MONOTONICITY AND INEQUALITIES FOR RATIO OF THE GENRALIZED LOGARITHMIC MEANS

FENG QI AND CHAO-PING CHEN

Abstract. Let $c > b > a > 0$ be real numbers. Then the function $f(r) = \frac{L_r(a,b)}{L_r(a,c)}$ is strictly decreasing on $(-\infty, \infty)$, where $L_r(a,b)$ denotes the generalized (extended) logarithmic mean of two positive numbers a and b.

1. Introduction

If $-\infty < p < \infty$ and a, b are two positive numbers, the generalized (extended) logarithmic mean $L_p(a,b)$ of a and b is defined for $a = b$ by $L_p(a,b) = a$ and for $a \neq b$ by

$$L_p(a,b) = \begin{cases} \left(\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, & p \neq -1, 0; \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1; \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases} \tag{1}$$

the case $p = -1$ is called the logarithmic mean of a and b, and will be written $L(a,b)$; while the case $p = 0$ is the identric mean of a and b, written $I(a,b)$.

This definition of the generalized logarithmic mean can be found in [2, p. 6] and [33, 34].

It is well known that if $r > 0$ is a real number, then for all natural numbers n

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{2}$$

2000 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Monotonicity, inequality, ratio, the generalized (extended) logarithmic mean, identric (exponential) mean.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, Doctor Fund of Jiaozuo Institute of Technology, CHINA.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

The first inequality in (2) is called H. Alzer's inequality [1], and the second one in (2) J. S. Martins' inequality [11]. The inequality between two ends of (2) is called Minc-Sathre's inequality [12].

There exists a very rich literature on inequality (2). Alzer's inequality has been generalized and extended, for example, in [4, 5, 6, 7, 10, 14, 15, 16, 17, 22, 24, 25, 30, 31, 32, 35, 36, 37]. So does Martins's inequality in [3, 5, 17, 21, 23, 25, 26, 27, 29, 35, 37, 38] and Minc-Sathre's inequality in [1, 5, 9, 18, 19, 20, 25, 27, 28], respectively.

Recently, F. Qi and B.-N. Guo proved in [15, 23] the following double inequality: Let $b > a > 0$ and $\delta > 0$, then for any positive real number r,

$$\frac{b}{b+\delta} < \left(\frac{\frac{1}{b-a}\int_a^b x^r \,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \,\mathrm{d}x} \right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{3}$$

The upper and lower bounds in (3) are the best possible, or more accurately say,

$$\lim_{r\to\infty} \left(\frac{\frac{1}{b-a}\int_a^b x^r \,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \,\mathrm{d}x} \right)^{1/r} = \frac{b}{b+\delta}, \tag{4}$$

$$\lim_{r\to 0} \left(\frac{\frac{1}{b-a}\int_a^b x^r \,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \,\mathrm{d}x} \right)^{1/r} = \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{5}$$

Inequality (3) can be taken for an integral form of (2).

It is easy to see that inequality (3) can be written for $r > 0$ as

$$\frac{b}{b+\delta} < \frac{L_r(a,b)}{L_r(a,b+\delta)} < \frac{I(a,b)}{I(a,b+\delta)}. \tag{6}$$

In this short note, we are about to extend the result presented by (3) to (5) which are established in [15, 23] by F. Qi and B.-N. Guo, and obtain the following

Theorem 1. *Let $c > b > a > 0$ be real numbers. Then the function*

$$f(r) = \frac{L_r(a,b)}{L_r(a,c)} \tag{7}$$

is strictly decreasing with $r \in (-\infty, \infty)$.

The following corollary is straightforward.

Corollary 1. *Let $c > b > a > 0$ be real numbers.*

(1) *For any real number* $r \in \mathbb{R}$,

$$\frac{b}{c} < \frac{L_r(a,b)}{L_r(a,c)} < 1. \tag{8}$$

The both bounds in (8) *are the best possible.*

(2) *For any positive real number* $r > 0$,

$$\frac{b}{c} < \frac{L_r(a,b)}{L_r(a,c)} < \frac{I(a,b)}{I(a,c)}. \tag{9}$$

The both bounds in (9) *are also the best possible.*

Remark 1. It is worthwhile pointing out that inequalities (3) and (9) are equivalent each other.

In [29] it was conjectured that the function

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r}\right)^{1/r} \tag{10}$$

is decreasing with r. Now it is still keep open. We can regard Theorem 1 as a solution to an integral form of the conjecture above.

2. Proof of Theorem 1

In order to verify Theorem 1, we shall make use of the following elementary lemma which can be found in [8, p. 395].

Lemma 1 ([8, p. 395])**.** *Let the second derivative of* $\phi(x)$ *be continuous with* $x \in (-\infty, \infty)$ *and* $\phi(0) = 0$. *Define*

$$g(x) = \begin{cases} \dfrac{\phi(x)}{x}, & x \neq 0; \\ \phi'(0), & x = 0. \end{cases} \tag{11}$$

Then $\phi(x)$ *is (strictly) convex if and only if* $g(x)$ *is (strictly) increasing with* $x \in (-\infty, \infty)$.

Remark 2. In [13, p. 18] a general conclusion was given: A function f is convex on $[a, b]$ if and only if $\frac{f(x)-f(x_0)}{x-x_0}$ is nondecreasing on $[a, b]$ for every point $x_0 \in [a, b]$.

Proof of Theorem 1. Define for $r \in (-\infty, \infty)$

$$\varphi(r) = \begin{cases} \ln\left(\dfrac{c-a}{b-a} \cdot \dfrac{b^{r+1} - a^{r+1}}{c^{r+1} - a^{r+1}}\right), & r \neq -1; \\ \ln\left(\dfrac{c-a}{b-a} \cdot \dfrac{\ln b - \ln a}{\ln c - \ln a}\right), & r = -1. \end{cases} \tag{12}$$

Then we have

$$\ln f(r) = \begin{cases} \dfrac{\varphi(r)}{r}, & r \neq 0, \\ \varphi'(0), & r = 0. \end{cases} \tag{13}$$

In order to prove that $\ln f(r)$ is strictly decreasing it suffices to show that φ is strictly concave in $(-\infty, \infty)$. Easy computation reveals that

$$\varphi(-1 - r) = \varphi(r - 1) + r \ln \frac{c}{b}, \tag{14}$$

which implies that $\varphi''(-r - 1) = \varphi''(r - 1)$, and then $\varphi(r)$ has the same concavity on both $(-\infty, -1)$ and $(-1, \infty)$. Hence, it is sufficient to prove that φ is strictly concave on $(-1, \infty)$.

A simple computation yields

$$\varphi''(r) = \frac{(a/c)^{r+1}[\ln(a/c)]^2}{[1 - (a/c)^{r+1}]^2} - \frac{(a/b)^{r+1}[\ln(a/b)]^2}{[1 - (a/b)^{r+1}]^2}. \tag{15}$$

Define for $0 < t < 1$

$$\omega(t) = \frac{t(\ln t)^2}{(1 - t)^2}. \tag{16}$$

Differentiation yields

$$(1 - t)t \ln t \frac{\omega'(t)}{\omega(t)} = (1 + t) \ln t + 2(1 - t) = -\sum_{n=2}^{\infty} \frac{n - 1}{n(n + 1)} t^{n+1} < 0, \tag{17}$$

which means that $\omega'(t) > 0$ for $0 < t < 1$. As a result of applying this conclusion in (15), we obtain $\varphi''(r) < 0$ for $r > -1$. Thus $\varphi(r)$ is strictly concave in $(-1, \infty)$. The proof is complete. □

References

[1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.

[2] P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, 1998.

[3] T. H. Chan, P. Gao and F. Qi, *On a generalization of Martins' inequality*, Monatsh. Math. (2003), in press. RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 12, 93–101. Available online at http://rgmia.vu.edu.au/v4n1.html.

[4] Ch.-P. Chen and F. Qi, *Notes on proofs of Alzer's inequality*, Octogon Math. Mag. (2003), accepted.

[5] Ch.-P. Chen, F. Qi, P. Cerone, and S. S. Dragomir, *Monotonicity of sequences involving convex and concave functions*, Math. Inequal. Appl. **6** (2003), no. 2, in press. RGMIA Res. Rep. Coll. **5** (2002), no. 1, Art. 1, 3–13. Available online at http://rgmia.vu.edu.au/v5n1.html.

[6] N. Elezović and J. Pečarić, *On Alzer's inequality*, J. Math. Anal. Appl. **223** (1998), 366–369.

[7] B.-N. Guo and F. Qi, *An algebraic inequality, II*, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 8, 55–61. Available online at http://rgmia.vu.edu.au/v4n1.html.

[8] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 2nd ed., Hunan Education Press, Changsha, China, 1993. (Chinese)

[9] J.-Ch. Kuang, *Some extensions and refinements of Minc-Sathre inequality*, Math. Gaz. **83** (1999), 123–127.

[10] Zh. Liu, *New generalization of H. Alzer's inequality*, Tamkang J. Math. **34** (2003), accepted.

[11] J. S. Martins, *Arithmetic and geometric means, an applications to Lorentz sequence spaces*, Math Nachr. **139** (1988), 281–288.

[12] H. Minc and L. Sathre, *Some inequalities involving* $(r!)^{1/r}$, Proc. Edinburgh Math. Soc. **14** (1964/65), 41–46.

[13] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[14] N. Ozeki, *On some inequalities*, J. College Arts Sci. Chiba Univ. **4** (1965), no. 3, 211–214. (Japanese)

[15] F. Qi, *An algebraic inequality*, J. Inequal. Pure Appl. Math. **2** (2001), no. 1, Art. 13. Available online at http://jipam.vu.edu.au/v2n1/006_00.html. RGMIA Res. Rep. Coll. **2** (1999), no. 1, Art. 8, 81–83. Available online at http://rgmia.vu.edu.au/v2n1.html.

[16] F. Qi, *Generalizations of Alzer's and Kuang's inequality*, Tamkang J. Math. **31** (2000), no. 3, 223–227. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Art. 12, 891–895. Available online at http://rgmia.vu.edu.au/v2n6.html.

[17] F. Qi, *Generalization of H. Alzer's inequality*, J. Math. Anal. Appl. **240** (1999), 294–297.

[18] F. Qi, *Inequalities and monotonicity of sequences involving* $\sqrt[n]{(n+k)!/k!}$, RGMIA Res. Rep. Coll. **2** (1999), no. 5, Art. 8, 685–692. Available online at http://rgmia.vu.edu.au/v2n5.html.

[19] F. Qi, *Inequalities and monotonicity of the ratio for the geometric means of a positive arithmetic sequence with arbitrary difference*, Tamkang. J. Math. **34** (2003), no. 3, in press.

[20] F. Qi, *Inequalities and monotonicity of the ratio for the geometric means of a positive arithmetic sequence with unit difference*, Internat. J. Math. Ed. Sci. Tech. **34** (2003), in press.

Australian Math. Soc. Gaz. **30** (2003), no. 2, in press. RGMIA Res. Rep. Coll. **6** (2003), suppl., Art. 2. Available online at http://rgmia.vu.edu.au/v6(E).html.

[21] F. Qi, *On a new generalization of Martins' inequality*, RGMIA Res. Rep. Coll. **5** (2002), no. 3, Art. 13, 527–538. Available online at http://rgmia.vu.edu.au/v5n3.html.

[22] F. Qi and L. Debnath, *On a new generalization of Alzer's inequality*, Internat. J. Math. Math. Sci. **23** (2000), no. 12, 815–818.

[23] F. Qi and B.-N. Guo, *An inequality between ratio of the extended logarithmic means and ratio of the exponential means*, Taiwanese J. Math. **7** (2003), no. 2, accepted. RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 8, 55–61. Available online at http://rgmia.vu.edu.au/v4n1.html.

[24] F. Qi and B.-N. Guo, *A lower bound for ratio of power means*, Internat. J. Math. Math. Sci. (2003), accepted. RGMIA Res. Rep. Coll. **5** (2002), no. 4, Art. 2. Available online at http://rgmia.vu.edu.au/v5n4.html.

[25] F. Qi and B.-N. Guo, *Monotonicity of sequences involving convex function and sequence*, RGMIA Res. Rep. Coll. **3** (2000), no. 2, Art. 14, 321–329. Available online at http://rgmia.vu.edu.au/v3n2.html.

[26] F. Qi and B.-N. Guo, *Monotonicity of sequences involving geometric means of positive sequences with logarithmical convexity*, RGMIA Res. Rep. Coll. **5** (2002), no. 3, Art. 10, 497–507. Available online at http://rgmia.vu.edu.au/v5n3.html.

[27] F. Qi and B.-N. Guo, *Some inequalities involving the geometric mean of natural numbers and the ratio of gamma functions*, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 6, 41–48. Available online at http://rgmia.vu.edu.au/v4n1.html.

[28] F. Qi and Q.-M. Luo, *Generalization of H. Minc and J. Sathre's inequality*, Tamkang J. Math. **31** (2000), no. 2, 145–148. RGMIA Res. Rep. Coll. **2** (1999), no. 6, Art. 14, 909–912. Available online at http://rgmia.vu.edu.au/v2n6.html.

[29] J. A. Sampaio Martins, *Inequalities of Rado-Popoviciu type*, In: Marques de Sá, Eduardo (ed.) et al. Mathematical studies. Homage to Professor Doctor Luís de Albuquerque. Coimbra: Universidade de Coimbra, Faculdade de Ciências e Tecnologia, Departamento de Matemática, (1994), 169–175.

[30] J. Sándor, *Comments on an inequality for the sum of powers of positive numbers*, RGMIA Res. Rep. Coll. **2** (1999), no. 2, 259–261. Available online at http://rgmia.vu.edu.au/v2n2.html.

[31] J. Sándor, *On an inequality of Alzer*, J. Math. Anal. Appl. **192** (1995), 1034–1035.

[32] J. Sándor, *On an inequality of Bennett*, General Mathematics (Sibiu), **3** (1995), no. 3-4, 121–125.

[33] K. B. Stolarsky, Generalizations of the logarithmic mean, Math. Mag. **48** (1975), 87–92.

[34] K. B. Stolarsky, The power and generalized logarithmic means, Amer. Math. Monthly, **87** (1980), 545–548.

[35] N. Towghi and F. Qi, *An inequality for the ratios of the arithmetic means of functions with a positive parameter*, RGMIA Res. Rep. Coll. **4** (2001), no. 2, Art. 15, 305–309. Available online at http://rgmia.vu.edu.au/v4n2.html.

[36] J. S. Ume, *An elementary proof of H. Alzer's inequality*, Math. Japon. **44** (1996), no. 3, 521–522.

[37] Z.-K. Xu, *On further generalization of an inequality of H. Alzer*, J. Zhejiang Norm. Univ. (Nat. Sci.) **25** (2002), no. 3, 217–220. (Chinese)

[38] Z.-K. Xu and D.-P. Xu, *A general form of Alzer's inequality*, Comput. Math. Appl. **44** (2002), no. 3-4, 365–373.

(F. Qi) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

E-mail address: qifeng@jzit.edu.cn or fengqi618@member.ams.org

URL: http://rgmia.vu.edu.au/qi.html

(Ch.-P. Chen) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA