A REFINEMENT OF JENSEN’S DISCRETE INEQUALITY FOR DIFFERENTIABLE CONVEX FUNCTIONS

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Abstract. A refinement of Jensen’s discrete inequality and applications for the celebrated Arithmetic Mean – Geometric Mean – Harmonic Mean inequality and Cauchy-Schwartz-Bunikowski inequality are pointed out.

1. Introduction

The following inequality is well known in literature as Jensen’s inequality:

\[ f \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i), \]

provided \( f : [a, b] \to \mathbb{R} \) is a convex function on \([a, b]\), \( x_i \in [a, b] \), and \( p_i \geq 0 \) with \( P_n := \sum_{i=1}^{n} p_i > 0 \).

Its central role in Analytic Inequality Theory is determined by the fact that many other fundamental results such as: the Arithmetic Mean – Geometric Mean – Harmonic Mean inequality, or the Hölder and Minkowski inequalities, or even the Ky Fan inequality may be obtained from Jensen’s inequality by appropriate choices of the function \( f \).

There is an extensive literature devoted to Jensen’s inequality concerning different generalizations, refinements, counterparts and converse results, see, for example [1] – [21].

The main aim of this paper is to point out a new refinement of this classical result. Two applications in connection with the celebrated A – G – H – means inequality and the Cauchy-Buniakowski-Schwartz inequality are mentioned as well.

2. A Refinement of Jensen’s Inequality

The following refinement of Jensen’s inequality holds.
Theorem 1. Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function on $(a, b)$ and $x_i \in (a, b), p_i \geq 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$. Then one has the inequality

\begin{equation}
\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right)
\geq \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) \right|
- \left| f' \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i \right| x_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right| \geq 0.
\end{equation}

Proof. Since $f$ is differentiable convex on $(a, b)$, then for each $x, y \in (a, b)$, one has the inequality

\begin{equation}
f(x) - f(y) \geq (x - y) f'(y).
\end{equation}

Using the properties of the modulus, we have

\begin{equation}
f(x) - f(y) - (x - y) f'(y) = |f(x) - f(y) - (x - y) f'(y)|
\geq ||f(x) - f(y)| - |x - y|| f'(y)||
\end{equation}

for each $x, y \in (a, b)$.

If we choose $y = \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j$ and $x = x_i, i \in \{1, \ldots, n\}$, then we have

\begin{equation}
f(x_i) - f \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) - \left( x_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) f' \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right)
\geq \left| f(x_i) - f \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) \right| - \left| x_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right| f' \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right)\right|
\end{equation}

for any $i \in \{1, \ldots, n\}$.

If we multiply (2.4) by $p_i \geq 0$, sum over $i$ from 1 to $n$, and divide by $P_n > 0$, we deduce

\begin{align*}
\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right)
&- \frac{1}{P_n} \sum_{i=1}^{n} p_i \left( x_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) f' \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) \\
&= \frac{1}{P_n} \sum_{i=1}^{n} p_i \left( f(x_i) - f \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) \right)
- \frac{1}{P_n} \sum_{i=1}^{n} p_i \left( x_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) f' \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right)
\end{align*}
\[
\sum_{i=1}^{n} p_i \left| f(x_i) - f \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) \right| - \left| x_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right| \cdot f' \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) \geq 0.
\]

Since
\[
\sum_{i=1}^{n} p_i \left( x_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right) = 0,
\]
the inequality (2.1) is proved.

In particular, we have the following result for unweighted means.

**Corollary 1.** With the above assumptions for \( f \) and \( x_i \), one has the inequality
\[
(2.5) \quad f \left( \frac{x_1 + \cdots + x_n}{n} \right) - f \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \geq \left| \frac{1}{n} \sum_{i=1}^{n} \left( x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right| \geq 0.
\]

**Remark 1.** Similar integral inequalities may be stated as well. We omit the details.

### 3. A Refinement of \( A - G - H \) Inequality

For a positive \( n \)-tuple \( \bar{x} = (x_1, \ldots, x_n) \) and \( \bar{p} = (p_1, \ldots, p_n) \) with \( p_i \geq 0 \) and \( \sum_{i=1}^{n} p_i =: P_n > 0 \), define
\[
A_n (\bar{p}, \bar{x}) := \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \quad \text{(the weighted arithmetic mean)},
\]
\[
G_n (\bar{p}, \bar{x}) := \left( \prod_{i=1}^{n} x_i^{p_i} \right)^{1/n} \quad \text{(the weighted geometric mean)},
\]
\[
H_n (\bar{p}, \bar{x}) := \frac{P_n}{\sum_{i=1}^{n} \frac{p_i}{x_i}} = \left[A_n \left( \bar{p}, \frac{1}{\bar{x}} \right) \right]^{-1} \quad \text{(the weighted harmonic mean)}.
\]

The following inequality
\[
(3.1) \quad A_n (\bar{p}, \bar{x}) \geq G_n (\bar{p}, \bar{x}) \geq H_n (\bar{p}, \bar{x})
\]
is well known in the literature as the Arithmetic Mean – Geometric Mean – Harmonic Mean (A – G – H)-means inequality.

Using Theorem 1, we may improve this result as follows.
Proposition 1. Suppose that \( \bar{x}, \bar{p} \) are as above. Then we have the inequality
\[
A_n(\bar{p}, \bar{x}) G_n(\bar{p}, \bar{x}) \geq \exp \left| A_n \left( \bar{p}, \left| \ln \left( \frac{\bar{x}}{A_n(\bar{p}, \bar{x})} \right) \right| \right) - A_n \left( \bar{p}, \left| \frac{x - A_n(\bar{p}, \bar{x})}{A_n(\bar{p}, \bar{x})} \right| \right) \right| \geq 1,
\]
where for a function \( h \), we denote \( h(\bar{x}) := (h(x_1), \ldots, h(x_n)) \).

Proof. Applying the inequality (2.1) for \( f(x) = -\ln x \), we get
\[
\ln \left[ A_n(\bar{p}, \bar{x}) \right] \geq \left| \frac{1}{n} \sum_{i=1}^{n} p_i \ln \left( \frac{x_i}{A_n(p, x)} \right) \right| - A_n^{-1}(p, x) \cdot \left| \frac{1}{n} \sum_{i=1}^{n} p_i |x_i - A_n(p, x)| \right| \geq 0,
\]
from where we get the desired inequality (3.2).

The following proposition also holds.

Proposition 2. Suppose that \( \bar{x}, \bar{p} \) are as above. Then we have the inequality:
\[
G_n(\bar{p}, \bar{x}) H_n(\bar{p}, \bar{x}) \geq \exp \left| A_n \left( \bar{p}, \left| \ln \left( H_n(\bar{p}, \bar{x}) \bar{x} \right) \right| \right) - A_n \left( \bar{p}, \left| \frac{H_n(\bar{p}, \bar{x}) - \bar{x}}{\bar{x}} \right| \right) \right| \geq 1,
\]

Proof. Follows by Proposition 1 on choosing \( \frac{1}{2} \bar{x} \) instead of \( \bar{x} \).

4. A Refinement of Cauchy-Buniakowski-Schwartz's Inequality

The following inequality is well known in the literature as the Cauchy-Buniakowski-Schwartz inequality:
\[
\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left( \sum_{i=1}^{n} a_i b_i \right)^2,
\]
for any \( a_i, b_i \in \mathbb{R} \ (i \in \{1, \ldots, n\}) \).

The following refinement of (4.1) holds.

Proposition 3. If \( a_i, b_i \in \mathbb{R}, \ i \in \{1, \ldots, n\} \), then one has the inequality:
\[
\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 \geq \frac{1}{\sum_{i=1}^{n} b_i^2} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_j b_j \right)^2 \left( \sum_{j=1}^{n} b_j^2 \right)^2 - 2 \sum_{k=1}^{n} \sum_{i=1}^{n} a_k b_k \sum_{i=1}^{n} |b_i| \sum_{j=1}^{n} b_j \frac{a_i b_i}{a_j b_j} \geq 0.
\]
Proof. If we apply Theorem \[1\] for \( f(x) = x^2 \), we get

\[
(4.3) \quad \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i^2 - \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right)^2 \geq \frac{1}{P_n} \sum_{i=1}^{n} p_i \left[ x_i^2 - \left( \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right)^2 \right]
\]

\[
-2 \left| \frac{1}{P_n} \sum_{k=1}^{n} p_k x_k \right| - \frac{1}{P_n} \sum_{i=1}^{n} p_i \left[ x_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right] \geq 0.
\]

If in (4.3), we choose \( p_i = b_i^2, x_i = \frac{a_i}{b_i}, i \in \{1, \ldots, n\} \), we get

\[
(4.4) \quad \sum_{i=1}^{n} \frac{a_i^2}{b_i^2} - \left( \sum_{i=1}^{n} \frac{a_i b_i}{b_i^2} \right)^2 \geq \left| \frac{1}{\sum_{i=1}^{n} b_i^2} \sum_{i=1}^{n} b_i^2 \right| - \frac{1}{\sum_{i=1}^{n} b_i^2} \left| \sum_{i=1}^{n} a_i b_i \right| - \frac{1}{\sum_{i=1}^{n} b_i^2} \left| \sum_{j=1}^{n} a_j b_j \right| \geq 0.
\]

which is clearly equivalent to (4.2). \[\blacksquare\]

References


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