On the Bombieri Inequality in Inner Product Spaces

ON THE BOMBIERI INEQUALITY IN INNER PRODUCT SPACES

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Abstract. New results related to the Bombieri generalisation of Bessel's inequality in inner product spaces are given.

1. Introduction

Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\leq i\leq n}$ are orthonormal vectors in the inner product space H, i.e., $(e_i,e_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well known in the literature as Bessel's inequality (see for example [8, p. 391]):

$$\sum_{i=1}^{n}|(x,e_i)|^2\leq\|x\|^2 \quad \text{for any } x\in H. \tag{1.1}$$

For other results related to Bessel's inequality, see [4] – [6] and Chapter XV in the book [8].

In 1971, E. Bombieri [3] (see also [8, p. 394]) gave the following generalisation of Bessel's inequality.

Theorem 1. *If $x,y_1,\dots,y_n$ are vectors in the inner product space $(H;(\cdot,\cdot))$, then the following inequality:*

$$\sum_{i=1}^{n}|(x,y_i)|^2\leq\|x\|^2\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|(y_i,y_j)|\right\}, \tag{1.2}$$

holds.

It is obvious that if $(y_i)_{1\leq i\leq n}$ are supposed to be orthonormal, then from (1.2) one would deduce Bessel's inequality (1.1).

Another generalisation of Bessel's inequality was obtained by A. Selberg (see for example [8, p. 394]):

Theorem 2. *Let $x,y_1,\dots,y_n$ be vectors in H with $y_i\neq 0$ $(i=1,\dots,n)$. Then one has the inequality:*

$$\sum_{i=1}^{n}\frac{|(x,y_i)|^2}{\sum_{j=1}^{n}|(y_i,y_j)|}\leq\|x\|^2. \tag{1.3}$$

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In this case, also, if $(y_i)_{1\le i\le n}$ are orthonormal, then from (1.3) one may deduce Bessel's inequality.

Another type of inequality related to Bessel's result, was discovered in 1958 by H. Heilbronn [7] (see also [8, p. 395]).

Theorem 3. *With the assumptions in Theorem 1, one has*

$$\sum_{i=1}^{n}|(x,y_i)|\le\|x\|\left(\sum_{i,j=1}^{n}|(y_i,y_j)|\right)^{\frac{1}{2}}. \tag{1.4}$$

If in (1.4) one chooses $y_i=e_i$ $(i=1,\dots,n)$, where $(e_i)_{1\le i\le n}$ are orthonormal vectors in H, then

$$\sum_{i=1}^{n}|(x,e_i)|\le\sqrt{n}\,\|x\|,\quad\text{for any}\quad x\in H. \tag{1.5}$$

In 1992 J.E. Pečarić [9] (see also [8, p. 394]) proved the following general inequality in inner product spaces.

Theorem 4. *Let $x,y_1,\dots,y_n\in H$ and $c_1,\dots,c_n\in\mathbb{K}$. Then*

$$\begin{aligned}\left|\sum_{i=1}^{n}c_i(x,y_i)\right|^2&\le\|x\|^2\sum_{i=1}^{n}|c_i|^2\left(\sum_{j=1}^{n}|(y_i,y_j)|\right)\\&\le\|x\|^2\sum_{i=1}^{n}|c_i|^2\max_{1\le i\le n}\left\{\sum_{j=1}^{n}|(y_i,y_j)|\right\}.\end{aligned} \tag{1.6}$$

He showed that the Bombieri inequality (1.2) may be obtained from (1.6) for the choice $c_i=\overline{(x,y_i)}$ (using the second inequality), the Selberg inequality (1.3) may be obtained from the first part of (1.6) for the choice

$$c_i=\frac{\overline{(x,y_i)}}{\sum_{j=1}^{n}|(y_i,y_j)|},i\in\{1,\dots,n\};$$

while the Heilbronn inequality (1.4) may be obtained from the first part of (1.6) if one chooses $c_i=\frac{\overline{(x,y_i)}}{|(x,y_i)|}$, for any $i\in\{1,\dots,n\}$.

For other results connected with the above ones, see [5] and [6].

2. Some Preliminary Results

We start with the following lemma which is also interesting in itself.

Lemma 1. *Let $z_1,\dots,z_n\in H$ and $\alpha_1,\dots,\alpha_n\in\mathbb{K}$. Then one has the inequality:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \tag{2.1}$$

$$
\leq \begin{cases}
\max\limits_{1\leq k\leq n}|\alpha_k|^2 \sum\limits_{i,j=1}^{n}|(z_i,z_j)|; \\[2ex]
\max\limits_{1\leq k\leq n}|\alpha_k| \left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^s\right)^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\[2ex]
\max\limits_{1\leq k\leq n}|\alpha_k| \sum\limits_{k=1}^{n}|\alpha_k| \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right); \\[2ex]
\left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n}|\alpha_i| \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^q\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n}|\alpha_i|^t\right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\qquad\qquad\qquad\qquad\qquad\qquad t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\[2ex]
\left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n}|\alpha_i| \max\limits_{1\leq i\leq n}\left\{\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}\right\}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\sum\limits_{k=1}^{n}|\alpha_k| \max\limits_{1\leq i\leq n}|\alpha_i| \sum\limits_{i=1}^{n}\left[\max\limits_{1\leq j\leq n}|(z_i,z_j)|\right]; \\[2ex]
\sum\limits_{k=1}^{n}|\alpha_k| \left(\sum\limits_{i=1}^{n}|\alpha_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n}\left[\max\limits_{1\leq j\leq n}|(z_i,z_j)|\right]^l\right)^{\frac{1}{l}}, \quad m>1,\ \frac{1}{m}+\frac{1}{l}=1; \\[2ex]
\left(\sum\limits_{k=1}^{n}|\alpha_k|\right)^2 \max\limits_{i,1\leq j\leq n}|(z_i,z_j)|.
\end{cases}
$$

Proof. We observe that

$$
\begin{aligned}
(2.2)\qquad \left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 &= \left(\sum_{i=1}^{n}\alpha_i z_i, \sum_{j=1}^{n}\alpha_j z_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\,(z_i,z_j) = \left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\,(z_i,z_j)\right| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i|\,|\alpha_j|\,|(z_i,z_j)| = \sum_{i=1}^{n}|\alpha_i|\left(\sum_{j=1}^{n}|\alpha_j|\,|(z_i,z_j)|\right) \\
&:= M.
\end{aligned}
$$

Using Hölder's inequality, we may write that

(2.3)
$$\sum_{j=1}^{n}|\alpha_j|\,|(z_i,z_j)|\leq\begin{cases}\max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{j=1}^{n}|(z_i,z_j)|\\[2ex]\left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\sum\limits_{k=1}^{n}|\alpha_k|\max\limits_{1\leq j\leq n}|(z_i,z_j)|\end{cases}$$

for any $i\in\{1,\ldots,n\}$, giving

$$M\leq\begin{cases}\max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\sum\limits_{j=1}^{n}|(z_i,z_j)|=:M_1;\\[2ex]\left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\sum\limits_{i=1}^{n}|\alpha_i|\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}:=M_p,\\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[1ex]\sum\limits_{k=1}^{n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\max\limits_{1\leq j\leq n}|(z_i,z_j)|=:M_\infty.\end{cases}\tag{2.4}$$

By Hölder's inequality we also have:

$$\sum_{i=1}^{n}|\alpha_i|\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)\tag{2.5}$$
$$\leq\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|\sum\limits_{i,j=1}^{n}|(z_i,z_j)|;\\[2ex]\left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^s\right)^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1;\\[2ex]\sum\limits_{i=1}^{n}|\alpha_i|\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right);\end{cases}$$

and thus

$$M_1\leq\begin{cases}\max\limits_{1\leq k\leq n}|\alpha_k|^2\sum\limits_{i,j=1}^{n}|(z_i,z_j)|;\\[2ex]\max\limits_{1\leq k\leq n}|\alpha_k|\left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^s\right)^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1;\\[2ex]\max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right);\end{cases}$$

and the first 3 inequalities in (2.1) are obtained.

By Hölder's inequality we also have:

$$M_p \le \left(\sum_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \times \begin{cases} \max\limits_{1\le i\le n} |\alpha_i| \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^t\right)^{\frac{1}{t}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{u}{q}}\right)^{\frac{1}{u}}, \quad t > 1,\ \frac{1}{t} + \frac{1}{u} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\le i\le n} \left\{\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}}\right\}; \end{cases}$$

and the next 3 inequalities in (2.1) are proved.

Finally, by the same Hölder inequality we may state that:

$$M_\infty \le \sum_{k=1}^{n} |\alpha_k| \times \begin{cases} \max\limits_{1\le i\le n} |\alpha_i| \sum\limits_{i=1}^{n} \left(\max\limits_{1\le j\le n} |(z_i, z_j)|\right); \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n} \left(\max\limits_{1\le j\le n} |(z_i, z_j)|\right)^l\right)^{\frac{1}{l}}, \quad m > 1,\ \frac{1}{m} + \frac{1}{l} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\le i,j\le n} |(z_i, z_j)|; \end{cases}$$

and the last 3 inequalities in (2.1) are proved. □

If we would like to have some bounds for $\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2$ in terms of $\sum_{i=1}^{n} |\alpha_i|^2$, then the following corollaries may be used.

Corollary 1. *Let $z_1, \dots, z_n$ and $\alpha_1, \dots, \alpha_n$ be as in Lemma 1. If $1 < p \le 2$, $1 < t \le 2$, then one has the inequality*

$$\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2 \le n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{k=1}^{n} |\alpha_k|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}} \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$.

Proof. Observe, by the monotonicity of power means, we may write that

$$\left(\frac{\sum_{k=1}^{n} |\alpha_k|^p}{n}\right)^{\frac{1}{p}} \le \left(\frac{\sum_{k=1}^{n} |\alpha_k|^2}{n}\right)^{\frac{1}{2}}; \quad 1 < p \le 2,$$
$$\left(\frac{\sum_{k=1}^{n} |\alpha_k|^t}{n}\right)^{\frac{1}{t}} \le \left(\frac{\sum_{k=1}^{n} |\alpha_k|^2}{n}\right)^{\frac{1}{2}}; \quad 1 < t \le 2,$$

from where we get

$$\left(\sum_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$
$$\left(\sum_{k=1}^{n}|\alpha_k|^t\right)^{\frac{1}{t}} \leq n^{\frac{1}{t}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}}.$$

Using the fifth inequality in (2.1), we then deduce (2.6). □

Remark 1. *An interesting particular case is the one for* $p = q = t = u = 2$*, giving*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq \sum_{k=1}^{n}|\alpha_k|^2\left(\sum_{i,j=1}^{n}|(z_i,z_j)|^2\right)^{\frac{1}{2}}. \tag{2.7}$$

Corollary 2. *With the assumptions of Lemma 1 and if* $1 < p \leq 2$*, then*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq n^{\frac{1}{p}}\sum_{k=1}^{n}|\alpha_k|^2\max_{1\leq i\leq n}\left[\left(\sum_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}\right], \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since

$$\left(\sum_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$

and

$$\sum_{k=1}^{n}|\alpha_k| \leq n^{\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$

then by the sixth inequality in (2.1) we deduce (2.8). □

In a similar fashion, one may prove the following two corollaries.

Corollary 3. *With the assumptions of Lemma 1 and if* $1 < m \leq 2$*, then*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq n^{\frac{1}{m}}\sum_{k=1}^{n}|\alpha_k|^2\left(\sum_{i=1}^{n}\left[\max_{1\leq j\leq n}|(z_i,z_j)|\right]^l\right)^{\frac{1}{l}}, \tag{2.9}$$

where $\frac{1}{m}+\frac{1}{l}=1$.

Corollary 4. *With the assumptions of Lemma 1, we have:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq n\sum_{k=1}^{n}|\alpha_k|^2\max_{1\leq i,j\leq n}|(z_i,z_j)|. \tag{2.10}$$

The following lemma may be of interest as well.

Lemma 2. *With the assumptions of Lemma 1, one has the inequalities*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\sum_{j=1}^{n}|(z_i,z_j)| \tag{2.11}$$

$$\leq \begin{cases} \sum_{i=1}^{n} |\alpha_i|^2 \max_{1\leq i\leq n} \left[\sum_{j=1}^{n} |(z_i, z_j)| \right]; \\ \\ \left(\sum_{i=1}^{n} |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left(\left(\sum_{j=1}^{n} |(z_i, z_j)| \right)^{q} \right)^{\frac{1}{q}}, \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \max_{1\leq i\leq n} |\alpha_i|^2 \sum_{i,j=1}^{n} |(z_i, z_j)|. \end{cases}$$

Proof. As in Lemma 1, we know that

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| |(z_i, z_j)|. \tag{2.12}$$

Using the simple observation that (see also [8, p. 394])

$$|\alpha_i| |\alpha_j| \leq \frac{1}{2} \left(|\alpha_i|^2 + |\alpha_j|^2 \right), \quad i, j \in \{1, \ldots, n\}$$

we have

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| |(z_i, z_j)| &\leq \frac{1}{2} \sum_{i,j=1}^{n} \left(|\alpha_i|^2 + |\alpha_j|^2 \right) |(z_i, z_j)| \\ &= \frac{1}{2} \left[\sum_{i,j=1}^{n} |\alpha_i|^2 |(z_i, z_j)| + \sum_{i,j=1}^{n} |\alpha_j|^2 |(z_i, z_j)| \right] \\ &= \sum_{i,j=1}^{n} |\alpha_i|^2 |(z_i, z_j)|, \end{aligned}$$

which proves the first inequality in (2.11).

The second part follows by Hölder's inequality and we omit the details. □

Remark 2. *The first part in (2.11) is the inequality obtained by Pečarić in* [9].

3. Some Pečarić Type Inequalities

We are now able to point out the following result which complements the inequality (1.6) due to J.E. Pečarić [9] (see also [8, p. 394]).

Theorem 5. *Let $x, y_1, \ldots, y_n$ be vectors of an inner product space $(H; (\cdot, \cdot))$ and $c_1, \ldots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \tag{3.1}$$

$$\leq \|x\|^2 \times \begin{cases} \max\limits_{1\leq k\leq n} |c_k|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j)| ; \\ \max\limits_{1\leq k\leq n} |c_k| \left(\sum\limits_{i=1}^{n} |c_i|^r\right)^{\frac{1}{r}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^s\right]^{\frac{1}{s}}, \quad r > 1,\ \frac{1}{r} + \frac{1}{s} = 1; \\ \max\limits_{1\leq k\leq n} |c_k| \sum\limits_{k=1}^{n} |c_k| \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right); \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} |c_i| \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^q\right)^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} |c_i|^t\right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 1,\ \frac{1}{t} + \frac{1}{u} = 1; \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} |c_i| \max\limits_{1\leq i\leq n} \left\{\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right\}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum\limits_{k=1}^{n} |c_k| \max\limits_{1\leq i\leq n} |c_i| \sum\limits_{i=1}^{n} \left[\max\limits_{1\leq j\leq n} |(y_i, y_j)|\right]; \\ \sum\limits_{k=1}^{n} |c_k| \left(\sum\limits_{i=1}^{n} |c_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n} \left[\max\limits_{1\leq j\leq n} |(y_i, y_j)|\right]^l\right)^{\frac{1}{l}}, \quad m > 1,\ \frac{1}{m} + \frac{1}{l} = 1; \\ \left(\sum\limits_{k=1}^{n} |c_k|\right)^2 \max\limits_{i,1\leq j\leq n} |(y_i, y_j)| . \end{cases}$$

Proof. We note that

$$\sum_{i=1}^{n} c_i (x, y_i) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i\right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \left\|\sum_{i=1}^{n} \overline{c_i} y_i\right\|^2 . \tag{3.2}$$

Finally, using Lemma 1 with $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \dots, n)$, we deduce the desired inequality (3.1). We omit the details. $\square$

The following corollaries may be useful if one needs bounds in terms of $\sum_{i=1}^{n} |c_i|^2$.

Corollary 5. *With the assumptions in Theorem 5 and if* $1 < p \leq 2$, $1 < t \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$, *one has the inequality:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{i=1}^{n} |c_i|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \tag{3.3}$$

and, in particular, for $p = q = t = u = 2$,

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right)^{\frac{1}{2}}. \tag{3.4}$$

The proof is similar to the one in Corollary 1.

Corollary 6. *With the assumptions in Theorem 5 and if* $1 < p \leq 2$, *then*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 n^{\frac{1}{p}} \sum_{k=1}^{n} |c_k|^2 \max_{1\leq i\leq n} \left[\sum_{j=1}^{n} |(y_i, y_j)|^q\right]^{\frac{1}{q}}, \tag{3.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The proof is similar to the one in Corollary 2.

The following two inequalities also hold.

Corollary 7. *With the above assumptions for* X, y_i, c_i *and if* $1 < m \leq 2$, *then*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 n^{\frac{1}{m}} \sum_{k=1}^{n} |c_k|^2 \left(\sum_{i=1}^{n} \left[\max_{1\leq j\leq n} |(y_i, y_j)|\right]^l\right)^{\frac{1}{l}}, \tag{3.6}$$

where $\frac{1}{m} + \frac{1}{l} = 1$.

Corollary 8. *With the above assumptions for* X, y_i, c_i, *one has*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 n \sum_{k=1}^{n} |c_k|^2 \max_{i,1\leq j\leq n} |(y_i, y_j)|. \tag{3.7}$$

Using Lemma 2, we may state the following result as well.

Remark 3. *With the assumptions of Theorem 5, one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \sum_{j=1}^{n} |(y_i, y_j)| \tag{3.8}$$

$$\leq \|x\|^2 \times \begin{cases} \sum\limits_{i=1}^{n} |c_i|^2 \max\limits_{1\leq i\leq n} \left[\sum\limits_{j=1}^{n} |(y_i, y_j)|\right]; \\ \\ \left(\sum\limits_{i=1}^{n} |c_i|^{2p}\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^q\right)^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \max\limits_{1\leq i\leq n} |c_i|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j)|; \end{cases}$$

that provide some alternatives to Pečarić's result (1.6).

4. Some Inequalities of Bombieri Type

In this section we point out some inequalities of Bombieri type that may be obtained from (3.1) on choosing $c_i = \overline{(x, y_i)}$ $(i = 1, \dots, n)$.

If the above choice was made in the first inequality in (3.1), then one would obtain:

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \max_{1\leq i\leq n} |(x, y_i)|^2 \sum_{i,j=1}^{n} |(y_i, y_j)|$$

giving, by taking the square root,

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\| \max_{1\leq i\leq n} |(x, y_i)| \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2}}, \quad x \in H. \tag{4.1}$$

If the same choice for c_i is made in the second inequality in (3.1), then one would get

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \leq \|x\|^2 \max_{1\leq i\leq n} |(x, y_i)| \left(\sum_{i=1}^{n} |(x, y_i)|^r\right)^{\frac{1}{r}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^s\right]^{\frac{1}{s}},$$

implying

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\| \max_{1\leq i\leq n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^r\right)^{\frac{1}{2r}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^s\right]^{\frac{1}{2s}}, \tag{4.2}$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $s > 1$.

The other inequalities in (3.1) will produce the following results, respectively

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\| \max_{1\leq i\leq n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|\right)^{\frac{1}{2}} \left[\max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right]; \tag{4.3}$$

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\| \max_{1\leq i\leq n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^p\right)^{\frac{1}{2p}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right]^{\frac{1}{2}}, \tag{4.4}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$;

$$\sum_{i=1}^{n}|(x,y_i)|^2 \le \|x\| \left(\sum_{i=1}^{n}|(x,y_i)|^p\right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n}|(x,y_i)|^t\right)^{\frac{1}{2t}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|(y_i,y_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{2u}}, \tag{4.5}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ t>1,\ \frac{1}{t}+\frac{1}{u}=1$;

$$\sum_{i=1}^{n}|(x,y_i)|^2 \le \|x\| \left(\sum_{i=1}^{n}|(x,y_i)|^p\right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n}|(x,y_i)|\right)^{\frac{1}{2}} \max_{1\le i\le n}\left\{\left(\sum_{j=1}^{n}|(y_i,y_j)|^q\right)^{\frac{1}{2q}}\right\}, \tag{4.6}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$;

$$\sum_{i=1}^{n}|(x,y_i)|^2 \le \|x\| \left[\sum_{i=1}^{n}|(x,y_i)|\right]^{\frac{1}{2}} \max_{1\le i\le n}|(x,y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n}\left[\max_{1\le j\le n}|(y_i,y_j)|\right]\right)^{\frac{1}{2}}; \tag{4.7}$$

$$\sum_{i=1}^{n}|(x,y_i)|^2 \le \|x\| \left[\sum_{i=1}^{n}|(x,y_i)|^m\right]^{\frac{1}{2m}} \left[\sum_{i=1}^{n}\left[\max_{1\le j\le n}|(y_i,y_j)|^l\right]\right]^{\frac{1}{2l}}, \tag{4.8}$$

where $m>1,\ \frac{1}{m}+\frac{1}{l}=1$; and

$$\sum_{i=1}^{n}|(x,y_i)|^2 \le \|x\| \sum_{i=1}^{n}|(x,y_i)| \max_{i,1\le j\le n}|(y_i,y_j)|^{\frac{1}{2}}. \tag{4.9}$$

If in the above inequalities we assume that $(y_i)_{1\le i\le n}=(e_i)_{1\le i\le n}$, where $(e_i)_{1\le i\le n}$ are orthonormal vectors in the inner product space $(H,(\cdot,\cdot))$, then from (4.1) – (4.9) we may deduce the following inequalities similar in a sense with Bessel's inequality:

$$\sum_{i=1}^{n}|(x,e_i)|^2 \le \sqrt{n}\,\|x\| \max_{1\le i\le n}\{|(x,e_i)|\}; \tag{4.10}$$

$$\sum_{i=1}^{n}|(x,e_i)|^2 \le n^{\frac{1}{2s}}\,\|x\| \max_{1\le i\le n}\left\{|(x,e_i)|^{\frac{1}{2}}\right\} \left(\sum_{i=1}^{n}|(x,e_i)|^r\right)^{\frac{1}{2r}}, \tag{4.11}$$

where $r>1,\ \frac{1}{r}+\frac{1}{s}=1$;

$$\sum_{i=1}^{n}|(x,e_i)|^2 \le \|x\| \max_{1\le i\le n}\left\{|(x,e_i)|^{\frac{1}{2}}\right\} \left(\sum_{i=1}^{n}|(x,e_i)|\right)^{\frac{1}{2}}; \tag{4.12}$$

$$\sum_{i=1}^{n}|(x,e_i)|^2 \le \sqrt{n}\,\|x\| \max_{1\le i\le n}\left\{|(x,e_i)|^{\frac{1}{2}}\right\} \left(\sum_{i=1}^{n}|(x,e_i)|^p\right)^{\frac{1}{2p}}, \tag{4.13}$$

where $p>1$;

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2\leq n^{\frac{1}{2u}}\left\|x\right\|\left(\sum_{i=1}^{n}\left|(x,e_i)\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{n}\left|(x,e_i)\right|^t\right)^{\frac{1}{2t}}, \tag{4.14}$$

where $p>1$, $t>1$, $\frac{1}{t}+\frac{1}{u}=1$;

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2\leq\left\|x\right\|\left(\sum_{i=1}^{n}\left|(x,e_i)\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{n}\left|(x,e_i)\right|\right)^{\frac{1}{2}},\quad p>1; \tag{4.15}$$

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2\leq\sqrt{n}\left\|x\right\|\left(\sum_{i=1}^{n}\left|(x,e_i)\right|\right)^{\frac{1}{2}}\max_{1\leq i\leq n}\left\{\left|(x,e_i)\right|^{\frac{1}{2}}\right\}; \tag{4.16}$$

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2\leq n^{\frac{1}{2l}}\left\|x\right\|\left[\sum_{i=1}^{n}\left|(x,e_i)\right|^m\right]^{\frac{1}{m}},\quad m>1,\ \frac{1}{m}+\frac{1}{l}=1; \tag{4.17}$$

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2\leq\left\|x\right\|\sum_{i=1}^{n}\left|(x,e_i)\right|. \tag{4.18}$$

Corollaries 5 − 8 will produce the following results which do not contain the Fourier coefficients in the right side of the inequality.

Indeed, if one chooses $c_i=\overline{(x,y_i)}$ in (3.3), then

$$\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^2\right)^2\leq\left\|x\right\|^2 n^{\frac{1}{p}+\frac{1}{t}-1}\sum_{i=1}^{n}\left|(x,y_i)\right|^2\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}},$$

giving the following Bombieri type inequality:

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2\leq n^{\frac{1}{p}+\frac{1}{t}-1}\left\|x\right\|^2\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \tag{4.19}$$

where $1<p\leq 2$, $1<t\leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{t}+\frac{1}{u}=1$.

If in this inequality we consider $p=q=t=u=2$, then

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2\leq\left\|x\right\|^2\left(\sum_{i,j=1}^{n}\left|(y_i,y_j)\right|^2\right)^{\frac{1}{2}}. \tag{4.20}$$

For a different proof of this result see also [6].

In a similar way, if $c_i=\overline{(x,y_i)}$ in (3.6), then

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2\leq n^{\frac{1}{m}}\left\|x\right\|^2\left(\sum_{i=1}^{n}\left[\max_{1\leq j\leq n}\left|(y_i,y_j)\right|\right]^l\right)^{\frac{1}{l}}, \tag{4.21}$$

where $m>1$, $\frac{1}{m}+\frac{1}{l}=1$.

Finally, if $c_i = \overline{(x, y_i)}$ $(i = 1, \dots, n)$, is taken in (3.7), then

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le n \|x\|^2 \max_{1 \le i,j \le n} |(y_i, y_j)| . \tag{4.22}$$

Remark 4. *Let us compare Bombieri's result*

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\|^2 \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} |(y_i, y_j)| \right\} \tag{4.23}$$

with our result

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\|^2 \left\{ \sum_{i,j=1}^{n} |(y_i, y_j)|^2 \right\}^{\frac{1}{2}} . \tag{4.24}$$

Denote

$$M_1 := \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} |(y_i, y_j)| \right\}$$

and

$$M_2 := \left[\sum_{i,j=1}^{n} |(y_i, y_j)|^2 \right]^{\frac{1}{2}} .$$

If we choose the inner product space $H = \mathbb{R}$, $(x, y) := xy$ and $n = 2$, then for $y_1 = a$, $y_2 = b$, $a, b > 0$, we have

$$M_1 = \max \left\{ a^2 + ab, ab + b^2 \right\} = (a + b) \max (a, b) ,$$

$$M_2 = \left(a^4 + a^2b^2 + a^2b^2 + b^4 \right)^{\frac{1}{2}} = a^2 + b^2.$$

Assume that $a \ge b$. Then $M_1 = a^2 + ab \ge a^2 + b^2 = M_2$, showing that, in this case, the bound provided by (4.24) is better than the bound provided by (4.23). If $(y_i)_{1 \le i \le n}$ are orthonormal vectors, then $M_1 = 1$, $M_2 = \sqrt{n}$, showing that in this case the Bombieri inequality (which becomes Bessel's inequality) provides a better bound than (4.24).

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