MONOTONICITY OF SEQUENCES INVOLVING GEOMETRIC MEANS OF POSITIVE SEQUENCES WITH LOGARITHMICAL CONVEXITY

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Abstract. Let \( f \) be a positive function such that \( \frac{x[f(x + 1)/f(x)]}{f} \) is increasing on \([1, \infty)\), then the sequence \( \left\{ \sqrt[n]{\prod_{i=1}^{n} f(i)/f(n + 1)} \right\}_{n=1}^{\infty} \) is decreasing. If \( f \) is a logarithmically concave and positive function defined on \([1, \infty)\), then the sequence \( \left\{ \sqrt[n]{\prod_{i=1}^{n} f(i)/\sqrt[n]{f(n)}} \right\}_{n=1}^{\infty} \) is increasing.

As consequences of these monotonicities, the lower and upper bounds for the ratio \( \sqrt[n]{\prod_{i=k+1}^{n+k} f(i)/\sqrt[n]{f(n+k+1)} f(i)} \) of the geometric mean sequence \( \left\{ \sqrt[n+k]{\prod_{i=k+1}^{n+k} f(i)} \right\}_{n=1}^{\infty} \) are obtained, where \( k \) is a nonnegative integer and \( m \) a natural number. Some applications are given.

1. Introduction

It is known that, for \( n \in \mathbb{N} \), the following double inequality were given in [6]:

\[
\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1,
\]

which can be rearranged as

\[
\frac{\sqrt[n]{\Gamma(1+r)}}{\sqrt[n+1]{\Gamma(2+r)}} < \frac{\sqrt[n]{\Gamma(2+r)}}{\sqrt[n+1]{\Gamma(1+r)}} \tag{2}
\]

and

\[
\frac{\sqrt[n]{\Gamma(1+r)}}{r} > \frac{\sqrt[n]{\Gamma(2+r)}}{r+1} \tag{3}
\]

2000 Mathematics Subject Classification. Primary 26D15; Secondary 26A48.

Key words and phrases. monotonicity, inequality, geometric mean, ratio, positive sequence, logarithmically concave, mathematical induction.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), China.

This paper was typeset using \( \LaTeX \).
In [1], the left inequality in (1) was refined by
\[
\frac{n}{n + 1} < \left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{1/r} < \frac{\sqrt[n]{n!}}{n^+\sqrt{(n + 1)!}} \tag{4}
\]
for all positive real numbers \( r \). Both bounds are the best possible.

Using analytic method and Stirling’s formula, in [10, 14, 16, 17], for \( n, m \in \mathbb{N} \) and \( k \) being a nonnegative integer, the author and others proved the following inequalities:
\[
\frac{n + k + 1}{n + m + k + 1} < \left( \prod_{i=k+1}^{n+k} i \right)^{1/n} / \left( \prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)} \leq \sqrt[\sqrt{n+k+1}]{n + m + k + \alpha}, \tag{5}
\]
the equality in (5) is valid for \( n = 1 \) and \( m = 1 \), which extend and refine those in (1).

There is a rich literature on refinements, extensions, and generalizations of the inequalities in (4), for examples, [2, 8, 9, 13, 19] and references therein. Note that the inequalities in (4) are direct consequences of a conjecture which states that the function \( \left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{1/r} \) is decreasing with \( r \). Please refer to [18].

In [11], using the ideas and method in [3, 5, 15] and the mathematical induction, the following inequalities were obtained.

**Theorem A.** Let \( k \) be a nonnegative integer, \( n \) and \( m \) positive integers, and \( \alpha \in [0, 1] \) a constant. Then
\[
\frac{n + k + 1}{n + m + k + 1} + \frac{\alpha}{n + m + k + 1} \leq \left( \prod_{i=k+1}^{n+k} (i + \alpha) \right)^{1/n} / \left( \prod_{i=k+1}^{n+m+k} (i + \alpha) \right)^{1/(n+m)} \leq \sqrt[\sqrt{n+k+1}]{n + m + k + \alpha}. \tag{6}
\]

If \( n = 1 \) and \( m = 1 \), then the equality in the right hand side inequality of (6) holds.

In [12], Theorem A was generalized to the following

**Theorem B.** For all nonnegative integers \( k \) and natural numbers \( n \) and \( m \), we have
\[
\frac{a(n + k + 1) + b}{a(n + m + k + 1) + b} < \left( \prod_{i=k+1}^{n+k} (ai + b) \right)^{\frac{1}{n}} \leq \sqrt[\sqrt{n+k+1}]{a(n + m + k + b)} \tag{7}
\]
where \( a \) is a positive constant, and \( b \) is a nonnegative constant. The equality in (7) is valid for \( n = 1 \) and \( m = 1 \).
In [4], the following monotonicity results for the gamma function were established: The function \( \Gamma(1 + \frac{1}{x})^x \) decreases with \( x > 0 \) and \( x[\Gamma(1 + \frac{1}{x})]^x \) increases with \( x > 0 \), which recover the inequalities in (1) which refer to integer values of \( r \). These are equivalent to the function \( \Gamma(1 + x)^\frac{1}{x} \) being increasing and \( \frac{\Gamma(1+x)^\frac{1}{x}}{x} \) being decreasing on \((0, \infty)\), respectively. In addition, it was proved that the function \( x^{1-\gamma}[\Gamma(1 + \frac{1}{x})^x] \) decreases for \( 0 < x < 1 \), where \( \gamma = 0.57721566 \cdots \) denotes the Euler’s constant, which is equivalent to \( \frac{\Gamma(1+x)^\frac{1}{x}}{x} \) being increasing on \((1, \infty)\).

In [14], the following monotonicity result was obtained: The function

\[
\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1}
\]

is decreasing in \( x \geq 1 \) for fixed \( y \geq 0 \). Then, for positive real numbers \( x \) and \( y \), we have

\[
\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}.
\]

Inequality (9) extends and generalizes inequality (5), since \( \Gamma(n+1) = n! \).

**Definition 1** ([7, p. 7]). A positive function \( f : I \to \mathbb{R} \), \( I \) an interval in \( \mathbb{R} \), is said to be logarithmically convex (log-convex, multiplicatively convex) if \( \ln f \) is convex, or equivalently if for all \( x, y \in I \) and all \( \alpha \in [0, 1] \),

\[
f(\alpha x + (1-\alpha)y) \leq f^\alpha(x)f^{1-\alpha}(y).
\]

It is said to be logarithmically concave (log-concave) if the inequality in (10) is reversed.

**Remark 1.** By \( f = \exp \ln f \), it follows that a logarithmically convex function is convex (but not conversely). This directly follows from (10), of course, since by the arithmetic-geometric inequality we have

\[
f^\alpha(x)f^{1-\alpha}(y) \leq f(x) + (1-\alpha)f(y).
\]

J. Pečarić told the author that a concave positive function is a logarithmically concave one affirmatively.

In this article, we will further generalize the inequalities in (7) and obtain the following
Theorem 1. Let \( f \) be an increasing, logarithmically convex and positive function defined on \([1, \infty)\). Then the sequence
\[
\left\{ \frac{\sqrt[\sum_{i=1}^{n} f(i)}}{f(n+1)} \right\}_{n=1}^{\infty}
\]
is decreasing. As a consequence, we have the following
\[
\frac{\sqrt[\sum_{i=k+1}^{n+k} f(i)}}{\sqrt[\sum_{i=k+1}^{n+m+k} f(i)}} \geq \frac{f(n+k+1)}{f(n+m+k+1)},
\]
where \( m \) is a natural number and \( k \) a nonnegative integer.

Corollary 1. Let \( \{a_i\}_{i=1}^{\infty} \) be an increasing, logarithmically convex, and positive sequence, then the sequence
\[
\left\{ \frac{\sqrt[n]{a_n!}}{a_{n+1}} \right\}_{n=1}^{\infty}
\]
is decreasing. As a consequence, we have the following
\[
\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} \geq \frac{a_{n+1}}{a_{n+m+1}},
\]
where \( m \) is a natural number and \( a_n! \) is the sequence factorial defined by \( \prod_{i=1}^{n} a_i \).

Theorem 2. Let \( f \) be a logarithmically concave and positive function defined on \([1, \infty)\). Then the sequence
\[
\left\{ \frac{\sqrt[\sum_{i=1}^{n} f(i)}}{\sqrt{f(n)}} \right\}_{n=1}^{\infty}
\]
is increasing. As a consequence, we have the following
\[
\sqrt[\sum_{i=k+1}^{n+k} f(i)]{\sqrt[\sum_{i=k+1}^{n+m+k} f(i)}} \leq \sqrt{\frac{f(n+k)}{f(n+m+k)}},
\]
where \( m \) is a natural number and \( k \) a nonnegative integer. The equality in (16) is valid for \( n = 1 \) and \( m = 1 \).

Corollary 2. Let \( \{a_i\}_{i=1}^{\infty} \) be a logarithmically concave and positive sequence. Then the sequence
\[
\left\{ \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} \right\}_{n=1}^{\infty}
\]
is increasing. Therefore, we have
\[
\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} \leq \sqrt[\sum_{i=1}^{n} a_i]{\sqrt[\sum_{i=1}^{n+m} a_i]},
\]
where \( m \) is a natural number and \( a_n! \) is the sequence factorial defined by \( \prod_{i=1}^{n} a_i \).

The equality in (18) is valid for \( n = 1 \) and \( m = 1 \).

At last, in Section 3, some applications of Theorem 1 and Theorem 2 are given and an open problem is proposed.

Remark 2. It is well known that the left hand side term in (12) or (16) is a ratio of two geometric means of sequence \( \{f(i)\}_{i=1}^{\infty} \).

2. PROOFS OF THEOREM 1 AND THEOREM 2

Proof of Theorem 1. The monotonicity of the sequence (11) and inequality (12) are equivalent to the following

\[
\left( \prod_{i=1}^{n} \frac{f(i)}{f(n+1)} \right)^{1/n} \geq \left( \prod_{i=1}^{n} \frac{f(i)}{f(n+2)} \right)^{1/(n+1)},
\]

\[
\iff \frac{1}{n} \sum_{i=1}^{n} \ln \left( \frac{f(i)}{f(n+1)} \right) \geq \frac{1}{n+1} \sum_{i=1}^{n+1} \ln \left( \frac{f(i)}{f(n+2)} \right),
\]

\[
\iff \frac{n}{n+1} \sum_{i=1}^{n+1} \ln \left( \frac{f(i)}{f(n+2)} \right) \leq \sum_{i=1}^{n} \ln \left( \frac{f(i)}{f(n+1)} \right). \tag{19}
\]

Since \( \ln x \) is concave on \((0, \infty)\), by definition of concaveness, it follows that, for \( 1 \leq i \leq n \),

\[
\frac{i}{n+1} \ln \left( \frac{f(i+1)}{f(n+2)} \right) + \frac{n-i+1}{n+1} \ln \left( \frac{f(i)}{f(n+2)} \right) \leq \ln \left( \frac{i f(i+1) + (n-i+1) f(i)}{(n+1) f(n+2)} \right) \tag{20}
\]

Since \( f \) is logarithmically convex, we have \( f(n)f(n+2) \geq \lfloor f(n+1) \rfloor^2 \). Hence, for all \( 1 \leq i \leq n \), from the function \( f \) being increasing, we have

\[
f(n)f(n+2) - \lfloor f(n+1) \rfloor^2 \geq \frac{1}{n} f(n) \lfloor f(n+1) - f(n+2) \rfloor
\]

\[
\iff \frac{(n+1)f(n+2)}{f(n+1)} - 1 \geq \frac{n f(n+1)}{f(n)}
\]

\[
\iff \frac{(n+1)f(n+2)}{f(n+1)} - (n+1) \geq \frac{n f(n+1)}{f(n)} - n \tag{21}
\]

\[
\iff \frac{(n+1)f(n+2)}{f(n+1)} - (n+1) \geq \frac{i f(i+1)}{f(i)} - i
\]
\[
\iff (i + 1) + (n - i + 1)f(i) \leq \frac{(n + 1)f(n + 2)}{f(n + 1)}
\iff \frac{i f(i + 1) + (n - i + 1)f(i)}{(n + 1)f(n + 2)} \leq \frac{f(i)}{f(n + 1)}.
\]

Combining the last line above with (20) yields
\[
\frac{i}{n + 1} \ln \frac{f(i + 1)}{f(n + 2)} + \frac{n - i + 1}{n + 1} \ln \frac{f(i)}{f(n + 2)} \leq \ln \frac{f(i)}{f(n + 1)}.
\tag{22}
\]

Summing up on both sides of (22) from 1 to \(n\) and simplifying reveals inequality (19). The proof is complete. \(\square\)

**Proof of Theorem 2.** The monotonicity of the sequence (15) and inequality (16) are equivalent to the following
\[
\sqrt[n]{\prod_{i=1}^{n} f(i)} \leq \frac{n}{n + 1} \sqrt[n+1]{\prod_{i=1}^{n+1} f(i)}
\iff \frac{1}{n} \sum_{i=1}^{n} \ln f(i) - \frac{1}{n + 1} \sum_{i=1}^{n+1} \ln f(i) \leq \frac{1}{2} [\ln f(n) - \ln f(n + 1)]
\iff (1 + \frac{1}{n}) \sum_{i=1}^{n} \ln f(i) - \frac{n + 1}{n + 1} \sum_{i=1}^{n+1} \ln f(i) \leq \frac{n + 1}{2} [\ln f(n) - \ln f(n + 1)]
\iff \frac{n + 1}{2} \ln f(n) - \frac{n - 1}{2} \ln f(n + 1) \geq \frac{1}{n} \sum_{i=1}^{n} \ln f(i).
\tag{23}
\]

For \(n = 1\), the equality in (23) holds.

Suppose inequality (23) is valid for some \(n > 1\). Since, by the inductive hypothesis
\[
\frac{1}{n + 1} \sum_{i=1}^{n+1} \ln f(i) = \frac{n}{n + 1} \left[ \frac{1}{n} \sum_{i=1}^{n} \ln f(i) + \frac{\ln f(n + 1)}{n + 1} \right] \leq \frac{n + 1}{n + 1} \left[ \frac{n + 1}{2} \ln f(n) - \frac{n - 1}{2} \ln f(n + 1) \right] + \frac{\ln f(n + 1)}{n + 1}
= \frac{n}{2} \ln f(n) - \frac{n - 2}{2} f(n + 1),
\]

by induction, it is sufficient to prove
\[
\frac{n}{2} \ln f(n) - \frac{n - 2}{2} \ln f(n + 1) \leq \frac{n + 2}{2} \ln f(n + 1) - \frac{n}{2} \ln f(n + 2)
\iff n \ln f(n) \leq 2n \ln f(n + 1) - n \ln f(n + 2)
\iff \ln[f(n)f(n + 2)] \leq \ln f^2(n + 1)
\[\iff f(n)f(n + 2) \leq f^2(n + 1),\]

this follows from the logarithmic concaveness of the function \(f\). The proof is complete. \(\Box\)

Remark 3. If the function \(f\) in Theorem 1 is differentiable, then we can give the following proof of Theorem 1 by Cauchy’s mean value theorem and mathematical induction.

Proof of Theorem 1 under condition such that \(f\) being differentiable. The monotonicity of the sequence \((11)\) and inequality \((12)\) are equivalent to

\[
\iff \frac{1}{n} \sum_{i=1}^{n} \ln f(i) - \frac{1}{n+1} \sum_{i=1}^{n+1} \ln f(i) \geq \ln f(n+1) - \ln f(n+2) \\
\iff \frac{1}{n} \sum_{i=1}^{n} \ln f(i) - \ln f(n+1) \geq (n+1) \left[ \ln f(n+1) - \ln f(n+2) \right] \\
\iff (n+2) \ln f(n+1) - (n+1) \ln f(n+2) \leq \frac{1}{n} \sum_{i=1}^{n} \ln f(i). \quad (24)
\]

For \(n = 1\), inequality \((24)\) can be rewritten as \(f(1)[f(3)]^2 \geq [f(2)]^3\). Since \(f\) is logarithmically convex and increasing, we have \(f(1)f(3) \geq [f(2)]^2\) and \(f(3) \geq f(2)\), respectively. Therefore, inequality \((24)\) holds for \(n = 1\).

Suppose inequality \((24)\) is valid for some \(n > 1\). Then, by inductive hypothesis, we have

\[
\frac{1}{n+1} \sum_{i=1}^{n+1} \ln f(i) = \frac{n}{n+1} \left[ \frac{1}{n} \sum_{i=1}^{n} \ln f(i) \right] + \frac{f(n+1)}{n+1} \\
\geq \frac{n}{n+1} \left[ (n+2) \ln f(n+1) - (n+1) \ln f(n+2) \right] + \frac{f(n+1)}{n+1} \\
= (n+1) \ln f(n+1) - n \ln f(n+2).
\]

hence, by induction, it is sufficient to prove the following

\[(n+1) \ln f(n+1) - n \ln f(n+2) \geq (n+3) \ln f(n+2) - (n+2) \ln f(n+3),\]

which can be rearranged as

\[(n+1)[ \ln f(n+1) - \ln f(n+2)] \geq (n+2)[ \ln f(n+2) - \ln f(n+3)],\]
further, since \( f \) is increasing,

\[
\frac{\ln f(n+2) - \ln f(n+1)}{\ln f(n+3) - \ln f(n+2)} \leq \frac{n+2}{n+1}.
\]  

(25)

Using Cauchy’s mean values applied to \( g(x) = \ln f(n+1+x) \) and \( h(x) = \ln f(n+2+x) \) for \( x \in [0,1] \) in inequality (25), it follows that there exists a point \( \xi \in (0,1) \) such that

\[
\frac{f'(n+1+\xi)}{f(n+1+\xi)} \cdot \frac{f(n+2+\xi)}{f'(n+2+\xi)} \leq \frac{n+2}{n+1}.
\]

Since the positive function \( f \) is logarithmically convex and differentiable, then 

\[
[\ln f(x)]' = \frac{f'(x)}{f(x)}
\]

is increasing. Thus

\[
\frac{f'(n+1+\xi)}{f(n+1+\xi)} \leq \frac{f'(n+2+\xi)}{f(n+2+\xi)},
\]

and then

\[
\frac{f'(n+1+\xi)}{f(n+1+\xi)} \cdot \frac{f(n+2+\xi)}{f'(n+2+\xi)} \leq 1 < \frac{n+2}{n+1}.
\]

Inequality (25) follows. The proof is complete. \( \square \)

3. Applications

3.1. The affine function \( f(x) = ax + b \) for \( x > -\frac{b}{a} \), where \( a > 0 \) and \( b \in \mathbb{R} \) are constants, is positive and logarithmically concave. From Theorem 2 applied to this affine function, the right hand side inequality in (7) follows.

3.2. From procedure of the proof of Theorem 1 and noticing inequality (21), we can establish the following more general results.

**Theorem 3.** Let \( f \) be a positive function such that \( x \left[ \frac{f(x+1)}{f(x)} - 1 \right] \) is increasing on \([1, \infty)\), then the sequence (11) decreases and inequality (12) holds.

**Corollary 3.** Let \( \{a_i\}_{i=1}^{\infty} \) be a positive sequence such that \( \left\{ i \left[ \frac{a_{i+1}}{a_i} - 1 \right] \right\}_{i=1}^{\infty} \) is increasing, then the sequence (13) decreases and inequality (14) holds.

3.3. The left hand side inequality in (7) follows from Corollary 3.
3.4. Applying Theorem 3 or Corollary 3 to \( f(x) = \Gamma(x+1) \) or \( a_i = i! \) respectively yields

\[
\frac{\prod_{i=2}^{n}(i + k)}{\prod_{i=2}^{n+m}(i + k)} \geq \frac{\prod_{i=k+1}^{n+k}(i!)}{\prod_{i=k+1}^{n+m+k}(i!)} \geq \frac{(n + k + 1)!}{(n + m + k + 1)!} = \prod_{i=1}^{m}(n + k + 1 + i). \tag{26}
\]

Similarly, we have

\[
\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(i!!)}} \geq \frac{(n + k + 1)!!}{(n + m + k + 1)!!}, \tag{27}
\]

\[
\frac{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(2i!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(2i!!)}} \geq \frac{[2(n + k + 1)]!!}{[2(n + m + k + 1)]!!}, \tag{28}
\]

\[
\frac{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}((2i - 1)!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}((2i - 1)!!)}} \geq \frac{[2(n + k) + 1]!!}{[2(n + m + k) + 1]!!}. \tag{29}
\]

Where \( n \) and \( m \) are natural numbers and \( k \) a nonnegative integer.

3.5. In Corollary 1, considering the sequence \( \{\ln a_i\}_{i=1}^{\infty} \) is increasing, convex, and positive, we obtain the following

**Corollary 4.** Let \( \{a_i\}_{i=1}^{\infty} \) be an increasing convex positive sequence and \( A_n = \frac{1}{n} \sum_{i=1}^{n} a_i \) an arithmetic mean. Then the sequence \( A_n - a_{n+1} \) decreases. This gives a lower bound for difference of two arithmetic means:

\[
A_n - A_{n+m} \geq a_{n+1} - a_{n+m+1}, \tag{30}
\]

where \( m \) is a natural number.

3.6. In Corollary 2, considering the sequence \( \{\ln a_i\}_{i=1}^{\infty} \) is concave and positive, we have

**Corollary 5.** Let \( \{a_i\}_{i=1}^{\infty} \) be a concave positive sequence and \( A_n = \frac{1}{n} \sum_{i=1}^{n} a_i \) an arithmetic mean. Then the sequence \( A_n - a_{n+1/2} \) increases. This implies an upper bound for difference of two arithmetic means:

\[
A_n - A_{n+m} \leq \frac{a_n - a_{n+m}}{2}, \tag{31}
\]

where \( m \) is a natural number.
3.7. For real numbers $b \geq 1$ and $c \geq 0$ such that $b^2 > 2c$, the function $x^2 + bx + c$ is logarithmically concave and satisfies conditions of Theorem 3, then we have

$$\frac{(n+k+1)^2 + b(n+k+1) + c}{(n+m+k+1)^2 + b(n+m+k+1) + c} \leq \frac{\sqrt[n+m+k+1]{\prod_{i=k+1}^{n+m+k} (i^2 + bi + c)}}{\sqrt[n+m+k+1]{\prod_{i=k+1}^{n+m+k} (i^2 + bi + c)}} \leq \sqrt{\frac{(n+k)^2 + b(n+k) + c}{(n+m+k)^2 + b(n+m+k) + c}},$$

(32)

where $m$ is a natural number and $k$ a nonnegative integer.

4. Open Problem

In the final, we pose the following open problem.

**Open Problem.** For any positive real number $z$, define $z! = z(z-1)\cdots\{z\}$, where $\{z\} = z - [z - 1]$, and $[z]$ denotes Gauss function whose value is the largest integer not more than $z$. Let $x > 0$ and $y \geq 0$ be real numbers, then

$$\frac{x + 1}{x + y + 1} \leq \frac{\sqrt{x!}}{\sqrt{x+y}!} \leq \sqrt{\frac{x}{x+y}}.$$ 

(33)

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