denote by $f^\#$ the function defined on $(0, \infty)$ by the equation $f^\#(x) := f\left(\frac{1}{x}\right)$. With this notation we have

$$(5.6) \quad I_{f^\#}(p, q) = \sum_{i=1}^n q_i f^\#\left(\frac{p_i}{q_i}\right) = \sum_{i=1}^n q_i f\left(\frac{q_i}{p_i}\right).$$

By the use of Corollary 1 we can state the following result for the f -divergences.

Proposition 5. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex and normalized and p, q two probability distributions such that $R := \max_{i \in \{1, \dots, n\}} \left\{ \frac{p_i}{q_i} \right\} < \infty$ and $r := \min_{i \in \{1, \dots, n\}} \left\{ \frac{p_i}{q_i} \right\} > 0$. If $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave where is defined, then we have*

$$(5.7) \quad R\Phi [I_{f^\#}(p, q) - f(D_{\chi^2}(q, p) + 1)] \geq \Phi [I_f(q, p)] \\ \geq r\Phi [I_{f^\#}(p, q) - f(D_{\chi^2}(q, p) + 1)].$$

Proof. Utilising the inequality (2.5) we have

$$(5.8) \quad R\Phi \left[\sum_{i=1}^n q_i f \left(\frac{q_i}{p_i} \right) - f \left(\sum_{i=1}^n \frac{q_i^2}{p_i} \right) \right] \geq \Phi \left[\sum_{i=1}^n p_i f \left(\frac{q_i}{p_i} \right) - f(1) \right] \\ \geq r\Phi \left[\sum_{i=1}^n q_i f \left(\frac{q_i}{p_i} \right) - f \left(\sum_{i=1}^n \frac{q_i^2}{p_i} \right) \right].$$

Since

$$\sum_{i=1}^n \frac{q_i^2}{p_i} = D_{\chi^2}(q, p) + 1,$$

then by (5.8) we deduce the desired result (5.7). \square

Finally, by the use of Corollary 4 we also have the following lower bound for the f -divergence:

Proposition 6. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex and normalized and p, q two probability distributions. If $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave where is defined, then we have:*

$$(5.9) \quad \Phi [I_f(q, p)] \\ \geq \max_{1 \leq i < j \leq n} \left\{ (p_i + p_j) \Phi \left[\frac{p_i f \left(\frac{q_i}{p_i} \right) + p_j f \left(\frac{q_j}{p_j} \right)}{p_i + p_j} - f \left(\frac{q_i + q_j}{p_i + p_j} \right) \right] \right\} \geq 0.$$

Remark 3. *If one chooses different examples of convex functions generating the particular divergences mentioned at the beginning of the section, that one can obtain various inequalities of interest. However the details are not presented here.*

REFERENCES

- [1] R. P. Agarwal and S. S. Dragomir, The property of supermultiplicity for some classical inequalities and applications. *Comput. Math. Appl.* **35** (1998), no. 6, 105–118.
- [2] R. Beran, Minimum Hellinger distance estimates for parametric models, *Ann. Statist.*, **5** (1977), 445-463.
- [3] I. Csiszár, Information-type measures of differences of probability distributions and indirect observations, *Studia Sci. Math. Hung.*, **2** (1967), 299-318.
- [4] I. Csiszár, Information-type measures of difference of probability functions and indirect observations, *Studia Sci. Math. Hungar.*, **2** (1967), 299-318.
- [5] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*, Academic Press, New York, 1981.
- [6] S.S. Dragomir, An improvement of Jensen’s inequality, *Bull. Math. Soc. Sci. Math. Roumanie*, **34(82)** (1990), No. 4, 291-296.
- [7] S.S. Dragomir, Some refinements of Ky Fan’s inequality, *J. Math. Anal. Appl.*, **163(2)** (1992), 317-321.
- [8] S.S. Dragomir, Some refinements of Jensen’s inequality, *J. Math. Anal. Appl.*, **168(2)** (1992), 518-522.
- [9] S.S. Dragomir, A further improvement of Jensen’s inequality, *Tamkang J. Math.*, **25(1)** (1994), 29-36.
- [10] S.S. Dragomir, A new improvement of Jensen’s inequality, *Indian J. Pure and Appl. Math.*, **26(10)** (1995), 959-968.
- [11] S.S. Dragomir, Bounds for the normalised Jensen functional. *Bull. Austral. Math. Soc.* **74** (2006), no. 3, 471–478.
- [12] S.S. Dragomir, A refinement of Jensen’s inequality with applications for f -divergence measures, Preprint, *Res. Rep. Coll.* **10(2007)**, Supp., Article 15.

- [13] S.S. Dragomir, Refinements of the Jensen integral inequality, *in preparation*.
- [14] S.S. Dragomir and N.M. Ionescu, Some converse of Jensen's inequality and applications. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), no. 1, 71–78. MR1325895 (96c:26012).
- [15] S.S. Dragomir and C.J. Goh, A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory, *Mathl. Comput. Modelling*, **24**(1996), No. 2, 1-11.
- [16] S.S. Dragomir, J. Pečarić and L.E. Persson, Properties of some functionals related to Jensen's inequality, *Acta Math. Hung.*, **70**(1-2) (1996), 129-143.
- [17] M. Kato, K.-S. Saito and T. Tamura, Sharp triangle inequality and its reverse in Banach spaces. *Math. Inequal. Appl.* **10** (2007), no. 2, 451–460.
- [18] J.N. Kapur, A comparative assessment of various measures of directed divergence, *Advances in Management Studies*, **3** (1984), No. 1, 1-16.
- [19] S. Kullback, *Information Theory and Statistics*, J. Wiley, New York, 1959.
- [20] S. Kullback and R.A. Leibler, On information and sufficiency, *Annals Math. Statist.*, **22** (1951), 79-86.
- [21] F. Liese and I. Vajda, *Convex Statistical Distances*, Teubner Verlag, Leipzig, 198
- [22] J. Pečarić and S.S. Dragomir, A refinements of Jensen inequality and applications, *Studia Univ. Babeş-Bolyai, Mathematica*, **24**(1) (1989), 15-19.
- [23] A. Rényi, On measures of entropy and information, *Proc. Fourth Berkeley Symp. Math. Statist. Prob.*, Vol. 1, University of California Press, Berkeley, 1961.
- [24] F. Topsøe, Some inequalities for information divergence and related measures of discrimination, *Res. Rep. Coll. RGMIA*, **2** (1) (1999), 85-98.7.
- [25] I. Vajda, *Theory of Statistical Inference and Information*, Kluwer, Boston, 1989.
- [26] P. M. Vasić and Ž. Mijašević, On an index set function connected with Jensen inequality. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No. **544–576** (1976), 110–112.

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