ON SOME ANALOGUES OF KY FAN-TYPE INEQUALITIES

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Abstract. We study the behavior of means under equal increments of their variables and we apply the results to Ky Fan-type inequalities and certain bounds for the differences of means. We also give a sharpening of Sierpiński's inequality and prove a Rado-type inequality.

1. Introduction

Let \( P_{n,r}(x) \) be the generalized weighted power means: \( P_{n,r}(x) = (\sum_{i=1}^{n} \omega_i x_i^r)^{\frac{1}{r}} \), where \( \omega_i > 0, 1 \leq i \leq n \) with \( \sum_{i=1}^{n} \omega_i = 1 \) and \( x = (x_1, x_2, \cdots, x_n) \). Here \( P_{n,0}(x) \) denotes the limit of \( P_{n,r}(x) \) as \( r \to 0^+ \). Unless specified, we always assume \( 0 \leq x_1 \leq x_2 \cdots \leq x_n, m = \min\{x_i\}, M = \max\{x_i\} \).

We denote \( \sigma_n = \sum_{i=1}^{n} \omega_i(x_i - A_n)^2 \).

To any given \( x, t \geq 0 \) we associate \( x' = (1 - x_1, 1 - x_2, \cdots, 1 - x_n) \), \( x_t = (x_1 + t, \cdots, x_n + t) \).

When there is no risk of confusion, we shall write \( P_{n,r} \) for \( P_{n,r}(x) \), \( P_{n,r,t} \) for \( P_{n,r}(x_t) \) and \( P'_{n,r} \) for \( P_{n,r}(x') \) if \( 1 - x_i \geq 0 \) for all \( i \).

We also define \( A_n = P_{n,1}, G_n = P_{n,0}, H_n = P_{n,-1} \) and similarly for \( A_{n,t}, G_{n,t}, H_{n,t}, A'_n, G'_n, H'_n \).

To simplify expressions, we define

\[
\Delta_{r,s,t,0} = \frac{P_{n,r,t} - P_{n,s,t}}{P_{n,r} - P_{n,s}}, \Delta'_{r,s} = \frac{P'_{n,r} - P'_{n,s}}{P_{n,r} - P_{n,s}}
\]

with \( \Delta_{r,s,t,0} = (\ln \frac{P_{n,r,t}}{P_{n,s,t}})/(\ln \frac{P_{n,r}}{P_{n,s}}) \). We also write \( \Delta_{r,s,t} \) for \( \Delta_{r,s,t,1} \). In order to include the case of equality for various inequalities in our discussions, for any given inequality, we define \( 0/0 \) to be the number which makes the inequality an equality.

Recently, the authors [8, 9] proved the following result:

Theorem 1.1. For \( r > s, m > 0, t \geq 0 \), the following inequalities are equivalent:

\[
\frac{r-s}{2m} \sigma_n \geq P_{n,r} - P_{n,s} \geq \frac{r-s}{2M} \sigma_n
\]

\[
\frac{M}{1-M} \geq \Delta_{r,s} \geq \frac{m}{1-m}
\]

\[
\frac{M}{t+m} \geq \Delta_{r,s,0} \geq \frac{m}{t+M}
\]

where in [1.3] we require \( M < 1 \).

D. Cartwright and M. Field [6] first proved the validity of (1.2) for \( r = 1, s = 0 \). For other extensions and refinements of (1.2), see [3], [7], [11], and [12]. (1.3) is commonly referred as the additive Ky Fan's inequality. We refer the reader to the survey article [2] and the references therein for an account of Ky Fan's inequality.

J. Aczél and Zs. Páles [1] proved \( \Delta_{1,s,t} \leq 1 \) for any \( s \neq 1 \). We can interpret their result as an assertion of the monotonicity of \( A_{n,t} - P_{n,s,t} \) as a function of \( t \). The asymptotic behavior of \( t(P_{n,r,t} - A_{n,t}) \) was studied by J. Brenner and B. Carlson [5] and in this paper, we will study the
monotonicities of \((t + M)(P_{n,r,t} - P_{n,s,t})\) and \((t + m)(P_{n,r,t} - P_{n,s,t})\) as functions of \(t\) for \(r = 1\) or \(s = 1\) and then apply the result to inequalities of the type \(1.2\).

The following inequality connecting three classical means \(\text{with } \omega_i = 1/n \text{ here}\) is due to P.F.Wang and W.L.Wang \(15\) (right-hand side inequality), H. Alzer, S. Ruscheweyh and L. Salinas \(4\) (left-hand side inequality):

\[
(1.5) \quad \left( \frac{H_n}{H_n'} \right)^{n-1} \frac{A_n}{A_n'} \leq \left( \frac{G_n}{G_n'} \right)^n \leq \left( \frac{A_n}{A_n'} \right)^{n-1} \frac{H_n}{H_n'}
\]

(1.5) was refined in \(8\) and in section \(5\) we will give a further refinement of the above inequality. We will also prove a Rado-type inequality in the last section.

2. A Few Lemmas

**Lemma 2.1.** Let \(J(x)\) be the smallest closed interval that contains all of \(x_i\) and \(f(x), g(x) \in C^2(J(x))\) be two twice differentiable functions, then

\[
\sum_{i=1}^{n} \omega_i f(x_i) - f(\sum_{i=1}^{n} \omega_i x_i) = \frac{f''(\xi)}{g''(\xi)}
\]

for some \(\xi \in J(x)\), provided that the denominator of the left-hand side is nonzero.

Lemma 2.1 and the following consequence of it are due to A.M.Mercer \(10\):

**Lemma 2.2.** For \(w > u, w \neq v, u \neq v, x_1 > 0\)

\[
\left| \frac{u}{w} - v \right| x_1^\omega - 1 \geq \left| \frac{P_{u,v} - P_{w,v}}{P_{u,w} - P_{w,v}} \right| x_1^\omega \geq \left| \frac{u}{w} - v \right| x_1^\omega
\]

with equality holding if and only if \(x_1 = \cdots = x_n\).

Apply Lemma 2.1 to \(f(x) = (t + x)^r, g(x) = x^r, r \neq 0\) and \(f(x) = \ln(t + x), g(x) = \ln x\) when \(r = 0\), we obtain

**Corollary 2.1.** For \(x_1 > 0\)

\[
\min\left(\frac{(t + x_1)^{r-2}}{x_1}, \frac{(t + x_1)^{r-2}}{x_1}\right) \leq \Delta_{r,1,t,r} \leq \max\left\{\left(\frac{t + x_1}{x_1}\right)^{r-2}, \left(\frac{t + x_1}{x_1}\right)^{r-2}\right\}
\]

We now give a generalization of the result of Aczél and Páles:

**Lemma 2.3.** Let \(r > s, t \geq 0, \alpha \leq 1\).

(i). For \(s \neq 1, \Delta_{1,s,t,\alpha} \leq 1\).

(ii). If \(\Delta_{r,s,t} \geq \frac{x_n}{t+x_n}, \text{ then } \Delta_{r,s,t,\alpha} \leq (\frac{x_n}{t+x_n})^{2-\alpha}\).

(iii). If \(\Delta_{r,s,t} \leq \frac{x_n}{t+x_n}, \text{ then } \Delta_{r,s,t,\alpha} \geq (\frac{x_n}{t+x_n})^{2-\alpha}\).

**Proof.** We will prove (i) for \(s < 1, \alpha \neq 0\), (ii) for \(0 < \alpha < 1\) and the other proofs are similar. For (i), let \(f(t) = A_{n,t}^\alpha - P_{n,s,t}^\alpha\), then

\[
f'(t) = \alpha(A_{n,t}^{\alpha-1} - P_{n,s,t}^{\alpha-1})\left(\frac{P_{n,s,t}}{P_{n,s,t}^{\alpha-1,1}}\right)^{1-s} \leq \alpha\left(\frac{A_{n,t}^{\alpha-1} - P_{n,s,t}^{\alpha-1}}{P_{n,s,t}^{\alpha-1,1}}\right)^{1-s} \leq 0, 0 < \alpha \leq 1
\]

The conclusion then follows from the monotonicity of \(f(t)\).

For (ii), let \(f(t) = (t + x_n)^{2-\alpha}(P_{n,r,t}^\alpha - P_{n,s,t}^\alpha), \text{ then it suffices to show } f'(0) \leq 0 \text{ or equivalently}

\[
(2 - \alpha)(P_{n,r}^\alpha - P_{n,s}^\alpha) \leq \alpha x_n(P_{n,s}^{\alpha-1,1-s} - P_{n,r}^{\alpha-1,1-s}) - P_{n,r}(P_{n,r}^{\alpha-1})^{1-r}
\]

We also have

\[
\frac{P_{n,s}^{1-\alpha}}{\alpha}(P_{n,r}^{\alpha} - P_{n,s}^{\alpha}) \leq P_{n,r} - P_{n,s} \leq x_n\left(\frac{P_{n,s}}{P_{n,s-1}}\right)^{1-s} - \left(\frac{P_{n,r}}{P_{n,r-1}}\right)^{1-r}
\]
where the first inequality above follows from the mean value theorem and the second inequality follows from $\Delta_{r,s,t} \leq \frac{x_a}{x_a+x_b}$. Similarly, by using the mean value theorem, we get

\[
(2.5) \quad \frac{P_n^{\alpha,r} - P_n^{\alpha,s}}{P_n^{\alpha,s} - P_n^{\alpha,t}} \leq \frac{\alpha}{1-\alpha} \frac{P_n^{\alpha,r}}{P_n^{\alpha,s}} \leq \frac{\alpha}{1-\alpha} x_n \left( \frac{P_n^{\alpha,r}}{P_n^{\alpha,s}} \right)^{1-r}
\]

where the last inequality follows from $P_n^{\alpha,r} = \sum_{i=1}^n \omega_i x_i^r \leq \sum_{i=1}^n \omega_i x_i x_i^{r-1} = x_n P_n^{\alpha,s}$. Now (ii) follows by rewriting (2.4), (2.5) as

\[
(2.6) \quad P_n^{\alpha,r} - P_n^{\alpha,s} \leq \alpha x_n \left( \frac{P_n^{\alpha,s}}{P_n^{\alpha,s-1}} \right)^{1-s} - \left( \frac{P_n^{\alpha,r}}{P_n^{\alpha,s-1}} \right)^{1-r}
\]

(2.7) \quad (1-\alpha)(P_n^{\alpha,r} - P_n^{\alpha,s}) \leq \alpha x_n \left( P_n^{\alpha,s} - P_n^{\alpha,s-1} \right) \left( \frac{P_n^{\alpha,r}}{P_n^{\alpha,s-1}} \right)^{1-r}

and adding (2.6) and (2.7).

The following two lemmas will be needed in section 5.

**Lemma 2.4.** Let $x, b, u, v, t$ be real numbers with $0 < x \leq b, u \geq 1, v \geq 1, t \geq 0$, then $f(u, v, x, b) \leq f(u, v, x + t, b + t)$ where

\[
f(u, v, x, b) = b^2 \left( \frac{u + v - 1}{ux +vb} + \frac{1}{x^2(u/x + v/b)} - \frac{1}{x} \right)
\]

with equality holding if and only if $x = b$ or $u = v = 1$ or $t = 0$.

**Proof.** Let $x < b, t > 0$ and $u > 1, v > 1$. Write $D(u, v, x, b, t) = f(u, v, x, b) - f(u, v, x + t, b + t),$ then

\[
D(u, v, x, b, t) = v(b - x) \left[ -\frac{(u - 1)b/x + (v - 1)}{(v + ux/b)(u + vx/b)} + \frac{(u - 1)(b + t)/(x + t) + (v - 1)}{(v + u(x + t)/(b + t))(u + v(x + t)/(b + t))} \right] < 0
\]

since $(x+t)/(b+t) \geq x/b$. Thus we conclude that $D(u, v, x, b, t) \leq 0$ for $0 < x \leq b, u \geq 1, v \geq 1$. □

We remark here from the proof of the Lemma 2.4, one finds $f(u, v, s, b) \leq 0$ and we have $D \leq 0$ as long as the condition $u + v \geq 2, u \geq 1, v \geq 0$ is satisfied, we don’t really need $v \geq 1$.

**Lemma 2.5.** Let $x, a, b, u, v, s, t$ be real numbers with $t \geq 0, 0 < x \leq a \leq b, u \geq 1, v \geq 1, u + v \geq 3$ and $0 \leq s \leq v$, then $g(u, s, v, x, a, b) \leq g(u, s, v, x + t, a + t, b + t)$ where

\[
g(u, s, v, x, a, b) = b^2 \left[ \frac{u + v - 1}{ux + sa + (v - s)b} + \frac{1}{x^2(u/x + s/a + (v - s)/b)} - \frac{1}{x} \right]
\]

with equality holding if and only if one of the following cases is true: 1. $x = a = b$; 2. $s = 0, x = b$; 3. $t = 0$.

**Proof.** We may assume $t > 0$ and let $M = \{(s, a) \in R^2 | 0 \leq s \leq v, x \leq a \leq b\}$. Furthermore, we define $H(s, a) = g(u, s, v, x, a, b) - g(u, s, v, x + t, a + t, b + t)$, where $(s, a) \in M$. It suffices to show $H(s, a) \leq 0$. Let $m = (s_0, a_0)$ be the point in which the absolute minimum of $H$ is reached. If $m$
is an interior point of $M$, then we obtain
\[
0 = \frac{1}{s} \frac{\partial H}{\partial a} - \frac{1}{a-b} \frac{\partial H}{\partial s}\bigg|_{(s,a) = (s_0,a_0)} = \frac{(b-a)b/x}{x^2(a+b/(s+x/a+(v-s)x/b))^2} - \frac{(b-a)(b+t)/(x+t)}{(x+t)(a+t)((s+x)/(a+t)+(v-s)(x+t)/(b+t))^2} > 0
\]
where the inequality follows from $b/x > (b+t)/(x+t), (x+t)/(a+t) > x/a$. Hence, $m$ is a boundary point of $M$, so that we get $m \in \{(s_0,x),(s_0,b),(0,a_0),(v,a_0)\}$. Using Lemma 2.4 we obtain $H(s_0,b) = H(0,a_0) = D(u,v,x,b,t) \leq 0$ and
\[
H(s_0,x) = D(u+s_0,v-s_0,x,b,t) \leq 0
\]
The above inequality follows from the remark after the proof of the Lemma 2.4 since here $v-s_0 \geq 0$ but may not exceed 1. Finally,
\[
H(v,a_0) = b^2/a_0^2 f(u,v,x,a_0) - (b+t)^2/(a_0+t)^2 f(u,v,x+t,a_0+t) \leq 0
\]
The above inequality holds since $f(u,v,x,a_0) \leq f(u,v,x+t,a_0+t) \leq 0$ by the remark after the proof of the Lemma 2.4 and $b/a_0 \geq (b+t)/(a_0+t)$. Thus if $(s,a) \in M$, then $H(s,a) \leq 0$.

The conditions for equality can be easily checked by using Lemma 2.3 and noticing the condition $u+v \geq 3$.

\section{The Main Theorem}

\begin{theorem}
For $t \geq 0, x_1 > 0, -1 \leq s \neq 1 \leq 2$
\begin{equation}
\frac{x_1}{t+x_1} \leq \Delta_{1,s,t} \leq \frac{x_n}{t+x_n}
\end{equation}
with equality holding if and only if $t = 0$ or $x_1 = \cdots = x_n$.
\end{theorem}

The case $s = 0$ has been treated in \cite{Peng Gao} so we will assume $s \neq 0$ and prove the left-hand side inequality of (3.1) and the other proofs are similar. For $0 < s < 1$, let
\[
D_n(x,t) = x_n(A_n - P_{n,s}) - (t+x_n)(A_{n,t} - P_{n,s,t})
\]
We want to show $D_n \geq 0$ here. We can assume $x_1 < x_2 < \cdots < x_n$ and prove by induction, the case $n = 1$ is clear so we will start with $n > 1$ variables assuming the inequality holds for $n - 1$ variables. Then
\[
\frac{\partial D_n}{\partial x_n} = (A_n - P_{n,s}) - (A_{n,t} - P_{n,s,t}) + \omega_n[(A_n - P_{1,s,s}^1 - s) - (A_{n,t} - P_{1,s,s}^1 - s(t+x_n)]]
\]
where the inequality follows from $\Delta_{1,s,t} \leq 1$. Now consider
\[
g(t) = P_{1,s,t}^1 - s(t+x_n)^s + P_{n,s,t} - 2t
\]
and have
\[
g'(t) = (1-s)(t+x_n)^s(P_{n,s,t})^1-s + s(P_{n,s,t})^{1-s}(P_{P_{n,s,t},1}-t) - 2
\]
where $y = \frac{t+x_n}{P_{n,s,t}} \geq 1$ and the inequality follows from $(P_{n,s,t}^1)_{1-s} \geq 1$. Note $h'(y) = 0$ has only one root $y = 1$, which implies $h(y) \geq \min{h(1), \lim_{y \to \infty} h(y)} = 0$. Thus $g'(t) \geq 0$, hence $g(t) \geq g(0) = P_{n,s} + P_{1,s,s}^1 - s$ and it follows $\frac{\partial D_n}{\partial x_n} \geq 0$ and by letting $x_n$ tend to $x_{n-1}$, we have $D_n \geq D_{n-1}$ (with weights $\omega_1, \cdots, \omega_{n-2}, \omega_{n-1} + \omega_n$) and thus the right-hand side inequality of (3.1) holds by induction. It is easy to see the equality holds if and only if $t = 0$ or $x_1 = \cdots = x_n$.\hfill\(\square\)
For $-1 \leq s < 0$, we have
\[
\frac{1}{\omega_1} \frac{\partial D_n}{\partial x_1} = -t - x_n \left( \frac{P_{n,s}}{x_1} \right)^{1-s} + (t + x_n) \left( \frac{P_{n,s,t}}{t + x_1} \right)^{1-s} := -t - f(x_1)
\]
Consider
\[
f'(x_1) = -(1 - s) \sum_{j=2}^{n} \omega_j \left( \frac{P_{n,s}}{x_1} \right)^{1-2s} \cdot \frac{x_n x_j}{x_1^{s+1}} - \left( \frac{P_{n,s,t}}{t + x_1} \right)^{1-2s} \left( \frac{(t + x_n)(t + x_j)^{s}}{(t + x_1)^{s+1}} \right) \leq 0
\]
The last inequality holds, since when $-1 \leq s < 0$, $j = 2, \ldots, n$, we have
\[
\left( \frac{P_{n,s}}{x_1} \right)^{1-2s} \geq \left( \frac{P_{n,s,t}}{t + x_1} \right)^{1-2s}, \quad \frac{x_n x_j}{x_1^{s+1}} \geq \frac{t + x_j}{t + x_1}, \quad \frac{x_n}{t + x_1} \cdot \left( \frac{x_j}{t + x_j} \right)^s \geq \left( \frac{x_j}{t + x_j} \right)^{1+s} \geq \left( \frac{x_1}{t + x_1} \right)^{1+s}
\]
Thus by a similar argument as above, we deduce $f(x_1) \geq -t$ and $\frac{\partial D_n}{\partial x_1} \leq 0$, which implies $D_n \geq 0$ with equality holding if and only if $t = 0$ or $x_1 = \cdots = x_n$.

For $1 < s \leq 2$, it suffices to show $\frac{\partial D_n}{\partial x_1} \leq 0$, which is equivalent to
\[
\frac{P_{n,s}^{s-1}}{x_n} \leq \frac{(P_{n,s}^{s-1} - P_{n,s-1}^{s-1})}{(P_n - A_n)}
\]
The above inequality follows from $\frac{P_{n,s}^{s-1}}{x_n} \leq x_n^{s-2}$ and Lemma 2.2 with $u = s - 1, v = s, w = 1$. □

4. Some Consequences of Theorem 3.1

**Corollary 4.1.** (1.2) holds for $r = 1, -1 \leq s < 1$ and $1 < r \leq 2, s = 1$.

*Proof.* This follows from Theorems 3.1 and 1.1. □

The above result was first proved by the author in [8], in fact it was shown there those are the only cases (1.2) can hold for $r = 1$ or $s = 1$. Thus by Theorem 1.1 we have

**Corollary 4.2.** (3.1) holds for all $t \geq 0$ if and only if $-1 \leq s \neq 1 \leq 2$.

**Corollary 4.3.** For $-1 \leq s < 1$

\[
(4.1) \quad \frac{x_1}{P_{n,s-1}^{1-s}} \leq \frac{(A_n - P_{n,s})}{(P_{n,s}^{1-s} - P_{n,s-1}^{1-s})} \leq \frac{x_n}{P_{n,s-1}^{1-s}}
\]

*Proof.* Theorem 3.1 implies $f(t) = (t + x_n)(A_n, t - P_{n,s,t})$ is a decreasing function of $t$ and $f'(0) \leq 0$ implies the right-hand side inequality of (4.1) and the proof of the left-hand side inequality of (4.1) is similar. □

By a change of variables $x_i \to 1/x_i$ and let $x_1 = m > 0$, the right-hand side inequality of (4.1) when $s = -1$ gives

\[
(4.2) \quad A_n - H_n \leq \frac{H_n}{x_1 A_n} \sigma_n
\]
a refinement of the left-hand side inequality of (1.2) for $r = 1, s = -1$. We note here one can use the method in [9] to give a direct proof of (4.2) and show the equality holds if and only if $x_1 = \cdots = x_n$. We will leave the details to the reader.
5. A sharpening of Sierpiński’s inequality

Theorem 5.1. For $0 < x_1 \leq \cdots \leq x_n, t \geq 0$, $q = \min\{\omega_i\}$

\begin{align}
(5.1) & \quad \left(\frac{x_n}{x_n + t}\right)^2 \geq \frac{(1 - q)\ln A_{n,t} + q\ln H_{n,t} - \ln G_{n,t}}{(1 - q)\ln A_n + q\ln H_n - \ln G_n} \geq \left(\frac{x_1}{x_1 + t}\right)^2 \\
(5.2) & \quad \left(\frac{x_n}{x_n + t}\right)^2 \geq \frac{\ln G_{n,t} - q\ln A_{n,t} - (1 - q)\ln H_{n,t}}{\ln G_n - q\ln A_n - (1 - q)\ln H_n} \geq \left(\frac{x_1}{x_1 + t}\right)^2
\end{align}

with equality holding if and only if $t = 0$ or $q = 1/2$ or $x_1 = \cdots = x_n$.

Proof. The proof uses the ideas in [4]. We will prove the left-hand side inequality of (5.1) and the proofs for other inequalities are similar. We may assume $t > 0$ being fixed and $q > 0, 0 < x = x_1, x_n = b$ with $x_1 < x_n$, we define

$$f_n(x_n, q) = x_n^2[(1 - q)\ln A_n + q\ln H_n - \ln G_n] - (x_n + t)^2[(1 - q)\ln A_{n,t} + q\ln H_{n,t} - \ln G_{n,t}]$$

where we regard $A_n, G_n, H_n, A_{n,t}, G_{n,t}, H_{n,t}$ as functions of $x_n = (x_1, \ldots, x_n)$. Then

$$g_n(x_2, \ldots, x_{n-1}) := \frac{1}{\omega_1}\frac{\partial f_n}{\partial x_1} = x_n^2\left(\frac{1 - q}{A_n} + \frac{qH_n}{x_1^2} - \frac{1}{x_1}\right) - (x_n + t)^2\left(\frac{1 - q}{A_{n,t}} + \frac{qH_{n,t}}{(x_1 + t)^2} - \frac{1}{x_1 + t}\right)$$

We want to show $g_n \leq 0$. Let $D = \{(x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-2}|0 < x \leq x_2 \leq \cdots \leq x_{n-1} \leq b\}$. Let $a = (a_2, \cdots, a_{n-1}) \in D$ be the point in which the absolute minimum of $g_n$ is reached. Next, we show that

$$a = (x, \ldots, x, a, \ldots, a, b, \ldots, b) \text{ with } x < a < b$$

where the numbers $x, a$, and $b$ appear $u, v, w$ times, respectively, with $u, v, w \geq 0, u + v + w = n - 2$.

Suppose not, this implies two components of $a$ have different values and are interior points of $D$. We denote these values by $a_k$ and $a_l$. Partial differentiation shows $a_l, a_l$ are the roots of

$$\frac{\partial h(x)}{\partial x} = B \frac{x^2}{x^2} - \frac{B'}{(x + t)^2} + C = 0$$

where

$$B = q\frac{H_n^2}{x_1^2}, B' = q\frac{H_n^2H_{n,t}(x_n + t)^2}{(x_1 + t)^2}, C = \frac{(1 - q)(x_n + t)^2}{A_{n,t}} - \frac{(1 - q)x_n^2}{A_n}$$

It’s easy to show $h'(x)$ only has one positive root, which implies $h(x)$ can have at most two distinct positive roots, but $\lim_{x \to 0} h(x) = \infty, \lim_{x \to \infty} h(x) = C < 0$ implies $h(x)$ can have at most one positive root. Thus (5.4) yields $a_k = a_l$. This contradicts our assumption that $a_k \neq a_l$. Thus (5.3) is valid and it suffices to show $g_n \leq 0$ for the cases $n = 2, 3$.

When $n = 2$, by setting $x_1 = x, x_2 = b, \omega_1/q = u, \omega_2/q = v, g_2 \leq 0$ follows from Lemma 2.4.

When $n = 3$, by setting $x_1 = x, x_2 = a, x_3 = b, \omega_1/q = u, \omega_2/q = s, \omega_3/q = v - s, g_3 \leq 0$ follows from Lemma 2.5.

Thus we have shown that $g_n = \frac{1}{\omega_1}\frac{\partial f_n}{\partial x_1} \leq 0$ with equality holding if and only if $n = 1$ or $n = 2, q = 1/2$. By letting $x_1$ tend to $x_2$, we have

$$f_n(x_n, q) \geq f_{n-1}(x_{n-1}, q) \geq f_{n-1}(x_{n-1}, q')$$

where $x_{n-1} = (x_2, \cdots, x_n)$ with weights $\omega_1 + \omega_2, \cdots, \omega_{n-1}, \omega_n$ and $q' = \min\{\omega_1 + \omega_2, \cdots, \omega_n\}$. Here we have used $\Delta_{1,-1,0} \leq \left(\frac{\Delta_n}{x_n}\right)^2$, which is a consequence of Theorem 3.1 and Lemma 2.3.

It then follows by induction that $f_n \geq f_{n-1} \cdots \geq f_2 = 0$ when $q = 1/2$ in $f_2$ or else $f_n \geq f_{n-1} \cdots \geq f_1 = 0$ and this completes the proof. \qed
By letting $t \to \infty$ in (5.1), (5.2), we recover the following result of the author [8], which can be regarded as sharpenings of Sierpiński’s inequality [13] for the weighted cases:

**Corollary 5.1.** For $0 < x_1 \leq \cdots \leq x_n$, $q = \min\{\omega_i\}$

\[
\begin{align*}
(5.5) & \quad \frac{1 - 2q}{2x_i^2} \sigma_n \geq (1 - q) \ln A_n + q \ln H_n - \ln G_n \geq \frac{1 - 2q}{2x_i^2} \sigma_n \\
(5.6) & \quad \frac{1 - 2q}{2x_i^2} \sigma_n \geq \ln G_n - q \ln A_n - (1 - q) \ln H_n \geq \frac{1 - 2q}{2x_i^2} \sigma_n
\end{align*}
\]

with equality holding if and only if $q = 1/2$ or $x_1 = \cdots = x_n$.

6. A Rado-Type Inequality

By letting $\omega_i = q_i/Q_n$, $Q_n = \sum_{i=1}^{n} q_i$, $q_i > 0$ (note for different $n$, $\omega_i$’s take different values), C.L. Wang [14] proved the following Rado-type inequality:

**Theorem 6.1.** If $x_i \in (0, 1/2]$, $i = 1, \cdots, n$, then

\[
Q_n(A_nG_n - A_n'G_n) \geq Q_{n-1}(A_{n-1}G_{n-1} - A_{n-1}'G_{n-1})
\]

We end the paper by giving an analogue of Wang’s theorem:

**Theorem 6.2.** For $t > 0$, $q_i > 0$, $i = 1, \cdots, n$

\[
Q_n(A_nG_{n,t} - A_n,tG_n) \geq Q_{n-1}(A_{n-1}G_{n-1,t} - A_{n-1,t}G_{n-1})(A_{n-1,t} - A_{n-1}) \frac{G_{n-1}}{G_{n-1,t} - G_{n-1}}
\]

**Proof.** Let $f(x_n) = Q_n(A_nG_{n,t} - A_n,G_n)$, by setting

\[
f'(x_n) = q_n(x_n + A_{n,t})(\frac{G_{n,t}}{\ell + x_n} - \frac{G_n}{x_n}) = 0
\]

we get $x_n = tG_{n-1}/(G_{n-1,t} - G_{n-1})$. Moreover, at this point

\[
f''(x_n) = \frac{q_nQ_{n-1}G_n}{Q_n} \frac{A_{n,t}}{x_n} \frac{A_n}{\ell + x_n} > 0
\]

and it is easy to see that $f(x_n)$ takes its absolute minimum at the point, which implies

\[
f(x_n) \geq f\left(\frac{tG_{n-1}}{G_{n-1,t} - G_{n-1}}\right) = Q_{n-1}(A_{n-1}G_{n-1,t} - A_{n-1,t}G_{n-1})(A_{n-1,t} - A_{n-1}) \frac{G_{n-1}}{G_{n-1,t} - G_{n-1}}
\]

for any $x_n \geq 0$, with equality holding if and only if $x_n = tG_{n-1}/(G_{n-1,t} - G_{n-1})$.

We note here by letting $t \to \infty$ in (6.2), we get back Rado’s inequality:

\[
Q_n(A_n - G_n) \geq Q_{n-1}(A_{n-1} - G_{n-1})
\]

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References


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