APPLICATIONS OF OSTROWSKI'S VERSION OF THE GRÜSS INEQUALITY FOR TRAPEZOID TYPE RULES

N.S. BARNETT AND S.S. DRAGOMIR

Abstract. Some applications of the Ostrowski inequality and a perturbed version of it for integral inequalities of the trapezoid type are given.

1. Introduction

In [1], A. Ostrowski proved the following inequality of the Grüss type,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{8}(b-a)(M-m)\left\|f'\right\|_{[a,b],\infty} \tag{1.1}$$

provided g is integrable on $[a,b]$ and satisfies the condition

$$-\infty<m\leq g(x)\leq M<\infty \text{ for a.e. } x\in[a,b]. \tag{1.2}$$

and f is absolutely continuous on $[a,b]$ with $f'\in L_\infty[a,b]$.

The constant $\frac{1}{8}$ is the best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

In this paper we present some applications of (1.1) as well as a perturbed version of it that can also be applied to create some useful integral inequalities.

2. Integral Inequalities

The following trapezoid type result for $n-$time differentiable functions has been obtained in [2].

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$. Then for any $x\in[a,b]$ one has the equality:*

$$\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right] + \frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt. \tag{2.1}$$

Some useful particular cases are as follows.

Date: May 25, 2002.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 41A55.

Key words and phrases. Ostrowski's inequality, Grüss type inequalities, Trapezoid type inequalities.

(1) For $n=1$, one can retrieve the identity (see for example [2])

$$\int_a^b f(t)\,dt = (x-a)\,f(a) + (b-x)\,f(b) + \int_a^b (x-t)\,f'(t)\,dt, \tag{2.2}$$

for each $x \in [a,b]$.

(2) For $n=2$, we have (see for example [2])

$$\begin{aligned}\int_a^b f(t)\,dt &= (x-a)\,f(a) + (b-x)\,f(b) \\ &\quad + \frac{1}{2}\left[(x-a)^2 f'(a) + (b-x)^2 f'(b)\right] + \frac{1}{2}\int_a^b (x-t)^2 f''(t)\,dt.\end{aligned} \tag{2.3}$$

If in (2.2) we choose $x = \frac{a+b}{2}$, then we get the trapezoid identity

$$\int_a^b f(t)\,dt = \frac{f(b)+f(a)}{2}(b-a) + \int_a^b \left(\frac{a+b}{2} - t\right) f'(t)\,dt, \tag{2.4}$$

while the same choice of x will produce, in (2.3), the following perturbed version of the trapezoid identity,

$$\begin{aligned}\int_a^b f(t)\,dt &= \frac{f(b)+f(a)}{2}(b-a) \\ &\quad + \frac{(b-a)^2}{8}\left[f'(a) - f'(b)\right] + \frac{1}{2}\int_a^b \left(t - \frac{a+b}{2}\right)^2 f''(t)\,dt.\end{aligned} \tag{2.5}$$

Consider now the following results.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$ and there exists the real numbers γ, Γ such that $-\infty < \gamma \leq f^{(n)}(x) \leq \Gamma < \infty$ for a.e. $x \in [a,b]$. Then we have the inequality:*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right. \\ &\qquad \left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!}\left[f^{(n-1)}; a, b\right]\right| \\ &\qquad \leq \frac{1}{8(n-1)!}(b-a)^2 (\Gamma - \gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{n-1}\end{aligned} \tag{2.6}$$

for any $x \in [a,b]$, where $[h; a, b] = \frac{h(b)-h(a)}{b-a}$ is the divided difference of h on $[a,b]$.

Proof. If we use Ostrowski's inequality (1.1) we may write

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^n\,dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right| \\ \leq\ & \frac{1}{8}(b-a)(\Gamma - \gamma)\, n \sup_{t \in [a,b]} |x-t|^{n-1} \\ =\ & \frac{n}{8}(b-a)(\Gamma - \gamma)\left[\max(x-a, b-x)\right]^{n-1} \\ =\ & \frac{n}{8}(b-a)(\Gamma - \gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{n-1}\end{aligned}$$

giving

$$\left|\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{n+1}\left[f^{(n-1)}; a, b\right]\right| \leq \frac{n}{8}(b-a)^2(\Gamma-\gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{n-1}. \tag{2.7}$$

If we divide this by $n!$ and use the representation (2.1), we obtain the desired inequality (2.6). ∎

Remark 1. *If $n = 1$ in (2.6), we deduce,*

$$\left|\int_a^b f(t)\,dt - (x-a) f(a) - (b-x) f(b) - (b-a)\left(x - \frac{a+b}{2}\right)[f; a, b]\right| \leq \frac{1}{8}(b-a)^2(\Gamma-\gamma),$$

which is clearly equivalent to the trapezoid inequality

$$\left|\int_a^b f(t)\,dt - \frac{f(b)+f(a)}{2}(b-a)\right| \leq \frac{1}{8}(b-a)^2(\Gamma-\gamma). \tag{2.8}$$

It has been shown in various papers that $\frac{1}{8}$ is a sharp constant (see [4]*, and* [3]*).*

Remark 2. *Further work has yet to be done on comparing, for any $x \in [a, b]$, the bounds provided by (2.6) and the bound*

$$\frac{\Gamma-\gamma}{2} \cdot \frac{1}{n!} I(x, n),$$

where

$$I(x, n) := \frac{1}{(n+1)\sqrt{2n+1}}\left\{n^2 (b-a)\left[(x-a)^{2n+1} + (b-x)^{2n+1}\right] + (2n+1)(x-a)(b-x)\left[(x-a)^n + (b-x)^n\right]^2\right\}^{\frac{1}{2}}$$

has been obtained in [2]*.*

The Ostrowski inequality (1.1) can also be applied in the following way.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \in L_\infty [a, b]$. Then we have the inequality:*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!}\left[f^{(n-1)}; a, b\right]\right| \tag{2.9}$$

$$\leq \begin{cases} \frac{1}{8n!}(b-a)^2\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^n \left\|f^{(n+1)}\right\|_{[a,b],\infty} & \text{if } n \text{ is even,} \\ \frac{1}{8n!}(b-a)^2\left[(x-a)^n + (b-x)^n\right]\left\|f^{(n+1)}\right\|_{[a,b],\infty} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For $n=2k$, consider $h_{2k}(t)=(x-t)^{2k}$. It is obvious that

$$\inf_{t\in[a,b]} h_{2k}(t)=0,$$

$$\begin{aligned}\sup_{t\in[a,b]} h_{2k}(t) &= \max\left[(x-a)^{2k},(b-x)^{2k}\right]=\left[\max(x-a,b-x)\right]^{2k}\\ &= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{2k}.\end{aligned}$$

If we now apply Ostrowski's inequality (1.1) for h_{2k} and $f^{(2k)}$, we deduce

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-t)^{2k} f^{(2k)}(t)\,dt-\frac{1}{b-a}\int_a^b (x-t)^{2k}\,dt\cdot\frac{1}{b-a}\int_a^b f^{(2k)}(t)\,dt\right|\\ &\qquad\leq \frac{1}{8}(b-a)\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{2k}\left\|f^{(2k+1)}\right\|_{[a,b],\infty},\end{aligned}$$

by a similar argument to that in Theorem 1, proving the first part of (2.9).

For $n=2k+1$, consider $h_{2k+1}(t)=(x-t)^{2k+1}$. Then

$$h_{2k+1}'(t)=-(2k+1)(x-t)^{2k},$$

showing that h_{2k+1} is decreasing on $[a,b]$, and thus

$$\inf_{t\in[a,b]} h_{2k+1}(t)=h_{2k+1}(b)=(x-b)^{2k+1}=-(b-x)^{2k+1}$$

and

$$\sup_{t\in[a,b]} h_{2k+1}(t)=h_{2k+1}(a)=(x-a)^{2k+1}.$$

Now apply Ostrowski's inequality (1.1) for h_{2k+1} and $f^{(2k+1)}$ we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-t)^{2k+1} f^{(2k+1)}(t)\,dt\right.\\ &\qquad\left.-\frac{1}{b-a}\int_a^b (x-t)^{2k+1}\,dt\cdot\frac{1}{b-a}\int_a^b f^{(2k+1)}(t)\,dt\right|\\ &\qquad\qquad\leq \frac{1}{8}(b-a)\left[(x-a)^{2k+1}+(b-x)^{2k+1}\right]\left\|f^{(2k+1)}\right\|_{[a,b],\infty},\end{aligned}$$

giving, by a similar procedure to that in Theorem 1, the second part of (2.9). ∎

3. A Perturbed Version

The following result holds.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ so that the derivative $f':[a,b]\to\mathbb{R}$ satisfies the condition:*

$$-\infty<\gamma\leq f'(x)\leq\Gamma<\infty \text{ for a.e. } x\in[a,b]. \tag{3.1}$$

If $g:[a,b]\to\mathbb{R}$ is such that

$$-\infty<m\leq g(x)\leq M<\infty \text{ for a.e. } x\in[a,b], \tag{3.2}$$

then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right.$$
$$\left.-\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx\right|\leq\frac{1}{16}(b-a)(M-m)(\Gamma-\gamma). \tag{3.3}$$

The constant $\frac{1}{16}$ *is best possible.*

Proof. We know that

$$\left|\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|$$
$$\leq\frac{1}{8}(b-a)(M-m)\|h'\|_{[a,b],\infty}, \tag{3.4}$$

provided $-\infty<m\leq g(x)\leq M<\infty$ for a.e. $x\in[a,b]$.

Choose $h(x)=f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)$, $\alpha\in\mathbb{R}$. Then

$$h'(x)=f'(x)-\frac{\gamma+\Gamma}{2}$$

and since

$$|h'(x)|\leq\frac{\Gamma-\gamma}{2},$$

for a.e. $x\in[a,b]$, we have

$$\left|\frac{1}{b-a}\int_a^b\left[f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)\right]g(x)\,dx\right.$$
$$\left.-\frac{1}{b-a}\int_a^b\left[f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)\right]dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|$$
$$\leq\frac{1}{16}(b-a)(M-m)(\Gamma-\gamma). \tag{3.5}$$

However,

$$\frac{1}{b-a}\int_a^b\left[f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)\right]g(x)\,dx$$
$$=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b(x-\alpha)\,g(x)\,dx$$

and

$$\frac{1}{b-a}\int_a^b\left[f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)\right]dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx$$
$$=\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx$$
$$-\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b(x-\alpha)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx.$$

By (3.5) we deduce,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b (x-\alpha)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx + \frac{\gamma+\Gamma}{2} \times \frac{1}{b-a}\int_a^b (x-\alpha)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{16}(b-a)(M-m)(\Gamma-\gamma). \tag{3.6}$$

Now, observe that

$$\begin{aligned}
&\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b (x-\alpha)\,g(x)\,dx \\
&\qquad - \frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b (x-\alpha)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx \\
&= \frac{\gamma+\Gamma}{2}\cdot\frac{1}{(b-a)^2}\left\{(b-a)\left[\int_a^b xg(x)\,dx - \alpha\int_a^b g(x)\,dx\right.\right. \\
&\qquad \left.\left. - \left[\int_a^b xdx - \alpha(b-a)\right]\int_a^b g(x)\,dx\right]\right\} \\
&= \frac{\gamma+\Gamma}{2}\cdot\frac{1}{(b-a)^2}\left[(b-a)\int_a^b xg(x)\,dx - \int_a^b xdx\cdot\int_a^b g(x)\,dx\right] \\
&= \frac{\gamma+\Gamma}{2}\left[\frac{1}{b-a}\int_a^b xg(x)\,dx - \frac{a+b}{2}\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right]
\end{aligned}$$

and by (3.6) we deduce the desired result.

The fact that $\frac{1}{16}$ is the best constant, follows by Ostrowski's result on choosing $\gamma = -\|f'\|_{[a,b],\infty}$, $\Gamma = \|f'\|_{[a,b],\infty}$. We omit the details. ∎

In what follows, an application of the above perturbed version of Ostrowski's inequality is given.

Using Lemma 1, we have the identity, (see also [2])

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\frac{1}{(k+1)!2^{k+1}}(b-a)^{k+1}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right] + \frac{1}{n!}\int_a^b\left(\frac{a+b}{2}-t\right)^n f^{(n)}(t)\,dt. \tag{3.7}$$

We can further state the following result.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$ and there exists the real numbers γ,Γ such* *that* $-\infty < \gamma \le f^{(n)}(x) \le \Gamma < \infty$ *for a.e.,* $x \in [a,b]$. *Then we have the inequality:*

$$
\begin{aligned}
(3.8)\quad & \Bigg| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \\
& - \frac{(b-a)^{n+1}}{2^n (n+1)!} \left[f^{(n-1)}; b, a \right] - \frac{(b-a)^{n+1}}{2^{n+1} n!} \cdot \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} \\
& + \frac{(b-a)^{n+1}}{2^{n+1} n!} \left[f^{(n-2)}; b, a \right] \Bigg| \\
& \le \begin{cases} \dfrac{(b-a)^{n+2}}{2^{n+5} n!} (\Gamma - \gamma) & \textit{if } n \textit{ is even,} \\[2ex] \dfrac{(b-a)^{n+2}}{2^{n+3} n!} (\Gamma - \gamma) & \textit{if } n \textit{ is odd.} \end{cases}
\end{aligned}
$$

Proof. The proof is by application of Theorem 3 for $g \to f^{(n)}$ and $f \to \left(\frac{a+b}{2} - \cdot\right)^n$.
We first observe that

$$
\int_a^b \left(\frac{a+b}{2} - t \right)^n dt = \frac{(b-a)^{n+1} \left[1 + (-1)^n \right]}{2^{n+1} (n+1)},
$$

$$
\frac{1}{b-a} \int_a^b f^{(n)}(t)\,dt = \left[f^{(n-1)}; b, a \right],
$$

$$
\gamma = \inf_{t \in [a,b]} \left(\frac{a+b}{2} - t \right)^n = \begin{cases} 0 & \text{if } n \text{ is even,} \\[2ex] \dfrac{(a-b)^n}{2^n} & \text{if } n \text{ is odd,} \end{cases}
$$

$$
\Gamma = \sup_{t \in [a,b]} \left(\frac{a+b}{2} - t \right)^n = \begin{cases} \dfrac{(b-a)^n}{2^n} & \text{if } n \text{ is even,} \\[2ex] \dfrac{(b-a)^n}{2^n} & \text{if } n \text{ is odd,} \end{cases}
$$

and then

$$
\frac{\gamma + \Gamma}{2} = \begin{cases} \dfrac{(b-a)^n}{2^{n+1}} & \text{if } n \text{ is even,} \\[2ex] 0 & \text{if } n \text{ is odd.} \end{cases}
$$

Also,

$$
\begin{aligned}
& \int_a^b \left(\frac{a+b}{2} - t \right) f^{(n)}(t)\,dt \\
& = f^{(n-1)}(t) \left(\frac{a+b}{2} - t \right) \Bigg|_a^b + \int_a^b f^{(n-1)}(t)\,dt \\
& = -(b-a) \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} + \left[f^{(n-2)}; b, a \right] (b-a).
\end{aligned}
$$

Consequently, when n is even, we have

$$\begin{aligned}&\left|\int_a^b\left(\frac{a+b}{2}-t\right)^n f^{(n)}(t)\,dt-\int_a^b\left(\frac{a+b}{2}-t\right)^n dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right.\\&\left.-\frac{(b-a)^n}{2^{n+1}}\left[(b-a)\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2}-\left[f^{(n-2)};b,a\right](b-a)\right]\right|\\&\leq\frac{1}{16}(b-a)^2\frac{(b-a)^n}{2^{n+1}}(\Gamma-\gamma),\end{aligned}$$

i.e.,

$$\begin{aligned}&\left|\int_a^b\left(\frac{a+b}{2}-t\right)^n f^{(n)}(t)\,dt-\frac{(b-a)^{n+1}}{2^n(n+1)}\left[f^{(n-1)};b,a\right]\right.\\&\left.-\frac{(b-a)^{n+1}}{2^{n+1}}\cdot\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2}+\frac{(b-a)^{n+1}}{2^{n+1}}\left[f^{(n-2)};b,a\right]\right|\\&\leq\frac{(b-a)^{n+2}}{2^{n+5}}(\Gamma-\gamma)\end{aligned}$$

from which we we obtain the first branch in (3.8).

When n is odd, we have

$$\begin{aligned}\left|\int_a^b\left(\frac{a+b}{2}-t\right)^n f^{(n)}(t)\,dt\right|&\leq\frac{1}{16}(b-a)^2(\Gamma-\gamma)\frac{(b-a)^n}{2^{n-1}}\\&=\frac{(b-a)^{n+2}(\Gamma-\gamma)}{2^{n+3}}\end{aligned}$$

from where we get the second branch of (3.8).

The theorem is thus proved. ∎

The second approach is incorporated in the following theorem.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $g^{(n)}$ is absolutely continuous on $f^{(n+1)}\in L_\infty[a,b]$ and assume that there exist constants $\phi,\Phi\in\mathbb{R}$ such that $-\infty<\phi\leq f^{(n+1)}(x)\leq\Phi<\infty$ for a.e. $x\in[a,b]$. Then we have the inequality*

$$\begin{aligned}(3.9)\quad&\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right.\\&\left.\times\begin{cases}-\dfrac{(b-a)^{n+1}}{2^n(n+1)!}\left[f^{(n-1)};b,a\right]\\+\dfrac{\phi+\Phi}{2}\cdot\dfrac{(b-a)^{n+2}}{2^{n+1}(n+2)n!}\end{cases}\right|\\&\leq\begin{cases}\dfrac{(b-a)^{n+2}}{2^{n+2}n!}(\Phi-\phi)&\text{if } n \text{ is even,}\\[2ex]\dfrac{(b-a)^{n+2}}{2^{n+3}n!}(\Phi-\phi)&\text{if } n \text{ is odd.}\end{cases}\end{aligned}$$

Proof. We apply Theorem 3 for the choices $g \to \left(\frac{a+b}{2} - \cdot\right)^n$ and $f \to f^{(n)}$.

We observe that

$$\int_a^b \left(t - \frac{a+b}{2}\right)\left(\frac{a+b}{2} - t\right)^n dt = -\int_a^b \left(\frac{a+b}{2} - t\right)^{n+1} dt$$
$$= \frac{(b-a)^{n+2}\left[(-1)^{n+1} + 1\right]}{2^{n+2}(n+2)}.$$

Then by (3.3) we get

$$\left| \int_a^b \left(\frac{a+b}{2} - t\right)^n f^{(n)}(t)\,dt - \int_a^b \left(\frac{a+b}{2} - t\right)^n dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt \right.$$
$$\left. - \frac{\phi + \Phi}{2}\int_a^b \left(t - \frac{a+b}{2}\right)\left(\frac{a+b}{2} - t\right)^n dt \right|$$
$$\leq \frac{1}{16}(b-a)^2(\Phi - \phi)\begin{cases} \dfrac{(b-a)^n}{2^n} & \text{if } n \text{ is even,} \\ \dfrac{(b-a)^n}{2^{n-1}} & \text{if } n \text{ is odd,} \end{cases}$$

that is,

$$\left| \int_a^b \left(\frac{a+b}{2} - t\right)^n f^{(n)}(t)\,dt - \frac{(b-a)^{n+1}\left[1 + (-1)^n\right]}{2^{n+1}(n+1)}\left[f^{(n-1)}; b, a\right] \right.$$
$$\left. + \frac{\phi + \Phi}{2} \cdot \frac{(b-a)^{n+2}\left[(-1)^{n+1} + 1\right]}{2^{n+2}(n+2)} \right| \leq \begin{cases} \dfrac{(b-a)^n}{2^{n+4}}(\Phi - \phi) & \text{if } n \text{ is even,} \\ \dfrac{(b-a)^n}{2^{n+3}}(\Phi - \phi) & \text{if } n \text{ is odd,} \end{cases}$$

and the inequality (3.9) is proved. ∎

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, MCMC 8001, Victoria, Australia.

E-mail address: `neil@matilda.vu.edu.au`

URL: `http://`

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`