Some Inequalities in Inner Product Spaces Related to the Generalized Triangle Inequality

SOME INEQUALITIES IN INNER PRODUCT SPACES RELATED TO THE GENERALIZED TRIANGLE INEQUALITY

S.S. DRAGOMIR, Y.J. CHO*, AND S.S. KIM

Abstract. In this paper we obtain some inequalities related to the generalized triangle and quadratic triangle inequalities for vectors in inner product spaces. Some results that employ the Ostrowski discrete inequality for vectors in normed linear spaces are also obtained.

1. Introduction

In Mathematical Analysis one of the most important inequalities is the (generalized) *triangle inequality* which states that, in a normed linear space $(X;\|\cdot\|)$ we have the inequality

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq \sum_{i=1}^{n}\|x_i\| \tag{1.1}$$

for any vectors $x_i \in X$, $i \in \{1,\ldots,n\}$.

In 1966, J.B. Diaz and F.T. Metcalf [3] proved the following reverse of the triangle inequality in inner product spaces:

Theorem 1 (Diaz-Metcalf, 1966, [3]). *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Suppose that the vectors $x_i \in H\setminus\{0\}$, $i \in \{1,\ldots,n\}$ satisfy*

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \qquad i \in \{1,\ldots,n\}. \tag{1.2}$$

Then

$$r\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.3}$$

where equality holds if and only if

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.4}$$

The following result with a natural geometrical meaning also holds:

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Theorem 2 (Dragomir, 2004, [5])**.** *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ and $\rho\in(0,1)$. If $x_i\in H$, $i\in\{1,\dots,n\}$ are such that*

$$\|x_i-a\|\leq\rho \quad \text{for each} \quad i\in\{1,\dots,n\}, \tag{1.5}$$

then we have the inequality

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\|\leq\left\|\sum_{i=1}^{n}x_i\right\|, \tag{1.6}$$

with equality if and only if

$$\sum_{i=1}^{n}x_i=\sqrt{1-\rho^2}\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.7}$$

The following result that provides a more intuitive condition for the vectors involved may be stated as well:

Theorem 3 (Dragomir, 2004, [5])**.** *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ and $M\geq m>0$. If $x_i\in H$, $i\in\{1,\dots,n\}$ are such that either*

$$\operatorname{Re}\langle Ma-x_i,x_i-ma\rangle\geq0 \tag{1.8}$$

or, equivalently,

$$\left\|x_i-\frac{M+m}{2}\cdot a\right\|\leq\frac{1}{2}(M-m) \tag{1.9}$$

holds for each $i\in\{1,\dots,n\}$, then we have the inequality

$$\frac{2\sqrt{mM}}{m+M}\sum_{i=1}^{n}\|x_i\|\leq\left\|\sum_{i=1}^{n}x_i\right\|, \tag{1.10}$$

or, equivalently,

$$(0\leq)\sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\|\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}}\left\|\sum_{i=1}^{n}x_i\right\|. \tag{1.11}$$

The equality holds in (1.10) (or in (1.11)) if and only if

$$\sum_{i=1}^{n}x_i=\frac{2\sqrt{mM}}{m+M}\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.12}$$

In inner product spaces $(H;\langle\cdot,\cdot\rangle)$ another important inequality for sequences of vectors is the following one

$$\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\leq\sum_{i=1}^{n}p_i\|x_i\|^2, \tag{1.13}$$

where $x_i\in H$, $i\in\{1,\dots,n\}$ and $p_i\geq0$ with $\sum_{i=1}^{n}p_i=1$. This can be seen as the quadratic version of the generalized triangle inequality.

A reverse of this inequality is incorporated in the following result:

Theorem 4 (Dragomir, 2000, [4])**.** *Consider an inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Suppose that the vectors $x_i,x,X\in H$, $i\in\{1,\dots,n\}$ satisfy the condition*

$$\operatorname{Re}\langle X-x_i,x_i-x\rangle\geq0 \quad \text{for all } i\in\{1,\dots,n\}, \tag{1.14}$$

or, equivalently,

$$\left\| x_i - \frac{x+X}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \textit{for all } i \in \{1, \dots, n\}, \tag{1.15}$$

then we have the inequality

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq \frac{1}{4} \|X - x\|^2 \tag{1.16}$$

for any $p_i \geq 0$, $i \in \{1, \dots, n\}$, *with* $\sum_{i=1}^{n} p_i = 1$.

The constant $\frac{1}{4}$ *is best possible in (1.16).*

For other results related to the generalised triangle inequality, see [1], [2], [7], [8] and [9], where further references are provided.

In this paper we obtain other inequalities related to the generalized triangle and quadratic triangle inequalities for vectors in inner product spaces. Some results that employ the Ostrowski discrete inequality for vectors in normed linear spaces are also obtained.

2. The Results

The following result that encompasses a condition depending on the probability distribution $p_i \geq 0$, $i \in \{1, \dots, n\}$ can be stated:

Theorem 5. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space,* $x_i \in H$, $i \in \{1, \dots, n\}$ *and* $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$. *If there exist constants* $r_i > 0$, $i \in \{1, \dots, n\}$, *so that*

$$\left\| x_i - \sum_{j=1}^{n} p_j x_j \right\| \leq r_i \quad \textit{for each } \; i \in \{1, \dots, n\}, \tag{2.1}$$

then

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq \sum_{i=1}^{n} p_i r_i^2. \tag{2.2}$$

The inequality (2.2) is sharp.

Proof. Taking the square in (2.1) we get

$$\|x_i\|^2 - 2 \operatorname{Re} \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \leq r_i^2 \tag{2.3}$$

for each $i \in \{1, \dots, n\}$.

Now, if we multiply (2.3) by $p_i \geq 0$ and sum over i from 1 to n, we obtain

$$\sum_{i=1}^{n} \|x_i\|^2 - 2 \operatorname{Re} \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \leq \sum_{i=1}^{n} p_i r_i^2 \tag{2.4}$$

and since

$$\operatorname{Re} \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle = \left\| \sum_{i=1}^{n} p_i x_i \right\|^2$$

we deduce from (2.4) the desired inequality (2.2).

Now, assume that $p_1, p_2 \in (0,1)$ with $p_1 + p_2 = 1$ and $x_1, x_2 \in H$ with $x_1 \neq x_2$. With this assumption, (2.1) becomes

$$\|x_1 - p_1 x_1 - p_2 x_2\| \leq r_1, \quad \|x_2 - p_1 x_1 - p_2 x_2\| \leq r_2$$

which obviously holds for $r_1 = p_2 \|x_1 - x_2\|$ and $r_2 = p_1 \|x_1 - x_2\|$.

Now, assume that there exists a constant $c > 0$ such that (2.2) holds with c, i.e.,

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq c \sum_{i=1}^{n} p_i r_i^2. \tag{2.5}$$

Since

$$\begin{aligned}
&p_1 \|x_1\|^2 + p_2 \|x_2\|^2 - \|p_1 x_1 + p_2 x_2\|^2 \\
&= p_1 (1 - p_1) \|x_1\|^2 + p_2 (1 - p_2) \|x_2\|^2 - 2 p_1 p_2 \operatorname{Re} \langle x_1, x_2 \rangle \\
&= p_1 p_2 \left(\|x_1\|^2 + \|x_2\|^2 - 2 \operatorname{Re} \langle x_1, x_2 \rangle \right) \\
&= p_1 p_2 \|x_1 - x_2\|^2
\end{aligned}$$

and

$$p_1 r_1^2 + p_2 r_2^2 = \left(p_1 p_2^2 + p_2 p_1^2 \right) \|x_1 - x_2\|^2 = p_1 p_2 \|x_1 - x_2\|^2,$$

hence (2.5) becomes

$$p_1 p_2 \|x_1 - x_2\|^2 \leq c p_1 p_2 \|x_1 - x_2\|^2,$$

implying that $c \geq 1$. This proves the sharpness of the inequality (2.2). ∎

The following reverse of the generalized triangle inequality also holds.

Theorem 6. *Let x_i, p_i and r_i, $i \in \{1, \dots, n\}$ be as in the statement of Theorem 5. Then*

$$(0 \leq) \sum_{i=1}^{n} p_i \|x_i\| - \left\| \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} p_i r_i^2}{\left\| \sum_{i=1}^{n} p_i x_i \right\|}, \tag{2.6}$$

provided that $\sum_{i=1}^{n} p_i x_i \neq 0$.

Proof. From (2.3) we obviously have

$$\|x_i\|^2 + \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \leq 2 \operatorname{Re} \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + r_i^2 \tag{2.7}$$

for any $i \in \{1, \dots, n\}$.

Utilising the elementary inequality $\alpha^2 + \beta^2 \geq 2\alpha\beta$, with $\alpha, \beta > 0$, we also have

$$2 \|x_i\| \left\| \sum_{j=1}^{n} p_j x_j \right\| \leq \|x_i\|^2 + \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \tag{2.8}$$

for any $i \in \{1, \dots, n\}$, which together with (2.7) produces

$$\|x_i\| \left\| \sum_{j=1}^{n} p_j x_j \right\| \leq \operatorname{Re} \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \frac{1}{2} r_i^2 \tag{2.9}$$

for any $i \in \{1, \dots, n\}$.

Now, if we multiply (2.9) by $p_i > 0$ and sum over i from 1 to n, we deduce

$$\sum_{i=1}^{n} p_i \|x_i\| \left\| \sum_{j=1}^{n} p_j x_j \right\| \leq \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} p_i r_i^2, \tag{2.10}$$

which, for $\sum_{i=1}^{n} p_i x_i \neq 0$, is equivalent to the desired result (2.6). ∎

Another result is as follows.

Theorem 7. *Let x_i, p_i and r_i be as in the statement of Theorem 5. Then*

$$0 \leq \left(\sum_{i=1}^{n} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} p_i r_i^2}{\left\| \sum_{i=1}^{n} p_i x_i \right\|}. \tag{2.11}$$

Proof. If we multiply the inequality (2.7) by $p_i \geq 0$ and sum over i from 1 to n, we deduce

$$\sum_{i=1}^{n} p_i \|x_i\|^2 + \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \leq 2 \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 + \sum_{i=1}^{n} p_i r_i^2 \tag{2.12}$$

for any $i \in \{1, \ldots, n\}$.

However,

$$2 \left(\sum_{i=1}^{n} p_i \|x_i\|^2 \right)^{\frac{1}{2}} \left\| \sum_{j=1}^{n} p_j x_j \right\| \leq \sum_{i=1}^{n} p_i \|x_i\|^2 + \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \tag{2.13}$$

and by (2.12) and (2.13), we deduce the desired result (2.11). ∎

Remark 1. *It is an open problem of whether the constant $\frac{1}{2}$ is best possible in (2.6) or in (2.11). Also, since*

$$\sum_{i=1}^{n} p_i \|x_i\| \leq \left(\sum_{i=1}^{n} p_i \|x_i\|^2 \right)^{\frac{1}{2}},$$

we observe that the inequality (2.11) is better than (2.6).

The following result also holds.

Theorem 8. *With the assumptions of Theorem 5 and if, in addition, we have*

$$(0 <) \max_{i=\overline{1,n}} \{r_i\} \leq \left\| \sum_{j=1}^{n} p_j x_j \right\| \quad \text{and} \quad x_i \neq 0 \quad \text{for all } i \in \{1, ..., n\},$$

then we obtain

$$\left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \leq \sum_{i=1}^{n} p_i r_i^2 + \mathrm{Re}^2 \left\langle \sum_{i=1}^{n} p_i \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} p_j x_j \right\rangle. \tag{2.14}$$

This implies the inequality

$$(0 \leq) \, 1 - \left\| \sum_{i=1}^{n} p_i \frac{x_i}{\|x_i\|} \right\|^2 \leq \frac{\sum_{i=1}^{n} p_i r_i^2}{\left\| \sum_{i=1}^{n} p_i x_i \right\|^2}. \tag{2.15}$$

Proof. From (2.3), we have

$$\|x_i\|^2 + \left[\left(\left\|\sum_{j=1}^{n} p_j x_j\right\|^2 - r_i^2\right)^{\frac{1}{2}}\right]^2 \leq 2\operatorname{Re}\left\langle x_i, \sum_{j=1}^{n} p_j x_j\right\rangle \tag{2.16}$$

for each $i \in \{1, \dots, n\}$ and since

$$2\|x_i\|\sqrt{\left\|\sum_{j=1}^{n} p_j x_j\right\|^2 - r_i^2} \leq \|x_i\|^2 + \left[\left(\left\|\sum_{j=1}^{n} p_j x_j\right\|^2 - r_i^2\right)^{\frac{1}{2}}\right]^2, \tag{2.17}$$

then by (2.16) and (2.17) we get

$$\|x_i\|\left(\left\|\sum_{j=1}^{n} p_j x_j\right\|^2 - r_i^2\right)^{\frac{1}{2}} \leq \operatorname{Re}\left\langle x_i, \sum_{j=1}^{n} p_j x_j\right\rangle \tag{2.18}$$

for each $i \in \{1, \dots, n\}$. If we multiply (2.14) by $\frac{p_i}{\|x_i\|} \geq 0$ and sum over i from 1 to n, we obtain

$$\sum_{i=1}^{n}\left(\left\|\sum_{j=1}^{n} p_j x_j\right\|^2 - r_i^2\right)^{\frac{1}{2}} p_i \leq \operatorname{Re}\left\langle \sum_{i=1}^{n} p_i \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} p_j x_j\right\rangle. \tag{2.19}$$

However,

$$\sum_{i=1}^{n} p_i\left(\left\|\sum_{j=1}^{n} p_j x_j\right\|^2 - r_i^2\right)^{\frac{1}{2}} \geq \left(\left\|\sum_{j=1}^{n} p_j x_j\right\|^2 - \sum_{i=1}^{n} p_i r_i^2\right)^{\frac{1}{2}} \tag{2.20}$$

and by (2.19) and (2.20), we then have

$$\left\|\sum_{j=1}^{n} p_j x_j\right\|^2 \leq \sum_{i=1}^{n} p_i r_i^2 + \operatorname{Re}^2\left\langle \sum_{i=1}^{n} p_i \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} p_j x_j\right\rangle,$$

which proves the desired inequality (2.14).

Now, on making use of the Schwarz inequality, we have

$$\operatorname{Re}^2\left\langle \sum_{i=1}^{n} p_i \frac{x_i}{\|x_i\|}, \sum_{j=1}^{n} p_j x_j\right\rangle \leq \left\|\sum_{i=1}^{n} p_i \frac{x_i}{\|x_i\|}\right\|^2 \left\|\sum_{j=1}^{n} p_j x_j\right\|^2,$$

which together with (2.14) produces the desired result (2.15). ∎

Theorem 9. *With the assumptions of Theorem 5 and if, in addition, we have*

$$(0 <) \max_{i=\overline{1,n}} \{r_i\} \leq \left\|\sum_{j=1}^{n} p_j x_j\right\|,$$

then

$$(0 \le) \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \le \frac{\sum_{i=1}^{n} p_i r_i^2 \|x_i\|^2}{\left\| \sum_{j=1}^{n} p_j x_j \right\|^2} \tag{2.21}$$

$$\left(\le \max_{i=\overline{1,n}} r_i^2 \frac{\sum_{i=1}^{n} p_i \|x_i\|^2}{\left\| \sum_{j=1}^{n} p_j x_j \right\|^2} \right).$$

Proof. From (2.18) we have that

$$\sum_{i=1}^{n} p_i \|x_i\| \left(\left\| \sum_{j=1}^{n} p_j x_j \right\|^2 - r_i^2 \right)^{\frac{1}{2}} \le \left\| \sum_{j=1}^{n} p_j x_j \right\|^2. \tag{2.22}$$

However,

$$\left[\sum_{i=1}^{n} p_i \|x_i\|^2 \left(\left\| \sum_{j=1}^{n} p_j x_j \right\|^2 - r_i^2 \right) \right]^{\frac{1}{2}} \tag{2.23}$$

$$\le \sum_{i=1}^{n} p_i \|x_i\| \left(\left\| \sum_{j=1}^{n} p_j x_j \right\|^2 - r_i^2 \right)^{\frac{1}{2}}$$

and since

$$\sum_{i=1}^{n} p_i \|x_i\|^2 \left(\left\| \sum_{j=1}^{n} p_j x_j \right\|^2 - r_i^2 \right) = \sum_{i=1}^{n} p_i \|x_i\|^2 \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 - \sum_{i=1}^{n} p_i \|x_i\|^2 r_i^2,$$

hence by (2.22) and (2.23) we obtain (2.21). ∎

3. Some Results Via Ostrowski's Inequality

In order to provide more convenient inequalities, the condition (2.1), namely

$$\left\| x_i - \sum_{j=1}^{n} p_j x_j \right\| \le r_i \quad \text{for all} \quad i \in \{1, \dots, n\}, \tag{3.1}$$

where $x_i \in H$, $p_i \ge 0$ with $i \in \{1, \dots, n\}$ and $\sum_{i=1}^{n} p_i = 1$ can be replaced with a number of discrete Ostrowski inequalities such as those obtained by Dragomir in [6].

For the sake of completeness, we recall them here in a single theorem.

Theorem 10 (Dragomir, 2002, [6]). *Let $(X, \|\cdot\|)$ be a normed linear space, $x_i \in X$ $(i \in \{1, \dots, n\})$ and $p_i \ge 0$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the* *inequalities:*

$$\left\|x_i-\sum_{j=1}^{n}p_jx_j\right\| \tag{3.2}$$

$$\leq \max_{k\in\{1,\dots,n-1\}}\|\Delta x_k\|\sum_{j=1}^{n}p_j\,|j-i|$$

$$\leq \max_{k\in\{1,\dots,n-1\}}\|\Delta x_k\|\times\begin{cases}\frac{n-1}{2}+\left|i-\frac{n+1}{2}\right|;\\[2ex] \left(\sum_{j=1}^{n}|j-i|^{p}\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n}p_j^{q}\right)^{\frac{1}{q}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\frac{n^2-1}{4}+\left(i-\frac{n+1}{2}\right)^2\right]\max\limits_{j\in\{1,\dots,n\}}\{p_j\}\end{cases}$$

and, for $\alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1$, *the inequalities:*

$$\left\|x_i-\sum_{j=1}^{n}p_jx_j\right\|\leq\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^{\alpha}\right)^{\frac{1}{\alpha}}\sum_{j=1}^{n}|i-j|^{\frac{1}{\beta}}\,p_j \tag{3.3}$$

$$\leq\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^{\alpha}\right)^{\frac{1}{\alpha}}\begin{cases}\left[\frac{n-1}{2}+\left|i-\frac{n+1}{2}\right|\right]^{\frac{1}{\beta}};\\[2ex] \left(\sum\limits_{j=1}^{n}|i-j|^{\frac{\delta}{\beta}}\right)^{\frac{1}{\delta}}\left(\sum\limits_{j=1}^{n}p_j^{\gamma}\right)^{\frac{1}{\gamma}},\\ \qquad\qquad \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1;\\[2ex] \sum\limits_{j=1}^{n}|i-j|^{\frac{1}{\beta}}\max\limits_{j\in\{1,\dots,n\}}\{p_j\}\end{cases}$$

and

$$\left\|x_i-\sum_{j=1}^{n}p_jx_j\right\|\leq\begin{cases}\max\{P_{i-1},1-P_i\}\sum\limits_{k=1}^{n-1}\|\Delta x_k\|\\[2ex] (1-p_i)\max\left\{\sum\limits_{k=1}^{i-1}\|\Delta x_k\|,\sum\limits_{k=i}^{n-1}\|\Delta x_k\|\right\}\end{cases} \tag{3.4}$$

$$\leq(1-p_i)\sum_{k=1}^{n-1}\|\Delta x_k\|$$

for each $i\in\{1,\dots,n\}$, *where* $\Delta x_k:=x_{k+1}-x_k$ *and* $P_m:=\sum_{i=1}^{m}p_i,\ k\in\{1,\dots,n-1\}$, $m\in\{1,\dots,n\}$ *and* $P_0:=0$.

Now, if we use Theorems 5 and 10 we can state the following result that provides upper bounds for the quantity

$$\sum_{i=1}^{n}p_i\|x_i\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2.$$

Proposition 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space, $x_i \in H$, $i \in \{1,\dots,n\}$ and $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$. Then we have the inequalities:*

$$\sum_{i=1}^n p_i \|x_i\|^2 - \left\|\sum_{i=1}^n p_i x_i\right\|^2 \tag{3.5}$$

$$\leq \max_{k\in\{1,\dots,n-1\}} \|\Delta x_k\|^2 \sum_{i=1}^n p_i \left(\sum_{j=1}^n p_j |j-i|\right)^2$$

$$\leq \max_{k\in\{1,\dots,n-1\}} \|\Delta x_k\|^2 \begin{cases} \sum_{i=1}^n p_i \left(\frac{n-1}{2} + \left|i - \frac{n+1}{2}\right|\right)^2; \\ \left(\sum_{j=1}^n p_j^q\right)^{\frac{2}{q}} \sum_{i=1}^n \left(\sum_{j=1}^n |j-i|^p\right)^{\frac{2}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max\limits_{j\in\{1,\dots,n\}} \left\{p_j^2\right\} \sum_{i=1}^n p_i \left[\frac{n^2-1}{4} + \left(i-\frac{n+1}{2}\right)^2\right]^2 \end{cases}$$

and, for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, the inequalities:

$$\sum_{i=1}^n p_i \|x_i\|^2 - \left\|\sum_{i=1}^n p_i x_i\right\|^2 \tag{3.6}$$

$$\leq \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^\alpha\right)^{\frac{2}{\alpha}} \sum_{i=1}^n p_i \left(\sum_{j=1}^n |i-j|^{\frac{1}{\beta}} p_j\right)^2$$

$$\leq \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^\alpha\right)^{\frac{2}{\alpha}} \begin{cases} \sum_{i=1}^n p_i \left[\frac{n-1}{2} + \left|i-\frac{n+1}{2}\right|\right]^{\frac{2}{\beta}}; \\ \left(\sum_{j=1}^n p_j^\gamma\right)^{\frac{2}{\gamma}} \sum_{i=1}^n p_i \left(\sum_{j=1}^n |i-j|^{\frac{\delta}{\beta}}\right)^{\frac{2}{\delta}}, \\ \qquad\qquad \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \max\limits_{j\in\{1,\dots,n\}} \left\{p_j^2\right\} \sum_{i=1}^n p_i \left(\sum_{j=1}^n |i-j|^{\frac{1}{\beta}}\right)^2 \end{cases}$$

and

$$\sum_{i=1}^n p_i \|x_i\|^2 - \left\|\sum_{i=1}^n p_i x_i\right\|^2 \tag{3.7}$$

$$\leq \begin{cases} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|\right)^2 \sum_{i=1}^n p_i \max\left\{P_{i-1}^2, (1-P_i)^2\right\} \\ \sum_{i=1}^n p_i (1-p_i)^2 \left(\max\left\{\sum_{k=1}^{i-1} \|\Delta x_k\|, \sum_{k=i}^{n-1} \|\Delta x_k\|\right\}\right)^2 \end{cases}$$

$$\leq \left(\sum_{k=1}^{n-1} \|\Delta x_k\|\right)^2 \sum_{i=1}^n p_i (1-p_i)^2.$$

Remark 2. *If one uses the other inequalities obtained in Theorems 6 – 9, then other similar results can be stated. However the details are not presented here.*

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