ON CARLEMAN-TYPE INEQUALITIES

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Abstract. We give a weighted version of an inequality of Redheffer, which he used to treat Carleman’s inequality. We then apply the result to get some new Carleman-type Inequalities.

1. Introduction

Throughout let \( a = (a_n)_{n \geq 1} \) be a nonnegative sequence with \( \sum_{n=1}^{\infty} a_n < \infty \). Let \( \Lambda_n = \sum_{i=1}^{n} \lambda_i, \lambda_i > 0 \) and \( G_n = (\prod_{i=1}^{n} a_i^{\lambda_i})^{1/\Lambda_n} \). The Carleman inequality asserts that

\[
\sum_{n=1}^{\infty} (\prod_{k=1}^{n} a_k)^{1/n} \leq e \sum_{n=1}^{\infty} a_n.
\]

We refer the reader to the survey article [6] and the references therein for an account of Carleman’s inequality. Among the various generalizations of Carleman’s inequality, we mention the result of E. Love, who proved for \( \alpha > 0, \beta \geq 1, \lambda_i = i^\alpha - (i - 1)^\alpha, \)

\[
\sum_{n=1}^{\infty} n^\beta (\prod_{i=1}^{n} a_i^{\alpha - (i-1)^\alpha})^{1/n^\alpha} \leq e^{\frac{\beta + 1}{\alpha}} \sum_{n=1}^{\infty} n^\beta a_n,
\]

and the constant \( e^{\frac{\beta + 1}{\alpha}} \) is best possible.

A remarkable proof of Carleman’s inequality was given by R.Redheffer in [7] by developing the method of "recurrent inequalities". Another proof was given by him in [8] and his result has been generalized by H.Alzer[4] and most recently by J. Pečarić and K. Stolarsky[6], who proved for \( b_n > 0, N \geq 1, \)

\[
\sum_{n=1}^{N} \Lambda_n (b_n - 1)G_n + \Lambda_N G_N \leq \sum_{n=1}^{N} \lambda_n G_n b_n^{\lambda_n}/\lambda_n.
\]

It’s our goal in this paper to give another weighted version of Redheffer’s treatment of Carleman’s inequality and use it to get some new Carleman-type Inequalities.

2. Lemmas

Lemma 2.1. Let \( \Lambda_k = \sum_{i=1}^{k} \lambda_i, \lambda_i > 0 \) and \( G_k = (\prod_{i=1}^{k} a_i^{\lambda_i})^{1/\Lambda_k} \), then for \( \mu_i > 0, n \geq 2, \)

\[
G_1 + \sum_{i=2}^{n} \left( \frac{\Lambda_i \mu_i}{\lambda_i} - \frac{\Lambda_i}{\lambda_{i+1}} \right) G_i + \frac{\Lambda_n \mu_n}{\lambda_n} G_n \leq (1 + \frac{\Lambda_1}{\lambda_2}) a_1 + \sum_{i=2}^{n} \frac{\lambda_i}{\lambda_i} a_i.
\]

Proof. This is essentially due to R.Redheffer[7]. We note for \( k \geq 2, \mu > 0, \eta > 0, \)

\[
\mu G_k - \eta a_k = G_{k-1}(\mu t - \eta \frac{\lambda_k}{\lambda_{k-1}}) \leq G_{k-1}(\frac{\Lambda_{k-1}}{\lambda_k} \eta^{\frac{\lambda_k}{\lambda_{k-1}}} (\mu \frac{\lambda_k}{A_k})^{\frac{\lambda_k}{\lambda_{k-1}}},
\]

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where $t^{\lambda_k} = a_k/G_{k-1}$ (compare this with the one on page 686 of [7]). By setting $\mu_k \lambda_k = \mu, \eta_k = \eta = \mu_k/\lambda_k$, we get

\begin{equation}
\frac{\lambda_k \mu_k G_k - a_k \mu_k^{\lambda_k}}{\lambda_k} \leq \frac{\lambda_{k-1}}{\lambda_k} G_k.
\end{equation}

The lemma then follows by adding (2.2) for $2 \leq k \leq n$ and $G_1 = a_1$, together.

\section*{Lemma 2.2.}

Let $f(x) \in C^3[a, b]$ and $f'''(x) \geq 0$ for $x \in [a, b]$. Then

\begin{equation}
f(b) - f(a) \geq f\left(\frac{a + b}{2}\right)(b - a).
\end{equation}

\textbf{Proof.} By Taylor’s expansion,

\begin{align*}
f(b) &= f\left(\frac{a + b}{2}\right) + f'\left(\frac{a + b}{2}\right)(b - \frac{a + b}{2}) + f''(\eta_1)(a - b)^2/4, \\
f(a) &= f\left(\frac{a + b}{2}\right) + f'\left(\frac{a + b}{2}\right)(a - \frac{a + b}{2}) + f''(\eta_2)(a - b)^2/4,
\end{align*}

where $a < \eta_2 < (a + b)/2 < \eta_1 < b$. The lemma then follows by noticing $f'''(x) \geq 0$ for $x \in [a, b]$. \hfill \Box

\section*{3. The Main Results}

\textbf{Theorem 3.1.} Assume the same conditions in Lemma 2.1 and let $f(x)$ be a real valued function defined for $x \geq 2$ such that $f(n) = \Lambda_n/\Lambda_n$ for $n \geq 2$ and $0 \leq f(x+1) - f(x) \leq 1/\alpha$ for some $\alpha > 0$. If $(1 + \lambda_1/\lambda_2) \leq e^{1/\alpha}$ for the same $\alpha$ then

\begin{equation}
\sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} a_i^{\lambda_i}\right)^{1/\Lambda_n} \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n.
\end{equation}

\textbf{Proof.} It suffices to prove the theorem for any integer $n \geq 2$. Set $\mu_i = f(i+1)/f(i)$ in Lemma 2.1

we get

\begin{align*}
\sum_{i=1}^{n} G_i &\leq \sum_{i=1}^{n-1} G_i + f(n+1)G_N \leq (1 + \frac{\lambda_1}{\lambda_2})a_1 + \sum_{i=2}^{n} a_i (1 + \frac{f(i+1) - f(i)}{f(i)} f(i)) \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n,
\end{align*}

by the conditions of the theorem and this completes the proof. \hfill \Box

Apply Theorem 3.1 to $\lambda_1 = 1, \lambda_i = \alpha^{i-1} - \alpha^{i-2}, i \geq 2$ for some $\alpha > 1$, then $f(x) = \alpha/(\alpha - 1)$ and we get

\textbf{Theorem 3.2.} For $\alpha > 1$,

\begin{equation}
\sum_{n=1}^{\infty} (a_1 \prod_{k=2}^{n} a_k^{\alpha^{k-1} - \alpha^{k-2}})^{1/\alpha^{n-1}} \leq (1 + \frac{1}{\alpha - 1})a_1 + \sum_{n=2}^{\infty} a_n.
\end{equation}

Apply Theorem 3.1 to $\lambda_i = \alpha^i, i \geq 1$ for some $\alpha > 0$, then $f(i+1) - f(i) = \alpha^{-i}$ and we get

\textbf{Theorem 3.3.} For $\alpha > 0, \sum_{n=1}^{\infty} e^{1/\alpha^{n}} a_n < \infty$,

\begin{equation}
\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{\alpha^{k-1}}\right)^{(\alpha^{n-1})/(\alpha-1)} \leq (1 + \frac{1}{\alpha})a_1 + \sum_{n=2}^{\infty} e^{1/\alpha^{n}} a_n \leq \sum_{n=1}^{\infty} e^{1/\alpha^{n}} a_n.
\end{equation}

The $\lambda_i$’s in Theorems 3.2, 3.3 are of the “exponential” type and now we consider the cases where the $\lambda_i$’s are of the “polynomial” type.
Theorem 3.4. For $\alpha \geq 2$,

\begin{equation}
\sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} a_k^{\alpha - (k-1)^\alpha} \right)^{1/n^\alpha} \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n.
\end{equation}

Proof. Apply Theorem 3.1 with $\lambda_i = i^\alpha - (i - 1)^\alpha, f(x) = x^\alpha / (x^\alpha - (x - 1)^\alpha), x \geq 2$. Note for $\alpha \geq 1$,

\[ 1 + \frac{1}{2^\alpha - 1} \leq 1 + \frac{1}{\alpha} \leq e^{1/\alpha}. \]

And $f(i + 1) - f(i) = f'(\xi), 2 \leq i < \xi < i + 1$, with

\[ 0 < f'(\xi) = \frac{\alpha \xi^{\alpha - 1}(\xi - 1)^{\alpha - 1}}{(\xi^\alpha - (\xi - 1)^\alpha)\alpha} \leq 1, \]

where the last inequality follows from Lemma 2.2 and the arithmetic-geometric inequality, since for $\alpha \geq 2$,

\[ \xi^\alpha - (\xi - 1)^\alpha \geq \alpha \left( \frac{\xi + (\xi - 1)^\alpha}{2} \right)^{\alpha - 1} \geq \alpha (\xi - 1)^{(\alpha - 1)/2}. \]

This completes the proof. \qed

We note the theorem implies \cite{1} for $\alpha \geq 2$(see page 40 in \cite{2}), and one should also be able to improve the range of $\alpha$ in the theorem.

Let $[x]$ denote the largest integer not exceeding the real number $x$. For $x > 1, \alpha \geq 0$, we define $[x]^{\alpha - 1} f(x) = \int_x^{x^{\alpha - 1} d[t]} = \lim_{\alpha \to 0} \int_x^{x^{\alpha - 1} d[t]}$. Note for any integer $n \geq 2$, $f(n) = \sum_{i=1}^{n} i^{\alpha - 1} n^{\alpha - 1}$. Apply Theorem 3.1 with this $f(x)$, $\lambda_i = i^{\alpha - 1}$ and note $1 + \frac{1}{2^\alpha - 1} \leq 1 + 1/\alpha \leq e^{1/\alpha}$ for $\alpha \geq 2$ and for $\alpha = 2$, $f(n) = (n + 1)/2$ for $\alpha = 3$, $f(n) = (n + 1) (2n + 1)/6n$; for $\alpha = 4$, $f(n) = (n + 1)^2 / 4n$. In either case, one verifies directly $f(i + 1) - f(i) \leq 1/\alpha$ which gives for $\alpha = 2, 3, 4$,

\begin{equation}
\sum_{n=1}^{\infty} \left( \prod_{i=1}^{n} a_i^{\alpha - 1} \right) \left( \sum_{i=1}^{n} i^{\alpha - 1} \right)^{1/\alpha} \leq e^{1/\alpha} \sum_{n=1}^{\infty} a_n.
\end{equation}

We don’t know in this case whether $f(i + 1) - f(i) \leq 1/\alpha$ holds in general. The case $i = 1$ implies it is necessary to have $\alpha \geq 2$. We note here by a result of G.Bennett and G. Jameson, we know $f(i + 1)/(i + 2) \leq f(i)/(i + 1)$\cite{4}. Hence $f(i + 1) - f(i) \leq f(i)/(i + 1) \leq (1 + 2\alpha^2)/(3 \cdot 2^\alpha - 1)$ for $i \geq 2$.

Now we let $p \neq 0, \frac{1}{p} + \frac{1}{q} = 1$ and let $l^p$ be the Banach space of all complex sequences $a = (a_n)_{n \geq 1}$ with norm

\[ ||a|| := \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty. \]

Corresponding to inequalities \cite{3} and \cite{5}, we have the following

\begin{equation}
\sum_{n=1}^{\infty} \left( \frac{1}{n^\alpha} \sum_{i=1}^{n} (i^\alpha - (i - 1)^\alpha) |a_i|^p \right) \leq \left( \frac{\alpha p}{\alpha p - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p,
\end{equation}

\begin{equation}
\sum_{n=1}^{\infty} \sum_{i=1}^{n} i^{\alpha - 1} |a_i|^p \leq \left( \frac{\alpha p}{\alpha p - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p.
\end{equation}

These two inequalities were announced to hold(see \cite{2}, page 40-41 and \cite{3}, page 407)whenever $\alpha > 0, p > 0, \alpha p > 1$. Replacing $|a_i|$ with $|a_i|^{1/p}$ and making $p \to \infty$ in \cite{6}, \cite{7} gives back \cite{3} and \cite{5} respectively. It is thus interesting to ask whether one can apply Redheffer’s method to give a proof of \cite{6} and \cite{7}.
References


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