Ky Fan Inequality and Bounds for Differences of Means II

KY FAN INEQUALITY AND BOUNDS FOR DIFFERENCES OF MEANS II

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Abstract. We study properties of Ky-Fan typed inequalities and their relations to certain bounds for the differences of means.

1. Introduction

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n}\omega_i x_i^r)^{\frac{1}{r}}$, where $\omega_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^{n}\omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Here $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \to 0^+$. Unless specified, we always assume $0 \leq x_1 \leq x_2 \cdots \leq x_n, m = \min\{x_i\}, M = \max\{x_i\}$, $r > s$, $q \in [0,1]$. We denote $\sigma_n = \sum_{i=1}^{n}\omega_i(x_i - A_n)^2$.

To any given $\mathbf{x}, t, \epsilon \geq 0$ we associate $\mathbf{x}(\epsilon) = (1-\epsilon x_1, 1-\epsilon x_2, \cdots, 1-\epsilon x_n), \mathbf{x}_t = (x_1+t, \cdots, x_n+t)$ and $\mathbf{x}' = \mathbf{x}(1)$. When there is no risk of confusion, We shall write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$, $P_{n,r,t}$ for $P_{n,r}(\mathbf{x}_t)$, $P^{'}_{n,r}$ for $P_{n,r}(\mathbf{x}')$ and $P_{n,r}(\epsilon)$ for $P_{n,r}(\mathbf{x}(\epsilon))$ if $1 - \epsilon x_i \geq 0$ for all i. We also define $A_n = P_{n,1}, G_n = P_{n,0}, H_n = P_{n,-1}$ and similarly for $A_{n,t}, G_{n,t}, H_{n,t}, A_n(\epsilon), G_n(\epsilon), H_n(\epsilon)$, $A'_n, G^{'}_n, H^{'}_n$.

Let $m > 0$, we consider the following bounds for the differences between power means:

$$\frac{r-s}{2m}\sigma_n \geq P_{n,r} - P_{n,s} \geq \frac{r-s}{2M}\sigma_n \tag{1.1}$$

We will refer to (1.1) as $I_{r,s,n}$. D.Cartwright and M.Field[6] first proved the validity of $I_{1,0,n}$, a refinement of the arithmetic-geometric mean inequality. For other extensions and refinements of (1.1), see [3], [8] and [10]. We note the constant $(r-s)/2$ is best possible.

There is a close relation between (1.1) and the following additive Ky Fan's inequality($m < M < 1$):

$$\frac{m}{1-m} < \frac{P'_{n,r} - P'_{n,s}}{P_{n,r} - P_{n,s}} < \frac{M}{1-M} \tag{1.2}$$

The case $r = 1, s = 0, x_n \leq 1/2$ in (1.2) is due to H. Alzer[4]. P. Mercer[12] showed Alzer's result follows from the result of Cartwright and Field. Recently, the author[9] showed that (1.1) and (1.2) are equivalent. We refer the reader to the survey article[2] and the references therein for an account of Ky Fan's inequality.

It is an open problem to determine all the pairs (r, s) so that $I_{r,s,n}$ is valid for all n. It is natural to try to reduce the problem to the case $n = 2$. In this paper, we will reduce the problem to the case $n = 3$ and in many situations $n = 2$.

A counterpart of (1.2) is the following result of J.Aczél and Zs. Pâles[1]("$\geq$" for $s < 1$,"$\leq$" for $s > 1$):

$$P_{n,s,t} - A_{n,t} \geq (\leq) P_{n,s} - A_n \tag{1.3}$$

J.Brenner and B. Carlson[5] studied the asymptotic behavior of $t(P_{n,r,t} - A_{n,t})$ and showed it is bounded. As an analogue to the relation between (1.1) and (1.2), we shall establish an equivalent relation between (1.1) and inequalities similar to (1.3) in this paper. We will also discuss certain asymptotic behaviors of the differences of means.

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2. The Main Theorem

Lemma 2.1. *If $I_{r,s,n}$ holds, then $0 \leq r+s \leq 3$ and $q^{1/r} - q^{1/s} \geq (r-s)q(1-q)/2$ for $s>0$, $q^{1/r} \geq (r-s)q(1-q)/2$ for $s \leq 0$. In particular, $r \geq 1, rs \leq 2$ and r is bounded above.*

Proof. The condition $0 \leq r+s \leq 3$ was proved in [9]. Now let $n=2$, write $\omega_1 = 1-q$, $\omega_2 = q$, $x_1 = 1$ and $x_2 = 1+t$ with $t \geq -1$. Let

$$D(t;r,s,q) = \frac{r-s}{2}\sigma_2 - P_{2,r} + P_{2,s} \tag{2.1}$$

For $t \geq 0$, $D(t;r,s,q) \geq 0$ implies the validity of the left-hand side inequality of (1.1) while for $-1 \leq t \leq 0$, $D(t;r,s,q) \leq 0$ implies the validity of the right-hand side inequality of (1.1).

By taking $t \to -1$ in (2.1) we get $q^{1/r} - q^{1/s} \geq (r-s)q(1-q)/2$ for $s>0$ and $q^{1/r} \geq (r-s)q(1-q)/2$ for $s \leq 0$. By letting $q \to 0$ in each case implies $r \geq 1$(note $r>0$ because of $r+s \geq 0$). By letting $q \to 1$ when $s>0$ gives $rs \leq 2$. Furthermore, $r+s \leq 3$ implies $r-s \geq 2r-3$. This then implies for $s \leq 0$, $q^{1/r} \geq (2r-3)q(1-q)/2$, which fails to hold when $r \to \infty$ for a fixed $q \neq 0$. Thus r is bounded above and this completes the lemma. $\square$

Lemma 2.1 allows us to focus on $0 \leq r+s \leq 3, r \geq 1$ when considering the validities of (1.1).

Lemma 2.2. *For $2 > r > s > 1, 0 \leq x_i \leq 1$*

$$P_{n,r}^{1-r} \geq P_{n,s}^{1-s} \tag{2.2}$$

Proof. This is equivalent to

$$(1-\frac{1}{r})\ln(\sum_{i=1}^{n}\omega_i x_i^r)^{-1} \geq (1-\frac{1}{s})\ln(\sum_{i=1}^{n}\omega_i x_i^s)^{-1}$$

and the above inequality holds since $1-\frac{1}{r} > 1-\frac{1}{s} > 0$ and $(\sum_{i=1}^n \omega_i x_i^r)^{-1} \geq (\sum_{i=1}^n \omega_i x_i^s)^{-1} \geq 1$. $\square$

Lemma 2.3. *$I_{r,s,n}$ holds for all n if and only if $I_{r,s,3}$ holds.*

Proof. We will prove the lemma for the left-hand side inequality of (1.1) and the proof for the right-hand side inequality of (1.1) is similar. We may assume $x_1 = 1 < x_n = b$,$x_i \in (1,b)$ and define two functions($\omega = (\omega_1, \omega_2, \cdots, \omega_n)$):

$$\begin{aligned} f(\omega, \mathbf{x}) &= x_1(P_{n,r} - P_{n,s}) - \frac{r-s}{2}\sigma_n \\ g(x) &= x_1(P_{n,r}^{1-r}x^r/r - P_{n,s}^{1-s}x^s/s) - (r-s)(x^2 - 2xA_n)/2 - \lambda \end{aligned}$$

where we define $x^0/0$ to be $\ln x$. Note here in the definition of $g(x)$, $P_{n,r}, P_{n,s}, A_n$ are not functions of x, they take values at some points $(\omega, \mathbf{x})$ to be specified and λ is also a constant to be specified.

We prove the lemma by induction on n. It suffices to show $f(\omega, \mathbf{x}) \leq 0$ on the region $R_n \times S_{n-2}$, where $R_n = \{(\omega_1, \omega_2, \cdots, w_n) : 0 \leq w_k \leq 1, 1 \leq k \leq n, \sum_{k=1}^n w_k = 1\}$ and $S_{n-2} = \{(x_2, \cdots, x_{n-1}) : x_k \in [1,b], 2 \leq k \leq n-1\}$. It suffices to show f takes its minimal value at $n \leq 3$. The base case of $n \leq 3$ is clear. Now assume $n \geq 4$.

There is a point $(\omega', \mathbf{x}')$ of $R_n \times S_{n-2}$ where f is minimized subject to the constraint $\sum_{k=1}^n \omega_k = 1$. If $x_i' = x_{i+1}'$ for some $1 \leq i \leq n-1$, by combining x_i' with x_{i+1}' and ω_i' with ω_{i+1}', we are back to the case of $n-1$ variables with different weights. Similarly, if $\omega_i' = 1$ for some i then we are back to the case $n=1$. If $\omega_i' = 0$ for some $i > 1$, we are back to the case $n-1$. If $\omega_1' = 0$, since

$$x_2'(P_{n-1,r} - P_{n-1,s}) - \frac{r-s}{2}\sigma_{n-1} \geq x_1'(P_{n,r} - P_{n,s}) - \frac{r-s}{2}\sigma_n$$

we are again back to the case $n-1$. So without loss of generality, from now on we may assume for $1 \leq i,j \leq n, i \neq j$, $\omega_i \neq 0, 1, x_i \neq x_j$ and this implies $(\omega', \mathbf{x}')$ is an interior point of $R_n \times S_{n-2}$.

Thus we may use the Lagrange multiplier method to obtain a real number λ so that at $(\omega', \mathbf{x}')$:

$$\frac{\partial f}{\partial w_i} = \lambda \frac{\partial}{\partial \omega_i}(\sum_{k=1}^{n} w_k - 1), \ \frac{1}{\omega_j}\frac{\partial f}{\partial x_j} = 0 \tag{2.3}$$

for all $1 \leq i \leq n$ and $2 \leq j \leq n-1$.

By (2.3), a computation shows each x'_k $(2 \leq k \leq n-1)$ is a common root of the equations $g(x)$ and $g'(x)$(where $P_{n,r}$ takes its value at $(\omega', \mathbf{x}')$, etc.). Now $n \geq 4$ implies $g(x)$ and $g'(x)$ have in common at least two distinct, positive roots,$1 < x'_2, x'_3 < b$. Moreover, $g(1) = g(b) = 0$ by (2.3) and it follows from Rolle's Theorem that there must be at least five positive roots of $g'(x)$ and thus at least three positive roots of $g'''(x)$. But

$$g'''(x) = (r-1)(r-2)P_{n,r}^{1-r}x^{r-3} - (s-1)(s-2)P_{n,s}^{1-s}x^{s-3}$$

has at most one positive root and this contradiction implies the lemma. □

Theorem 2.1. *For $r \geq 1, 3 \geq r+s \geq 0$, except possibly for the left-hand side inequality of* (1.1) *when $r > 2$,* (1.1) *holds for all n if and only if it holds for $n = 2$.*

Proof. To facilitate the discussion, we will assume without loss of generality that for the left-hand side inequality of (1.1), $x_1 = 1$ and $x_i \geq 1, 2 \leq i \leq n$. For the right-hand side of (1.1), $x_1 = 1$ and $0 \leq x_i \leq 1, 2 \leq i \leq n$. The functions f, g used in the proof are defined as in Lemma 2.3.

We consider the right-hand side inequality of (1.1) first. Here we write $\mathbf{x}_{m,t} = (1, \cdots, x_m, t, \cdots, t)$ for some $m \leq n-1$, $x_i > 0, 1 \leq i \leq m$ and define $f(\omega, \mathbf{x}_{m,0})$ to be $\lim_{t \to 0} f(\omega, \mathbf{x}_{m,t})$. One checks that this makes f continuous on the boundary of $R_n \times S'_{n-1}$ and differentiable in the interior part of $R_n \times S'_{n-1}$, where R_n is defined as in the proof of Lemma 2.3 and $S'_{n-1} = \{(x_2, \cdots, x_n) : x_k \in [0,1], 2 \leq k \leq n\}$.

As in the proof of Lemma 2.3, we may assume for $1 \leq i, j \leq n, i \neq j$, $\omega_i \neq 0, 1, x_i \neq x_j$. It also follows from the proof of Lemma 2.3 and the definition of $f(\omega, \mathbf{x}_{m,0})$ that it suffices to show $f \geq 0$ with $n = 3, x_3 = 0$. If $s \leq 0$, this becomes $P_{3,r} - \frac{r-s}{2}\sigma_3 \geq 0$ and one using the method in the proof of Lemma 2.3 can easily show that $P_{n,r} - \frac{r-s}{2}\sigma_n \geq 0$ holds if and only if it holds for $n = 2$. Since $P_{2,r} - \frac{r-s}{2} \geq P_{2,r} - P_{2,s} - \frac{r-s}{2}$, this shows that we only need to check $I_{r,s,2}$.

Now we let $s > 0$. If $1 \leq r \leq 2, s \leq 1$, $g'''(x)$ has no positive root and by similar arguments as in the proof of Lemma 2.3, it suffices to check $I_{r,s,2}$. For $2 < r \leq 3, s < 1$, it suffices to check $I_{r,s,3}$ with $x_3 = 0$ by similar reasons as above. We allow $x_1 \leq 1$ here and f is defined on $R_3 \times [0,1]^3$. Suppose f takes its absolute minimum at $1 \geq x_1 > x_2 > x_3 = 0$, one checks in this case $\lim_{x \to 0} g'(x) = -\infty$. Also from the proof of Lemma 2.3, we know $g'(x)$ has exactly three positive roots, all less than x_1. If this really is the case then we can deduce that $g'(x_1) > 0$ and

$$\frac{\partial f}{\partial x_1} = g'(x_1) + P_{n,r} - P_{n,s} > 0$$

which implies by decreasing x_1 while fixing $x_3 = 0$ and x_2, we can decrease the value of f, a contradiction. Thus it also suffices to check $I_{r,s,2}$ in this case.

Similarly, when $1 < s < r < 2, n = 3$, $g''(x)$ has exactly two positive roots, both less than 1, since $\lim_{x \to 0} g''(x) < 0$, we must have $g''(1) < 0$, however by lemma 2.2,

$$g''(1) = (r-1)(P_{n,r}^{1-r} - 1) - (s-1)(P_{n,s}^{1-s} - 1) \geq 0$$

a contradiction, so it also suffices to check $I_{r,s,2}$ in this case.

Now for the left-hand side inequality of (1.1), we may also assume for $1 \leq i, j \leq n, i \neq j$, $\omega_i \neq 0, 1, x_i \neq x_j$. In this case, it suffices to show $f \leq 0$.

If $1 \leq r \leq 2, s \leq 1$, $g'''(x)$ has no positive root and it suffices to check $I_{r,s,2}$. If $1 < s < r < 2$, we allow $1 \leq x_1 \leq x_2 \leq x_3 \leq b$ in this case and f is defined on $R_3 \times [1,b]^3$. Suppose f takes its absolute minimum at $1 \leq x_1 < x_2 < x_3 = b$, one checks in this case $g'(x)$ has exactly three positive roots, all lie in (x_1, b). Since $\lim_{x\to\infty} g'(x) < 0$, if $g'(b) > 0$, $g'(x)$ will have another root in $(b, +\infty)$, contradicting the fact $g'''(x)$ only has one positive root. Thus $g'(b) < 0$, which implies $g'(x_1) > 0$. But

$$\frac{\partial f}{\partial x_1} = g'(x_1) + P_{n,r} - P_{n,s} > 0$$

which implies by increasing x_1 while fixing x_2, x_3, we can increase the value of f, a contradiction. Thus it also suffices to check $I_{r,s,2}$ in this case. □

Corollary 2.1. *Let $0 \le r + s \le 3$. If $1 \le r \le 2, s \le 1$, $I_{r,s,n}$ holds. For $r > 2, 1/2 \le s < 1$, the right-hand side inequality of (1.1) holds. In all the cases the equality holds if and only if $x_1 = x_2 = \cdots = x_n$.*

Proof. The case $1 \le r \le 2, s \le 1$ is a result of the author[9], we leave the proof to the reader since it is similar to the one we will give below. We note the condition for equality is given here while it was missing in [9].

For the case $r > 2, 1/2 \le s < 1$, it suffices to prove the case $n = 2$ by Theorem 2.1. Let $x_1 = x, x_2 = 1, \omega_1 = q, \omega_2 = 1 - q$ and we rewrite $I_{r,s,2}$ as

$$F(x) = (qx^r + 1 - q)^{1/r} - (qx^s + 1 - q)^{1/s} - \frac{(r-s)q(1-q)}{2}(x-1)^2$$

We need to show $F(x) \ge 0$ for $0 \le x \le 1$. We may also assume $q \ne 0, 1$ and calculation yields

$$(q(1-q))^{-1}F''(x) = (r-1)(qx^r + 1 - q)^{\frac{1-2r}{r}}x^{r-2} + (1-s)(qx^s + 1 - q)^{\frac{1-2s}{s}}x^{s-2} - (r-s) \tag{2.4}$$

We define $H(q)$ to be the right-hand side expression in (2.4). Now for $r > 2, 1/2 \le s < 1$,

$$\begin{aligned} H'(q) &= (r-1)(1-2r)(q+(1-q)x^{-r})^{\frac{1-3r}{r}}x^{-r-1}\frac{1-x^{-r}}{r} \\ &\quad -(s-1)(1-2s)(q+(1-q)x^{-s})^{\frac{1-3s}{s}}x^{-s-1}\frac{1-x^{-s}}{s} \ge 0 \end{aligned}$$

Thus $H(q) \ge H(0) = (r-1)x^{r-2} - (s-1)x^{s-2} - (r-s) := h(x)$. By setting $h'(x) = 0$ we get $x^{r-s} = \frac{(1-s)(2-s)}{(1-r)(2-r)} \ge 1$, since the last inequality is equivalent to $(r+s-3)(r-s) \le 0$. Thus $h(x) \ge \min\{h(1), \lim_{x\to 0} h(0)\} = 0$.

Thus $F''(x) \ge 0$ for $x \le 1$ and $F(x) \ge 0$ for $x \in [0,1]$ follows by considering the Taylor expansion of $F(x)$ at $x = 1$ with the observation $F(1) = F'(1) = 0$. It is also easy to see that the equality holds if and only if $x_1 = \cdots = x_n$ and this completes the proof. □

3. Relations Among Ky-Fan typed Inequalities

Let $n \ge 2, 0 < x_1 < x_n$, consider the following inequalities:

$$D(x_1)\sigma_n > P_{n,r} - P_{n,s} > D(x_n)\sigma_n \tag{3.1}$$

where

$$D(t) = \frac{(P_{n,r}/t)^{1-r} - (P_{n,s}/t)^{1-s}}{2(t - A_n)}$$

The case $r = 1, s = 0$ in (3.1) is a result of A. Mercer[11] and the author[8] proved (3.1) for $r = 1, 0 < s < 1$. It is easy to see that in those cases (3.1) refine (1.1).

We note here $D(x_1), D(x_n) \ge 0$ but $D(x_1) \ge D(x_n)$ does not hold in general. For example, when $r > s > 1$, the choice $n = 2, x_1 = 0, x_2 = 1, \omega_1 \ne 0, 1$ leads to $D(x_2) > D(x_1) = 0$. The same reason shows (3.1) does not always hold. Nor does (3.1) in general give refinements of (1.1). For example, if $D(x_n) \ge (r-s)(2x_n)^{-1}$, the choice $s > 0, n = 2, x_1 = 0, x_2 = 1, \omega_1 = q \to 0$ will lead to $rs \le 1$ which shows $D(x_n) \ge (r-s)(2x_n)^{-1}$ does not hold for $r = 2, s = 1$. It is interesting to know whether certain lower bounds(respectively, upper bounds) for the differences of means will imply certain upper bounds(respectively,lower bounds) for the differences. By using (3.1), we have:

Proposition 3.1. *Let $n \geq 2, 0 < x_1 < x_n$. If either side of* (3.1) *inequality holds, then the opposite side inequality of* (1.1) *holds.*

Proof. We will show the left-hand side inequality of (1.1) follows from the right-hand side inequality of (3.1) and the other proof is similar. consider

$$f(\mathbf{x}) = (P_{n,r} - P_{n,s})/\sigma_n$$

then by our assumption

$$\frac{\sigma_n^2}{\omega_n}\frac{\partial f}{\partial x_n} = ((P_{n,r}/x_n)^{1-r} - (P_{n,s}/x_n)^{1-s})\sigma_n - 2(P_{n,r} - P_{n,s})(x_n - A_n) < 0$$

Thus by letting x_n tend to x_{n-1}, x_{n-1} tend to x_{n-2} and noticing

$$\lim_{x_2 \to x_1}(P_{2,r} - P_{2,s})/\sigma_2 = (r-s)/(2x_1)$$

We get

$$(P_{n,r} - P_{n,s})/\sigma_n \leq (r-s)/(2x_1)$$

which is the desired conclusion. □

We note here from the proof of Proposition 3.1, if one assumes $P_{n,r} - P_{n,s} < D(x_n)\sigma_n$ instead, one will get $P_{n,r} - P_{n,s} \geq (r-s)\sigma_n/(2x_1)$.

We now consider a general form of (1.1)($\alpha \leq 2$):

$$\frac{r-s}{2m^{2-\alpha}}\sigma_n \geq \frac{P_{n,r}^\alpha - P_{n,s}^\alpha}{\alpha} \geq \frac{r-s}{2M^{2-\alpha}}\sigma_n \tag{3.2}$$

where we define $(P_{n,u}^0 - P_{n,v}^0)/0 = \ln(P_{n,u}/P_{n,v})$, the limit of $(P_{n,u}^\alpha - P_{n,v}^\alpha)/\alpha$ as $\alpha \to 0$. We will refer to (3.2) as $I_{r,s,\alpha,n}$(so $I_{r,s,1,n} = I_{r,s,n}$). We note in [9], the following theorem was proved:

Theorem 3.1. *For $x_1 > 0$, $q = \min\{\omega_i\}$*

$$\frac{1-2q}{2x_1^2}\sigma_n \geq (1-q)\ln A_n + q\ln H_n - \ln G_n \geq \frac{1-2q}{2x_n^2}\sigma_n \tag{3.3}$$

$$\frac{1-2q}{2x_1^2}\sigma_n \geq \ln G_n - q\ln A_n - (1-q)\ln H_n \geq \frac{1-2q}{2x_n^2}\sigma_n \tag{3.4}$$

with equality holding if and only if $q = 1/2$ or $x_1 = \cdots = x_n$.

Note the right-hand side inequality of (3.3) is equivalent to

$$(1-q)(\ln A_n - \ln G_n - \frac{1}{2x_n^2}\sigma_n) - q(\ln G_n - \ln H_n - \frac{1}{2x_n^2}\sigma_n) \geq 0$$

From this we deduce

Corollary 3.1. *$I_{1,0,0,n}$ is equivalent to $I_{0,-1,0,n}$.*

In fact, if we assume that $I_{r,s,\alpha,n}$ holds if and only if $I_{r,s,\alpha,2}$ holds, we then have

Proposition 3.2. *If $I_{r,s,\alpha,n}$ holds if and only if $I_{r,s,\alpha,2}$ holds, then $I_{r,s,0,n}$ is equivalent to $I_{-s,-r,0,n}$ and* (3.3) *is equivalent to* (3.4).

Proof. A change of variables $x_1 \to 1/x_2, x_2 \to 1/x_1$ when $n = 2$ yields the desired conclusion. □

We remark here the following result of J. Chen and Z.Wang[7] is also invariant under the change $(r,s) \to (-s,-r)$:

Theorem 3.2. *For arbitrary $n, r > s, x_i \in (0, 1/2]$, $P'_{n,r}/P'_{n,s} \leq P_{n,r}/P_{n,s}$ holds if and only if $|r+s| \leq 3, 2^s/s \geq 2^r/r$ when $s > 0, s2^s \leq r2^r$ when $r < 0$.*

4. An Equivalence Relation

Proposition 4.1. *For $t \geq 0$,(1.1) is equivalent to*

$$\frac{x_1}{t+x_n} \leq \frac{P_{n,r,t}-P_{n,s,t}}{P_{n,r}-P_{n,s}} \leq \frac{x_n}{t+x_1} \tag{4.1}$$

Proof. The argument is similar to the one in [9] and we leave it to the reader. □

By Corollary 2.1, (4.1) holds for r=1, s=0 and now we give a refinement of this case:

Proposition 4.2. *For $t \geq 0$, $n \geq 2, x_1 < x_n$*

$$\frac{x_1}{t+x_1} \leq \frac{A_{n,t}-G_{n,t}}{A_n-G_n} \leq \frac{x_n}{t+x_n} \tag{4.2}$$

with equality holding if and only if $t=0$.

Proof. We will prove the left-hand side inequality of (4.2) and the proof for right-hand side inequality is similar. Let

$$D_n(\mathbf{x}) = x_n(A_n-G_n)-(t+x_n)(A_{n,t}-G_{n,t}) \tag{4.3}$$

We want to show $D_n \geq 0$ here. We can assume $0 \leq x_1 < x_2 < \cdots < x_n$ and prove by induction, the case $n=1$ is clear so we will start with $n>1$ variables assuming the inequality holds for $n-1$ variables. Then

$$\frac{\partial D_n}{\partial x_n} = (1+\omega_n)[(A_n-G_n)-(A_{n,t}-G_{n,t})] \geq 0$$

where the last inequality follows from (1.3). Thus $\frac{\partial D_n}{\partial x_n} \geq 0$ and by letting x_n tend to x_{n-1}, we have $D_n \geq D_{n-1}$(with weights $\omega_1, \cdots, \omega_{n-2}, \omega_{n-1}+\omega_n$) and thus the right-hand side inequality of (4.3) holds by induction. Since $n \geq 2$ here, it is easy to see the equality holds if and only if $t=0$. □

We note here (4.2) gives a refinement of (1.3) for $r=1, s=0$. In fact it implies $f(t) = (t+x_n)(A_{n,t}-G_{n,t})$ is a decreasing function of t while $g(t) = (t+x_1)(A_{n,t}-G_{n,t})$ is an increasing function of t. Thus $f'(0) \leq 0, g'(0) \geq 0$ and we get

$$\frac{x_1}{H_n}(G_n-H_n) \leq A_n-G_n \leq \frac{x_n}{H_n}(G_n-H_n) \tag{4.4}$$

By using the left-hand side inequalities of (1.1) and (4.4), we obtain a refinement of a result of A. Mercer[10]:

$$G_n-H_n \leq \frac{H_n}{2x_1^2}\sigma_n \tag{4.5}$$

It is not hard to show $t(P_{n,r,t}-P_{n,s,t}) \to (r-s)\sigma_n/2$ as $t \to \infty$. We can consider higher orders of the behavior, for example

$$-\lim_{t\to\infty} t[(t+x_n)(P_{n,r,t}-P_{n,s,t})-(r-s)\sigma_n/2] = \lim_{\epsilon\to 0}[\frac{(1-\epsilon x_n)(P_{n,r}(\epsilon)-P_{n,s}(\epsilon))}{\epsilon^3}-\frac{(r-s)\sigma_n}{2\epsilon}]$$

$$= \frac{r-s}{6}[(3-2r-2s)A_n^3+3(r+s-2)A_nP_{n,2}^2+(3-r-s)P_{n,3}^3-3x_n\sigma_n] := E(\mathbf{x}) \tag{4.6}$$

Proposition 4.3. *For $0 \leq r+s \leq 3$, $E(\mathbf{x}) \leq 0$ with equality holding if and only if $x_1 = \cdots = x_n$.*

Proof. We may assume $x_n = 1$ and $0 \leq x_i \leq 1$. We leave for the reader to check by applying the method in the proof of Theorem 2.1, it suffices to prove (4.6) for $n=2$. By setting $\omega_1 = q, \omega_2 = 1-q, x_1 = x, x_2 = 1$, we rewrite (4.6) as

$$\begin{aligned} f(x) &= (3-2r-2s)(qx+1-q)^3+3(r+s-2)(qx+1-q)(qx^2+1-q) \\ &\quad +(3-r-s)(qx^3+1-q)-3q(1-q)(x-1)^2 \end{aligned}$$

One verifies that $f(1) = f'(1) = f''(1) = 0$ and

$$(6q)^{-1}f'''(x) = (3-2r-2s)q^2 + 3(r+s-2)q + (3-r-s) := g(q)$$

If $r+s=0$ or 3, one checks directly $f'''(x) \geq 0$. For $0 < r+s < 3$, if $g'(q) = 0$ has no root in $[0,1]$, then $g(q) \geq \min\{g(0), g(1)\} = 0$ for $0 \leq q \leq 1$. Otherwise let q_0 be the only root of $g'(q)$ in $[0,1]$. Notice $g'(1) = -(r+s) < 0$ and $g(0) = 3-r-s > 0, g(1) = 0$. These imply q_0 is a local maximum point of $g(q)$. So again $g(q) \geq \min\{g(0), g(1)\} = 0$ for $0 \leq q \leq 1$. Now $f(x) \leq 0$ follows by considering the Taylor expansion of $f(x)$ at $x = 1$. It is also easy to see that the equality holds if and only if $x_1 = \cdots = x_n$ and this completes the proof. □

We remark here by (1.3), $A_n - P_{n,s} \geq A_{n,t} - P_{n,s,t}$ hold for all $s \leq 1$. On the other hand, for $0 < x_1, x_n \leq 1/2$, a result of the author[9] shows $A_n - P_{n,s} \geq A'_n - P'_{n,s}$ holds for $-1 \leq s < 1$.Now for $s < 0$, let $n = 2, x_1 \to 0, x_2 = 1/2, \omega_1 = 1-q, \omega_2 = q$ and then let $q \to 0$, we get $1 - s \geq 2^{-s}$, which shows $A_n - P_{n,s} \geq A'_n - P'_{n,s}$ holds if and only if $-1 \leq s < 1$. Similarly, by letting $q \to 1$, we know $P_{n,r} - A_n \geq P'_{n,r} - A'_n$ holds if and only if $1 < r \leq 2$.

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