NEW TAYLOR-LIKE EXPANSIONS FOR FUNCTIONS OF TWO VARIABLES AND ESTIMATES OF THEIR REMAINDERS

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Abstract. In this article, a generalisation of Sard’s inequality for Appell polynomials is obtained. Estimates for the remainder are also provided.

1. Introduction

Let \( x \in [a, b] \) and \( y \in [c, d] \). If \( f(x, y) \) is a function of two variables we shall adopt the following notation for partial derivatives of \( f(x, y) \):

\[
\begin{align*}
  f^{(i,j)}(x, y) & \triangleq \frac{\partial^{i+j} f(x, y)}{\partial x^i \partial y^j}, \\
  f^{(0,0)}(x, y) & \triangleq f(x, y), \\
  f^{(i,j)}(\alpha, \beta) & \triangleq f^{(i,j)}(x, y)|_{(x,y)=(\alpha,\beta)}
\end{align*}
\]

for \( 0 \leq i, j \in \mathbb{N} \) and \( (\alpha, \beta) \in [a, b] \times [c, d] \).

A. H. Stroud has pointed out in [6] that one of the most important tools in the numerical integration of double integrals is the following Taylor’s formula [6, p. 138 and p. 157] due to A. Sard [5]:

**Theorem A.** If \( f(x, y) \) satisfies the condition that all the derivatives \( f^{(i,j)}(x, y) \) for \( i + j \leq m \) are defined and continuous on \( [a, b] \times [c, d] \), then \( f(x, y) \) has the expansion

\[
\begin{align*}
  f(x, y) &= \sum_{i+j \leq m} \frac{(x-a)^i (y-c)^j}{i! j!} f^{(i,j)}(a, c) \\
  &\quad + \sum_{j \leq q} \frac{(y-c)^j}{j!} \int_a^x \frac{(x-u)^{m-j-1}}{(m-j-1)!} f^{(m-j,j)}(u, c) \, du \\
  &\quad + \sum_{i \leq p} \frac{(x-a)^i}{i!} \int_c^y \frac{(y-v)^{m-i-1}}{(m-i-1)!} f^{(i,m-i)}(a, v) \, dv \\
  &\quad + \int_a^x \int_c^y \frac{(x-u)^{p-1} (y-v)^{q-1}}{(p-1)! (q-1)!} f^{(p,q)}(u, v) \, du \, dv,
\end{align*}
\]

where \( i, j \) are nonnegative integers; \( p, q \) are positive integers; and \( m \triangleq p + q \geq 2 \).

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Key words and phrases. Appell polynomials, Bernoulli polynomials, Euler polynomials, generalized Taylor’s formula, Taylor-like expansion, double integral, remainder.

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Essentially, the representation (2) is used for obtaining the fundamental Kernel Theorems and Error Estimates in numerical integration of double integrals [6, p. 142, p. 145 and p. 158] and has both theoretical and practical importance in the domain as a whole.

**Definition 1.** A sequence of polynomials \( \{ P_i(x) \}_{i=0}^{\infty} \) is called *harmonic* [4] if it satisfies the recursive formula

\[
P'_i(x) = P_{i-1}(x)
\]

for \( i \in \mathbb{N} \) and \( P_0(x) = 1 \).

A slightly different concept that specifies the connection between the variables is the following one.

**Definition 2.** We say that a sequence of polynomials \( \{ P_i(t, x) \}_{i=0}^{\infty} \) satisfies the *Appell condition* [2] if

\[
\frac{\partial P_i(t, x)}{\partial t} = P_{i-1}(t, x)
\]

and \( P_0(t, x) = 1 \) for all defined \( (t, x) \) and \( n \in \mathbb{N} \).

It is well-known that the Bernoulli polynomials \( B_i(t) \) can be defined by the following expansion

\[
\frac{xe^{tx}}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i(t)}{i!} x^i, \quad |x| < 2\pi, \quad t \in \mathbb{R}.
\]

It can be shown that the polynomials \( B_i(t), i \in \mathbb{N} \), are uniquely determined by the two formulae

\[
B'_i(t) = iB_{i-1}(t), \quad B_0(t) = 1; \quad \text{and} \quad B_i(t + 1) - B_i(t) = it^{i-1}.
\]

The Euler polynomials can be defined by the following expansion

\[
\frac{2e^{tx}}{e^x + 1} = \sum_{i=0}^{\infty} \frac{E_i(t)}{i!} x^i, \quad |x| < \pi, \quad t \in \mathbb{R}.
\]

It can also be shown that the polynomials \( E_i(t), i \in \mathbb{N} \), are uniquely determined by the two properties

\[
E'_i(t) = iE_{i-1}(t), \quad E_0(t) = 1; \quad \text{and} \quad E_i(t + 1) + E_i(t) = 2t^i.
\]

For further details about Bernoulli polynomials and Euler polynomials, please refer to [1, 23.1.5 and 23.1.6].

There are many examples of Appell polynomials. For instance, for \( i \) a nonnegative integer, \( \theta \in \mathbb{R} \) and \( \lambda \in [0, 1] \),

\[
P_{i,\lambda}(t) \equiv P_{i,\lambda}(t; x; \theta) = \frac{[t - (\lambda \theta + (1 - \lambda)x)]^i}{i!},
\]

\[
P_{i,B}(t) \equiv P_{i,B}(t; x; \theta) = \frac{(x - \theta)^i}{i!} B_i\left(\frac{t - \theta}{x - \theta}\right) ([4]),
\]

\[
P_{i,E}(t) \equiv P_{i,E}(t; x; \theta) = \frac{(x - \theta)^i}{i!} E_i\left(\frac{t - \theta}{x - \theta}\right) ([4]).
\]

In [4], the following generalized Taylor’s formula was established.
Theorem B. Let \( \{P_i(x)\}_{i=0}^{\infty} \) be a harmonic sequence of polynomials. Further, let \( I \subset \mathbb{R} \) be a closed interval and \( a \in I \). If \( f : I \to \mathbb{R} \) is any function such that \( f^{(n)}(x) \) is absolutely continuous for some \( n \in \mathbb{N} \), then, for any \( x \in I \), we have

\[
 f(x) = f(a) + \sum_{k=0}^{m} (-1)^{k+1} [P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)] + R_n(f; a, x),
\]

where

\[
 R_n(f; a, x) = (-1)^n \int_a^x P_n(t)f^{(n+1)}(t) \, dt.
\]

The fundamental aim of this article is to obtain a generalisation of the Taylor-like formula (2) for Appell polynomials and to study its impact on the numerical integration of double integrals.

2. Two New Taylor-like Expansions

Following a similar argument to the proof of Theorem 2 in [4], we obtain the following result.

Theorem 1. If \( g : [a, b] \to \mathbb{R} \) is such that \( g^{(n-1)} \) is absolutely continuous on \([a, b]\), then we have the generalised integration by parts formula for \( x \in [a, b] \)

\[
\int_a^b g(t) \, dt = \sum_{k=1}^{n} (-1)^{k+1} [P_k(b, x)g^{(k-1)}(b) - P_k(a, x)g^{(k-1)}(a)]
\]

\[
+ (-1)^n \int_a^b P_n(t, x)g^{(n)}(t) \, dt.
\]

Proof. By integration by parts we obtain, on using the Appell condition (4),

\[
(-1)^n \int_a^b P_n(t, x)g^{(n)}(t) \, dt
\]

\[
= (-1)^n P_n(t, x)g^{(n-1)}(t) \bigg|_a^b + (-1)^{n-1} \int_a^b P_{n-1}(t, x)g^{(n-1)}(t) \, dt
\]

\[
= (-1)^n \left[ P_n(b, x)g^{(n-1)}(b) - P_n(a, x)g^{(n-1)}(a) - \int_a^b P_{n-1}(t, x)g^{(n-1)}(t) \, dt \right].
\]

Clearly, the same procedure can be used for the term \( \int_a^b P_{n-1}(t, x)g^{(n-1)}(t) \, dt \). Therefore, formula (16) follows from successive integration by parts.

Theorem 2. Let \( D \) be a domain in \( \mathbb{R}^2 \) and the point \( (a, c) \in D \). Also, let \( \{P_i(t, x)\}_{i=0}^{\infty} \) and \( \{Q_j(s, y)\}_{j=0}^{\infty} \) be two Appell polynomials. If \( f : D \to \mathbb{R} \) is such that \( f^{(i,j)}(x, y) \) are continuous on \( D \) for all \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \), then

\[
f(x, y) = f(a, c) + C(f, P_m, Q_n) + D(f, P_m, Q_n) + S(f, P_m, Q_n) + T(f, P_m, Q_n),
\]

where

\[
C(f, P_m, Q_n) = \sum_{i=1}^{m} (-1)^{i+1} [P_i(x, x)f^{(i,0)}(x, c) - P_i(a, x)f^{(i,0)}(a, c)]
\]

\[
+ \sum_{j=1}^{n} (-1)^{j+1} [Q_j(y, y)f^{(0,j)}(a, y) - Q_j(c, y)f^{(0,j)}(a, c)],
\]
\textbf{Proof.} Let $P_m(t, x)$ be an Appell polynomial. Applying formula (14) to the function $f(x, y)$ with respect to variable $x$ yields
\begin{equation}
  f(x, y) = f(a, y) + \sum_{i=1}^{m} (-1)^{i+1} \left[ P_i(x, x) f^{(i,0)}(x, y) - P_i(a, x) f^{(i,0)}(a, y) \right] \\
  + (-1)^m \int_{a}^{x} P_m(t, x) f^{(m+1,0)}(t, y) \, dt. \tag{23}
\end{equation}

Similarly, for the functions $f^{(i,0)}(x, y)$, $f^{(i,0)}(a, y)$, $f^{(m+1,0)}(t, y)$ and $f(a, y)$, we have
\begin{equation}
  f^{(i,0)}(x, y) = f^{(i,0)}(x, c) + (-1)^n \int_{c}^{y} Q_n(s, y) f^{(i,n+1)}(x, s) \, ds \\
  + \sum_{j=1}^{n} (-1)^{j+1} \left[ Q_j(y, y) f^{(i,j)}(x, y) - Q_j(c, y) f^{(i,j)}(x, c) \right], \tag{24}
\end{equation}
\begin{equation}
  f^{(i,0)}(a, y) = f^{(i,0)}(a, c) + (-1)^n \int_{c}^{y} Q_n(s, y) f^{(i,n+1)}(a, s) \, ds \\
  + \sum_{j=1}^{n} (-1)^{j+1} \left[ Q_j(y, y) f^{(i,j)}(a, y) - Q_j(c, y) f^{(i,j)}(a, c) \right], \tag{25}
\end{equation}
\begin{equation}
  f^{(m+1,0)}(t, y) = f^{(m+1,0)}(t, c) + (-1)^n \int_{c}^{y} Q_n(s, y) f^{(m+1,n+1)}(t, s) \, ds \\
  + \sum_{j=1}^{n} (-1)^{j+1} \left[ Q_j(y, y) f^{(m+1,j)}(t, y) - Q_j(c, y) f^{(m+1,j)}(t, c) \right], \tag{26}
\end{equation}

\begin{align*}
  D(f, P_m, Q_n) &= \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(x, x) \left[ Q_j(y, y) f^{(i,j)}(x, y) - Q_j(c, y) f^{(i,j)}(x, c) \right] \\
  &\quad - \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(a, x) \left[ Q_j(y, y) f^{(i,j)}(a, y) - Q_j(c, y) f^{(i,j)}(a, c) \right], \tag{20}
\end{align*}

\begin{align*}
  S(f, P_m, Q_n) &= (-1)^m \int_{a}^{x} P_m(t, x) f^{(m+1,0)}(t, c) \, dt + (-1)^n \int_{c}^{y} Q_n(s, y) f^{(0,n+1)}(a, s) \, ds \\
  &\quad + \sum_{i=1}^{m} (-1)^{n+i+1} \int_{c}^{y} Q_n(s, y) \left[ P_i(x, x) f^{(i,n+1)}(x, s) - P_i(a, x) f^{(i,n+1)}(a, s) \right] \, ds \\
  &\quad + \sum_{j=1}^{n} (-1)^{m+j+1} \int_{a}^{x} P_m(t, x) \left[ Q_j(y, y) f^{(m+1,j)}(t, y) - Q_j(c, y) f^{(m+1,j)}(t, c) \right] \, dt \tag{21}
\end{align*}

\begin{align*}
  T(f, P_m, Q_n) &= (-1)^{m+n} \int_{a}^{x} \int_{c}^{y} P_m(t, x) Q_n(s, y) f^{(m+1,n+1)}(t, s) \, ds \, dt. \tag{22}
\end{align*}
\[ f(a, y) = f(a, c) + (-1)^n \int_c^y Q_n(s, y) f^{(0,n+1)}(a, s) \, ds \]
\[ + \sum_{j=1}^n (-1)^{j+1} [Q_j(y, y) f^{(0,j)}(a, y) - Q_j(c, y) f^{(0,j)}(a, c)] . \]  

Substituting formulae (24)–(27) into (23) produces

\[ f(x, y) = f(a, c) + \sum_{i=1}^m (-1)^{i+1} \left[ P_i(x, x) f^{(i,0)}(x, c) - P_i(a, x) f^{(i,0)}(a, c) \right] \]
\[ + \sum_{j=1}^n (-1)^{j+1} [Q_j(y, y) f^{(0,j)}(a, y) - Q_j(c, y) f^{(0,j)}(a, c)] \]
\[ + \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j} P_i(x, x) [Q_j(y, y) f^{(i,j)}(x, y) - Q_j(c, y) f^{(i,j)}(x, c)] \]
\[ - \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j} P_i(a, x) [Q_j(y, y) f^{(i,j)}(a, y) - Q_j(c, y) f^{(i,j)}(a, c)] \]
\[ + (-1)^m \int_a^x P_m(t, x) f^{(m+1,0)}(t, c) \, dt + (-1)^n \int_c^y Q_n(s, y) f^{(0,n+1)}(a, s) \, ds \]
\[ + \sum_{i=1}^m (-1)^{n+i+1} \int_c^y Q_n(s, y) \left[ P_i(x, x) f^{(i,n+1)}(x, s) - P_i(a, x) f^{(i,n+1)}(a, s) \right] \, ds \]
\[ + \sum_{j=1}^n (-1)^{m+j+1} \int_a^x P_m(t, x) [Q_j(y, y) f^{(m+1,j)}(t, y) - Q_j(c, y) f^{(m+1,j)}(t, c)] \, dt \]
\[ + (-1)^{m+n} \int_a^x \int_c^y P_m(t, x) Q_n(s, y) f^{(m+1,n+1)}(t, s) \, ds \, dt. \]

The proof of Theorem 2 is complete. \(\blacksquare\)

Remark 1. If we take

\[ P_i(t, x) = P_m, \lambda(t, x; a), \quad Q_j(s, y) = Q_j, \mu(s, y; c) \]  

for \(0 \leq i \leq m, 0 \leq j \leq n\) and \(\lambda, \mu \in [0, 1]\) in Theorem 2, then the expressions simplify to give, on using (11),

\[ C(f, P_m, Q_n) = \sum_{i=1}^m \frac{(x-a)^i}{i!} \left[(1 - \lambda)^i f^{(i,0)}(a, c) + \lambda^i f^{(i,0)}(x, c) \right] \]
\[ + \sum_{j=1}^n \frac{(y-c)^j}{j!} \left[(1 - \mu)^j f^{(0,j)}(a, c) + \mu^j f^{(0,j)}(a, y) \right], \]  

\[ D(f, P_m, Q_n) = \sum_{i=1}^m \sum_{j=1}^n \frac{\lambda^i (x-a)^i (y-c)^j}{i! \cdot j!} \left[\mu^j f^{(i,j)}(x, y) + (1 - \mu)^j f^{(i,j)}(x, c) \right] \]
Theorem 3. Let \( \{P_i(t, x)\}_{i=0}^{\infty} \) and \( \{Q_j(s, y)\}_{j=0}^{\infty} \) be two Appell polynomials and \( f : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) such that \( f^{(i,j)}(x, y) \) are continuous on \([a, b] \times [c, d]\) for all \(0 \leq i \leq m\) and \(0 \leq j \leq n\). We then have

\[
\int_a^b \int_c^d f(t, s) \, ds \, dt = A(f, P_m, Q_n) + B(f, P_m, Q_n) + R(f, P_m, Q_n),
\]

where

\[
A(f, P_m, Q_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(a, b) \left[ Q_j(d, d) f^{(i-1,j-1)}(a, d) - Q_j(c, d) f^{(i-1,j-1)}(a, c) \right]
\]

\[
- \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(c, b) \left[ Q_j(d, d) f^{(i-1,j-1)}(b, d) - Q_j(c, d) f^{(i-1,j-1)}(b, c) \right],
\]

\[
\int_a^b \int_c^d f(t, s) \, ds \, dt = (-1)^m \int_a^b \frac{[t - (\lambda a + (1 - \lambda)x)]^m}{m!} f^{(m+1,0)}(t, c) \, dt
\]

\[
+ (-1)^n \int_c^d \frac{[s - (\mu c + (1 - \mu)y)]^n}{n!} f^{(0,n+1)}(a, s) \, ds
\]

\[
+ \sum_{i=1}^{m} \int_c^d \frac{[\mu c + (1 - \mu)y - s]^n (x - a)^i}{n! \cdot i!} \left[ (\lambda - 1)^i f^{(i,n+1)}(a, s) - \lambda^i f^{(i,n+1)}(x, s) \right] \, ds
\]

\[
+ \sum_{j=1}^{n} \int_a^b \frac{[\lambda a + (1 - \lambda)x - t]^m (y - c)^j}{m! \cdot j!} \left[ (\mu - 1)^j f^{(m+1,j)}(t, c) - \mu^j f^{(m+1,j)}(t, y) \right] \, dt,
\]

and

\[
- \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(a, b) \left[ Q_j(d, d) f^{(i-1,j-1)}(a, d) - Q_j(c, d) f^{(i-1,j-1)}(a, c) \right]
\]

\[
- \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(c, b) \left[ Q_j(d, d) f^{(i-1,j-1)}(b, d) - Q_j(c, d) f^{(i-1,j-1)}(b, c) \right].
\]

Notice that, taking \( \lambda = 0 \) and \( \mu = 0 \) in (29), then we can deduce Theorem A from Theorem 2. Other choices of Appell type polynomials will provide generalizations of Theorem A.
\[ B(f, P_m, Q_n) = \sum_{j=1}^{n} (-1)^j Q_j(c, d) \int_{a}^{b} f^{(0,j-1)}(t, c) \, dt \]

\[ - \sum_{j=1}^{n} (-1)^j Q_j(d, d) \int_{a}^{b} f^{(0,j-1)}(t, d) \, dt \]

\[ + \sum_{i=1}^{m} (-1)^i P_i(a, b) \int_{c}^{d} f^{(i-1,0)}(a, s) \, ds \]

\[ - \sum_{i=1}^{m} (-1)^i P_i(b, b) \int_{c}^{d} f^{(i-1,0)}(b, s) \, ds \]

and

\[ R(f, P_m, Q_n) = (-1)^{m+n} \int_{a}^{b} \int_{c}^{d} P_m(t, b)Q_n(s, d)\, ds \, dt. \]  

**Proof.** Using the generalized integration by parts formula consecutively yields

\[ \int_{a}^{b} \int_{c}^{d} P_m(t, b)Q_n(s, d) f^{(m,n)}(t, s) \, ds \, dt \]

\[ = \int_{a}^{b} P_m(t, b) \left[ \int_{c}^{d} Q_n(s, d) f^{(m,n)}(t, s) \, ds \right] \, dt \]

\[ = (-1)^m \int_{a}^{b} P_m(t, b) \left\{ \int_{c}^{d} f^{(m,0)}(t, s) \, ds \right\} \, dt \]

\[ + \sum_{j=1}^{n} (-1)^j \left[ Q_j(d, d) f^{(m,j-1)}(t, d) - Q_j(c, d) f^{(m,j-1)}(t, c) \right] \, dt \]

\[ = (-1)^m \int_{a}^{b} \int_{c}^{d} P_m(t, b) f^{(m,0)}(t, s) \, ds \, dt \]

\[ + \sum_{j=1}^{n} (-1)^{m+j} Q_j(d, d) \int_{a}^{b} P_m(t, b) f^{(m,j-1)}(t, d) \, dt \]

\[ - \sum_{j=1}^{n} (-1)^{m+j} Q_j(c, d) \int_{a}^{b} P_m(t, b) f^{(m,j-1)}(t, c) \, dt \]

\[ = (-1)^m \int_{c}^{d} (-1)^n \left\{ \int_{a}^{b} f(t, s) \, dt \right\} \]

\[ + \sum_{i=1}^{m} (-1)^i \left[ P_i(b, b) f^{(i-1,0)}(b, s) - P_i(a, b) f^{(i-1,0)}(a, s) \right] \, ds \]

\[ + \sum_{j=1}^{n} (-1)^{n+j} Q_j(d, d) \left\{ (-1)^m \left[ \int_{a}^{b} f^{(0,j-1)}(t, d) \, dt \right. \right. \]

\[ + \sum_{i=1}^{m} (-1)^j \left( P_i(b, b) f^{(i-1,j-1)}(b, d) - P_i(a, b) f^{(i-1,j-1)}(a, d) \right) \]
\[- \sum_{j=1}^{n} (-1)^{n+j} Q_j(c, d) \left\{ (-1)^{m} \left[ \int_{a}^{b} f^{(0,j-1)}(t, c) \, dt \right. \right. \\
+ \left. \left. \sum_{i=1}^{m} (-1)^{i} \left( P_i(b, b) f^{(i-1,j-1)}(b, c) - P_i(a, b) f^{(i-1,j-1)}(a, c) \right) \right] \right\} \\
= (-1)^{m+n} \int_{a}^{b} \int_{c}^{d} f(t, s) \, ds \, dt \\
+ \sum_{i=1}^{m} (-1)^{m+n+i} \int_{c}^{d} \left[ P_i(b, b) f^{(i-1,0)}(b, s) - P_i(a, b) f^{(i-1,0)}(a, s) \right] \, ds \\
+ \sum_{j=1}^{n} (-1)^{m+n+j} Q_j(d, d) \int_{a}^{b} f^{(0,j-1)}(t, d) \, dt \\
+ \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n+i+j} P_i(b, b) Q_j(d, d) f^{(i-1,j-1)}(b, d) \\
- \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n+i+j} P_i(a, b) Q_j(d, d) f^{(i-1,j-1)}(a, d) \\
- \sum_{j=1}^{n} (-1)^{m+n+j} Q_j(c, d) \int_{a}^{b} f^{(0,j-1)}(t, c) \, dt \\
+ \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n+i+j} P_i(a, b) Q_j(c, d) f^{(i-1,j-1)}(a, c) \\
- \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n+i+j} P_i(b, b) Q_j(c, d) f^{(i-1,j-1)}(b, c) \\
= (-1)^{m+n} \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(b, b) \left[ Q_j(d, d) f^{(i-1,j-1)}(b, d) \right. \\
- Q_j(c, d) f^{(i-1,j-1)}(b, c) \right] \\
+ (-1)^{m+n} \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} P_i(a, b) \left[ Q_j(c, d) f^{(i-1,j-1)}(a, c) \right. \\
- Q_j(d, d) f^{(i-1,j-1)}(a, d) \right] \\
+ (-1)^{m+n} \sum_{i=1}^{m} (-1)^{i} P_i(b, b) \int_{c}^{d} f^{(i-1,0)}(b, s) \, ds \\
- (-1)^{m+n} \sum_{i=1}^{m} (-1)^{i} P_i(a, b) \int_{c}^{d} f^{(i-1,0)}(a, s) \, ds \\
+ (-1)^{m+n} \sum_{j=1}^{n} (-1)^{j} Q_j(d, d) \int_{a}^{b} f^{(0,j-1)}(t, d) \, dt \]
- \((-1)^{m+n} \sum_{j=1}^{n} (-1)^j Q_j(c, d) \int_a^b f^{(0,j-1)}(t, c) \, dt
+ \int_a^b \int_c^d f(t, s) \, ds \, dt.

The proof of Theorem 3 is complete. •

Remark 2. As usual, let \( B_i, i \in \mathbb{N} \), denote Bernoulli numbers.
From properties (6) and (7), (9) and (10) of the Bernoulli and Euler polynomials respectively, we can easily obtain that, for \( i \geq 1 \),

\[
B_{i+1}(0) = B_{i+1}(1) = B_{i+1}, \quad B_1(0) = -B_1(1) = -\frac{1}{2}, \quad \text{and (38)}
\]

and, for \( j \in \mathbb{N} \),

\[
E_j(0) = -E_j(1) = -\frac{2}{j+1} (2^{j+1} - 1) B_{j+1}. \quad \text{(39)}
\]

It is also a well known fact that \( B_{2i+1} = 0 \) for all \( i \in \mathbb{N} \).

As an example, taking \( P_i(t, x) = P_{i,B}(t, x; a) \) and \( Q_j(s, y) = P_{j,E}(s, y; c) \) from (12) and (13) for \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \) in Theorem 3 and using (38) and (39) yields

\[
A(f, P_m, Q_n) = \sum_{i=1}^{m} \sum_{j=2}^{n} \frac{(a-b)^i(c-d)^j}{i! \cdot j!} \cdot \frac{2(2^{j+1} - 1)}{j+1} B_i B_{j+1} \times [f^{(i-1,j-1)}(a, d) + f^{(i-1,j-1)}(a, c) - f^{(i-1,j-1)}(b, d) - f^{(i-1,j-1)}(b, c)]
+ (b-a) \sum_{i=1}^{m} \frac{(2^{i+1} - 1)(c-d)^j}{(i+1)!} B_{j+1} \times [f^{(i-1,0)}(a, d) + f^{(i-1,0)}(a, c) + f^{(i-1,0)}(b, d) + f^{(i-1,0)}(b, c)], \quad \text{(40)}
\]

\[
B(f, P_m, Q_n) = \frac{2}{} \sum_{j=1}^{n} \frac{(1-2^{j+1})(c-d)^j}{(j+1)!} B_{j+1} \int_a^b [f^{(0,j-1)}(t, c) + f^{(0,j-1)}(t, d)] \, dt
+ \sum_{j=2}^{n} \frac{(a-b)^j}{j!} B_j \int_c^d [f^{(i-1,0)}(a, s) - f^{(i-1,0)}(b, s)] \, ds
+ \frac{b-a}{2} \int_c^d [f(a, s) + f(b, s)] \, ds, \quad \text{(41)}
\]

and

\[
R(f, P_m, Q_n) = \left(\frac{a-b)^m(c-d)^n}{m! \cdot n!} \int_a^b \int_c^d B_m \left(\frac{t-a}{b-a}\right) E_n \left(\frac{s-c}{d-c}\right) f^{(m,n)}(t, s) \, ds \, dt. \quad \text{(42)}
\]
3. Estimates of the Remainders

In this section, we will give some estimates for the remainders of expansions in Theorem 2 and Theorem 3.

We firstly need to introduce some notation. For a function $\ell : [a, b] \times [c, d] \rightarrow \mathbb{R}$, then for any $x, y \in [a, b]$, $u, v \in [c, d]$ we define

$$\|\ell\|_{[x,y]\times[u,v],\infty} := \text{ess sup} \{ |\ell(t,s)| \},$$

$$t \in [x, y] \text{ or } [y, x] \text{ and } s \in [u, v] \text{ or } [v, u]$$

and

$$\|\ell\|_{[x,y]\times[u,v],p} := \left( \int_{x}^{y} \int_{u}^{v} |h(t,s)|^{p} \, ds \, dt \right)^{\frac{1}{p}}, \quad p \geq 1.$$

The following result establishing bounds for the remainder in the Taylor-like formula (18) holds.

**Theorem 4.** Assume that $\{P_i(t,x)\}_{i=0}^{\infty}$, $\{Q_j(s,y)\}_{j=0}^{\infty}$ and $f$ satisfy the assumptions of Theorem 2. Then we have the representation (18) and the remainder satisfies the estimate

$$|T(f, P_m, Q_n)| \leq \begin{cases} 
\|P_m(\cdot, x)\|_{[a,x],\infty} \|Q_n(\cdot, y)\|_{[c,y],\infty} \|f^{(m+1,n+1)}\|_{[a,x]\times[c,y],1}, \\
\|P_m(\cdot, x)\|_{[a,x],p} \|Q_n(\cdot, y)\|_{[c,y],p} \|f^{(m+1,n+1)}\|_{[a,x]\times[c,y],p}, \\
\text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\
\|P_m(\cdot, x)\|_{[a,x],1} \|Q_n(\cdot, y)\|_{[c,y],1} \|f^{(m+1,n+1)}\|_{[a,x]\times[c,y],\infty}. 
\end{cases}$$

(43)

The proof follows in a straightforward fashion on using Hölder’s inequality applied for the integral representation of the remainder $T(f, P_m, Q_n)$ provided by equation (22). We omit the details.

The integral remainder in the cubature formula (34) may be estimated in the following manner.

**Theorem 5.** Assume that $\{P_i(t,x)\}_{i=0}^{\infty}$, $\{Q_j(s,y)\}_{j=0}^{\infty}$ and $f$ satisfy the assumptions in Theorem 3. Then one has the cubature formula (34) and, the remainder $R(f, P_m, Q_n)$ satisfies the estimate:

$$|R(f, P_m, Q_n)| \leq \begin{cases} 
\|P_m(\cdot, b)\|_{[a,b],\infty} \|Q_n(\cdot, d)\|_{[c,d],\infty} \|f^{(m,n)}\|_{[a,b]\times[c,d],1}, \\
\|P_m(\cdot, b)\|_{[a,b],p} \|Q_n(\cdot, d)\|_{[c,d],p} \|f^{(m,n)}\|_{[a,b]\times[c,d],p}, \\
\text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\
\|P_m(\cdot, b)\|_{[a,b],1} \|Q_n(\cdot, d)\|_{[c,d],1} \|f^{(m,n)}\|_{[a,b]\times[c,d],\infty}. 
\end{cases}$$

(44)

**Remark 1.** If we consider the particular instances of Appell polynomials provided by (11), (12) and (13), then a number of particular formulae may be obtained. Their remainder may be estimated by the use of Theorems 4 and 5, providing a 2-dimensional version of the results in [4].
For instance, if we consider from (11),

\[ P_{m,\lambda}(t, x; a) = \frac{(t - (\lambda a + (1 - \lambda) x))^m}{m!} \]  

(45)

\[ Q_{n,\mu}(s, y; c) = \frac{(s - (\mu c + (1 - \mu) y))^n}{n!} \]  

(46)

then we obtain the following result.

**Theorem 6.** Let \( \{P_{m,\lambda}(t, x; a)\}_{m=0}^{\infty} \) and \( \{Q_{n,\mu}(s, y; c)\}_{n=0}^{\infty} \) and \( f \) satisfy the assumptions of Theorem 2. Then we have the representation (18) and the remainder satisfies for \( a \leq x, c \leq y, \) the estimate

\[
|T(f, P_{m,\lambda}, Q_{n,\mu})| \leq \begin{cases} 
\frac{(x-a)^m(y-c)^n}{m!n!} \lambda_\infty & |f^{(m+1,n+1)}|_{[a,x] \times [c,y], 1}, \\
\frac{1}{m!} \left[ \frac{(x-a)^{m+1}(y-c)^{n+1}}{(m+1)(n+1)} \right] \frac{1}{\eta} \lambda_p \mu_p & |f^{(m+1,n+1)}|_{[a,x] \times [c,y], q},
\end{cases}
\]  

where \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \);

\[
(47)
\]

where

\[ \lambda_1 = \left[ \lambda^{m+1} + (1 - \lambda)^{m+1} \right], \quad \lambda_p = \left[ \lambda^{mq+1} + (1 - \lambda)^{mq+1} \right]^{\frac{1}{p}} \text{ and } \lambda_\infty = \left[ \frac{1}{2} + \lambda - \frac{1}{2} \right]^m.
\]

and similar for \( \mu_1, \mu_p \) and \( \mu_\infty \)

**Proof.** Utilizing equations (45) and (46) and using Hölder’s inequality for double integrals and the properties of the modulus on equation (22), then we have that

\[
\left| \int_a^x \int_c^y T(f, P_{m,\lambda}, Q_{n,\mu}) \right| = \left| \int_a^x \int_c^y P_{m,\lambda}(t, x; a) Q_{n,\mu}(s, y; c) f^{(m+1,n+1)} ds dt \right| \leq \int_a^x \int_c^y \sup_{(t,s) \in [a,x] \times [c,y]} \left| P_{m,\lambda}(t, x; a) Q_{n,\mu}(s, y; c) \right| \left| f^{(m+1,n+1)} \right| ds dt \\
\leq \left( \int_a^x \int_c^y \left| P_{m,\lambda}(t, x; a) Q_{n,\mu}(s, y; c) \right|^q ds dt \right)^{\frac{1}{q}} \left( \int_a^x \int_c^y \left| f^{(m+1,n+1)} \right|^p ds dt \right)^{\frac{1}{p}} \quad \left( \frac{(x-a)^m(y-c)^n}{m!n!} \lambda_\infty \right),
\]  

(48)

Now, the result in equation (48) can be further simplified by the application of equations (45) and (46), given that,

\[ \alpha = (1 - \lambda) x + \lambda a \quad \text{and} \quad \beta = (1 - \mu) y + \mu c. \]
It then follows
\[
sup_{(t,s)\in[a,x] \times [c,y]} |P_{m,\lambda}(t, x; a)Q_{n,\mu}(s, y; c)|
= \sup_{t\in[a,c]} |P_{m,\lambda}(t, x; a)| \sup_{s\in[c,y]} |Q_{n,\mu}(s, y; c)|
= \max \left\{ \frac{(x-a)^m}{m!}, \frac{(x-a)^m}{m!} \right\} \times \max \left\{ \frac{(y-c)^n}{n!}, \frac{(y-c)^n}{n!} \right\}
= \frac{(x-a)^m (y-c)^n}{m!n!} \left[ \max\{(1-\lambda), \lambda\} \right]^m \times \left[ \max\{(1-\mu), \mu\} \right]^n
\]
giving the first inequality in (47) where we have used the fact that
\[
\max \{X, Y\} = \frac{X+Y}{2} + \frac{|X-Y|}{2}.
\]
Further, we have
\[
\left( \int_a^x \int_c^y |P_{m,\lambda}(t, x; a)Q_{n,\mu}(s, y; c)|^q \, ds \, dt \right)^{\frac{1}{q}}
= \left( \int_a^x |P_{m,\lambda}(t, x; a)|^q \, dt \right)^{\frac{1}{q}} \left( \int_c^y |Q_{n,\mu}(s, y; c)|^q \, ds \, dt \right)^{\frac{1}{q}}
= \frac{1}{m!n!} \left[ \int_a^\alpha (\alpha-t)^mq \, dt + \int_{\alpha}^x (t-\alpha)^mq \, dt \right]^{\frac{1}{q}}
\times \left[ \int_c^{\beta} (\beta-s)^nq \, ds + \int_{\beta}^y (s-\beta)^nq \, ds \right]^{\frac{1}{q}}
= \frac{1}{m!n!} \left[ \frac{(x-a)^{mq+1} (y-c)^{nq+1}}{(mq+1)(nq+1)} \right]^{\frac{1}{q}} \lambda_p \mu_p
\]
producing the second inequality in (47).
Finally,
\[
\int_a^x \int_c^y |P_{m,\lambda}(t, x; a)Q_{n,\mu}(s, y; c)| \, dt \, ds
= \int_a^x \left| \frac{(t-\alpha)^m}{m!} \right| \, dt \int_c^y \left| \frac{(s-\beta)^n}{n!} \right| \, ds
= \left[ \int_a^\alpha \frac{(\alpha-t)^m}{m!} \, dt + \int_{\alpha}^x \frac{(t-\alpha)^m}{m!} \, dt \right] \times \left[ \int_c^{\beta} \frac{(\beta-s)^n}{n!} \, ds + \int_{\beta}^y \frac{(s-\beta)^n}{n!} \, ds \right]
= \frac{(x-a)^{m+1} (y-c)^{n+1}}{(m+1)! (n+1)!} \left[ (1-\lambda)^{m+1} + \lambda^{m+1} \right] \times \left[ (1-\mu)^{n+1} + \mu^{n+1} \right]
gives the last inequality in (47). Thus the theorem is completely proved. 

Remark 2. By taking $\lambda = \mu = 0$ or 1, we recapture the result obtained by G. Hanna et al. in [3].

In a similar fashion, we can state the remainder $R(f, P_{m,\lambda}, Q_{n,\mu})$ estimate in the cubature formula (34) as in the following

**Theorem 7.** Let $\{P_{m,\lambda}(t, x; a)\}_{m=0}^{\infty}, \{Q_{n,\mu}(s, y)\}_{n=0}^{\infty}$ and $f$ satisfy the assumptions of Theorem 3, then the remainder $R(f, P_{m,\lambda}, Q_{n,\mu})$ estimate in the cubature formula (34) satisfies the following

$$|R(f, P_{m,\lambda}, Q_{n,\mu})| \leq \begin{cases} \frac{(b-a)^m(d-c)^n}{m!n!} \lambda \mu \|f(m,n)\|_{[a,b] \times [c,d], 1}, \\ \frac{1}{m!n!} \left[ \frac{(b-a)^{mq+1}(d-c)^{nq+1}}{(mq+1)(nq+1)} \right]^{\frac{1}{q}} \lambda \mu \|f(m,n)\|_{[a,b] \times [c,d], q}, \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$\frac{(b-a)^{m+1}(d-c)^{n+1}}{(m+1)!(n+1)!} \lambda_1 \mu_1 \|f(m,n)\|_{[a,b] \times [c,d], \infty}.$$ (49)

The proof is similar to the one in Theorem 6 applied on the interval $[a, b] \times [c, d]$, and we omit the details.

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**References**
