BOUNDING THE ČEBYŠEV FUNCTIONAL FOR SEQUENCES OF VECTORS IN NORMED LINEAR SPACES

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Abstract. Some new bounds for Čebyšev functional for sequences of vectors in normed linear spaces are pointed out.

1. Introduction

Consider the Čebyšev functional defined for $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\mathbf{x} = (x_1, ..., x_n) \in X^n$, where X is a linear space over the real or complex number field $\mathbb{K}$:

$$T_n(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{x}) := P_n \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \cdot \sum_{i=1}^{n} p_i x_i, \tag{1.1}$$

where $P_n := \sum_{i=1}^{n} p_i$.

The following Grüss type inequalities for sequences in normed linear spaces hold.

Theorem 1. *Let $(X, \|.\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n, \mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} p_i = 1$ and $\mathbf{x} = (x_1, ..., x_n) \in X^n$. Then one has the inequalities*

$$\|T_n(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{x})\| \leq \begin{cases} \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right] \max_{1\leq j\leq n-1} |\Delta\alpha_j| \max_{1\leq j\leq n-1} \|\Delta x_j\|, [1]; \\ \frac{1}{2} \sum_{i=1}^{n} p_i (1 - p_i) \sum_{j=1}^{n-1} |\Delta\alpha_j| \sum_{j=1}^{n-1} \|\Delta x_j\|, [3]; \\ \sum_{1\leq i<j\leq n} p_i p_j (j - i) \left(\sum_{j=1}^{n-1} |\Delta\alpha_j|^p\right)^{1/p} \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^q\right)^{1/q}, \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, [2]. \end{cases} \tag{1.2}$$

The constant 1 in the first branch, $\frac{1}{2}$ in the second branch and 1 in the third branch are best possible in the sense that they cannot be replaced by smaller constants.

The following particular inequalities for unweighted means hold as well, where $T_n(\boldsymbol{\alpha}, \mathbf{x})$ is defined as follows:

$$T_n(\boldsymbol{\alpha}, \mathbf{x}) := \frac{1}{n} \sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i.$$

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Corollary 1. *With the assumptions of Theorem 1 for $X, \boldsymbol{\alpha}$ and $\mathbf{x}$, we have*

$$\|T_n(\boldsymbol{\alpha},\mathbf{x})\| \leq \begin{cases} \frac{1}{12}\left(n^2-1\right)\max_{1\leq j\leq n-1}|\Delta\alpha_j|\max_{1\leq j\leq n-1}\|\Delta x_j\|, [1]; \\ \\ \frac{1}{2}\cdot\left(1-\frac{1}{n}\right)\sum_{j=1}^{n-1}|\Delta\alpha_j|\sum_{j=1}^{n-1}\|\Delta x_j\|, [3]; \\ \\ \frac{1}{6}\frac{n^2-1}{n}\left(\sum_{j=1}^{n-1}|\Delta\alpha_j|^p\right)^{1/p}\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^q\right)^{1/q}, \\ \\ p>1, \frac{1}{p}+\frac{1}{q}=1, [2]. \end{cases} \tag{1.3}$$

Here the constants $\frac{1}{12}, \frac{1}{2}$ and $\frac{1}{6}$ are best possible in the sense that they cannot be replaced by smaller constants.

For applications to estimate the p-moments of guessing mappings, see [1]. For applications in approximating the discrete Fourier transform, the discrete Mellin transform as well as some applications for polynomials and Lipschitzian mappings, see [2] and [3].

For classical results related the Čebyšev functional, see [4], [5], [6], [7], [8], [10] and [12]. For more recent results, see [12], [13], [14], [15], [9] and [11].

2. The Identities

The first result is embodied in the following

Theorem 2. *Let $\mathbf{p}=(p_1,...,p_n), \mathbf{a}=(a_1,...,a_n)$ be n-tuples of real or complex numbers and $\mathbf{x}=(x_1,...,x_n)$ an n-tuple of vectors in the linear space X. If we define*

$$P_i := \sum_{k=1}^{i} p_k, \bar{P}_i := P_n - P_i, i \in \{1,...,n-1\},$$

$$A_i(\mathbf{p}) := \sum_{k=1}^{i} p_k a_k, \bar{A}_i(\mathbf{p}) := A_n(\mathbf{p}) - A_i(\mathbf{p}), i \in \{1,...,n-1\},$$

then we have the identity

$$\begin{aligned} T_n(\mathbf{p};\mathbf{a},\mathbf{x}) &= \sum_{i=1}^{n-1}\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix}\cdot\Delta x_i \\ &= P_n\sum_{i=1}^{n-1}P_i\left(\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{P_i}\right)\cdot\Delta x_i \left(\text{if } P_i\neq 0, i\in\{1,...,n\}\right) \\ &= \sum_{i=1}^{n-1}P_i\bar{P}_i\left(\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i}-\frac{A_i(\mathbf{p})}{P_i}\right)\cdot\Delta x_i\left(\text{if } P_i,\bar{P}_i\neq 0, i\in\{1,...,n-1\}\right); \end{aligned} \tag{2.1}$$

where $\Delta x_i := x_{i+1}-x_i$ $(i\in\{1,...,n-1\})$ is the forward difference.

Proof. We use the following well known summation by parts formula

$$\sum_{l=p}^{q-1} d_l\Delta v_l = d_l v_l\Big|_p^q - \sum_{l=p}^{q-1} v_{l+1}\Delta d_l, \tag{2.2}$$

where d_l are real or complex numbers, and v_l are vectors in a linear space, $l = p, ..., q$ $(q > p; p, q$ are natural numbers$)$.

If we choose in (2.2), $p = 1, q = n, d_i = P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})$ and $v_i = x_i$ $(i \in \{1, ..., n-1\})$, then we get

$$\begin{aligned}
&\sum_{i=1}^{n-1} (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \cdot \Delta x_i \\
= \ & [P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})] \cdot x_i|_1^n - \sum_{i=1}^{n-1} \Delta (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \cdot x_{i+1} \\
= \ & [P_n A_n(\mathbf{p}) - P_n A_n(\mathbf{p})] \cdot x_n - [P_1 A_n(\mathbf{p}) - P_n A_1(\mathbf{p})] \cdot x_1 \\
& - \sum_{i=1}^{n-1} [P_{i+1} A_n(\mathbf{p}) - P_n A_{i+1}(\mathbf{p}) - P_i A_n(\mathbf{p}) + P_n A_i(\mathbf{p})] \cdot x_{i+1} \\
= \ & P_n p_1 a_1 x_1 - p_1 A_n(\mathbf{p}) x_1 - \sum_{i=1}^{n-1} (p_{i+1} A_n(\mathbf{p}) - P_n p_{i+1} a_{i+1}) \cdot x_{i+1} \\
= \ & P_n p_1 a_1 x_1 - p_1 A_n(\mathbf{p}) x_1 - A_n(\mathbf{p}) \sum_{i=1}^{n-1} p_{i+1} x_{i+1} + P_n \sum_{i=1}^{n-1} p_{i+1} a_{i+1} x_{i+1} \\
= \ & P_n \sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i x_i \\
= \ & T_n(\mathbf{p}; \mathbf{a}, \mathbf{x}),
\end{aligned}$$

which produce the first identity in (2.1).

The second and the third identities are obvious and we omit the details. □

Before we prove the second result, we need the following lemma providing an identity that is interesting in itself as well.

Lemma 1. *Let* $\mathbf{p} = (p_1, ..., p_n)$ *and* $\mathbf{a} = (a_1, ..., a_n)$ *be* n*-tuples of real or complex numbers. Then we have the equality*

$$\det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} = \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j, \tag{2.3}$$

for each $i \in \{1, ..., n-1\}$.

Proof. Define, for $i \in \{1, ..., n-1\}$,

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j.$$

We have

$$K(i) = \sum_{j=1}^{i} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}} \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}} \cdot \Delta a_j \tag{2.4}$$
$$= \sum_{j=1}^{i} P_j\bar{P}_i \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_i\bar{P}_j \cdot \Delta a_j$$
$$= \bar{P}_i\sum_{j=1}^{i} P_j \cdot \Delta a_j + P_i\sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j.$$

Using the summation by parts formula, we have

$$\sum_{j=1}^{i} P_j \cdot \Delta a_j = P_j \cdot a_j\big|_1^{i+1} - \sum_{j=1}^{i} (P_{j+1} - P_j) \cdot a_{j+1} \tag{2.5}$$
$$= P_{i+1}a_{i+1} - p_1a_1 - \sum_{j=1}^{i} p_{j+1} \cdot a_{j+1}$$
$$= P_{i+1}a_{i+1} - \sum_{j=1}^{i+1} p_j \cdot a_j$$

and

$$\sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j = \bar{P}_j \cdot a_j\big|_{i+1}^{n} - \sum_{j=i+1}^{n-1} \left(\bar{P}_{j+1} - \bar{P}_j\right) \cdot a_{j+1} \tag{2.6}$$
$$= \bar{P}_na_n - \bar{P}_{i+1}a_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j) \cdot a_{j+1}$$
$$= -\bar{P}_{i+1}a_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1}.$$

Using (2.5) and (2.6) we have

$$K(i) = \bar{P}_i\left(P_{i+1}a_{i+1} - \sum_{j=1}^{i+1} p_j \cdot a_j\right) + P_i\left(\sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_{i+1}a_{i+1}\right)$$
$$= \bar{P}_iP_{i+1}a_{i+1} - P_i\bar{P}_{i+1}a_{i+1} - \bar{P}_i\sum_{j=1}^{i+1} p_j \cdot a_j + P_i\sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1}$$
$$= \left[(P_n - P_i)P_{i+1} - P_i(P_n - P_{i+1})\right]a_{i+1}$$
$$+P_i\sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i\sum_{j=1}^{i+1} p_j \cdot a_j$$

$$
\begin{aligned}
&= P_n p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j \\
&= \left(P_i + \bar{P}_i\right) p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j \\
&= P_i \sum_{j=i+1}^{n-1} p_j \cdot a_j - \bar{P}_i \sum_{j=1}^{i} p_j \cdot a_j = P_i \bar{A}_i(\mathbf{p}) - \bar{P}_i A_i(\mathbf{p}) \\
&= \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix};
\end{aligned}
$$

and the identity is proved. □

We are able now to state and prove the second identity for the Čebyšev functional

Theorem 3. *With the assumptions of Theorem 2, we have the equality*

$$
T_n(\mathbf{p}; \mathbf{a}, \mathbf{x}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j \cdot \Delta x_i. \tag{2.7}
$$

The proof is obvious by Theorem 2 and Lemma 1.

Remark 1. *The identity* (2.7)*, for n-tuples of real numbers, was stated without a proof in paper* [12]*. It also may be found for the same sequences in* [9, p. 281]*, again without a proof. In the second place mentioned above there is a misprint for the index of $\bar{P}$ which, instead of* $\max\{i,j\} + 1$, *should be* $\max\{i,j\}$.

3. Some New Inequalities

The following result holds

Theorem 4. *Let $(X, \|.\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$, $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{K}^n$, $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ and $\mathbf{x} = (x_1, ..., x_n) \in X^n$. Then one has the inequalities*

$$
\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \le \begin{cases} \max_{1 \le i \le n-1} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right| \cdot \sum_{j=1}^{n-1} \|\Delta x_j\|; \\[2ex] \left(\sum_{i=1}^{n-1} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right|^q \right)^{1/q} \cdot \left(\sum_{j=1}^{n-1} \|\Delta x_j\|^p \right)^{1/p} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \sum_{i=1}^{n-1} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right| \cdot \max_{1 \le j \le n-1} \|\Delta x_j\|. \end{cases} \tag{3.1}
$$

All the inequalities in (3.1) *are sharp in the sense that the constants* 1 *cannot be replaced by smaller constants.*

Proof. Using the first identity in (2.1), we have

$$
\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \le \sum_{i=1}^{n} \left| \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \right| \|\Delta x_i\|.
$$

Using Hölder's inequality, we deduce the desired result (3.1).

Let prove, for instance, that the constant 1 in the second inequality is best possible.

Assume, for $C>0$, we have that

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq C\left(\sum_{i=1}^{n-1}\left|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right|^q\right)^{1/q}\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^p\right)^{1/p} \tag{3.2}$$

for $p>1, \frac{1}{p}+\frac{1}{q}=1, n\geq 2$.

If we choose $n=2$, then we get

$$T_2(\mathbf{p};\mathbf{a},\mathbf{x}) = p_1p_2(a_2-a_1)(x_2-x_1).$$

Also, for $n=2$,

$$\left(\sum_{i=1}^{n-1}\left|\det\begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p})\end{pmatrix}\right|^q\right)^{1/q} = |p_1p_2|\,|a_2-a_1|$$

and

$$\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^p\right)^{1/p} = \|x_2-x_1\|.$$

Then by (3.2), holding for $n=2, p_1,p_2>0, a_1\neq a_2, x_2\neq x_1$, we deduce $C\geq 1$, proving that 1 is the best possible constant in that inequality. □

The following corollary for the uniform distribution of the probability $\mathbf{p}$ holds.

Corollary 2. *With the assumptions of Theorem 4 for* $\mathbf{a}$ *and* $\mathbf{x}$, *we have the inequalities*

$$\begin{aligned} &\|T_n(\mathbf{a},\mathbf{x})\| \\ \leq\ & \frac{1}{n^2}\times\begin{cases} \max_{1\leq i\leq n-1}\left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}a_k & \sum_{k=1}^{n}a_k\end{pmatrix}\right|\cdot\sum_{j=1}^{n-1}\|\Delta x_j\|; \\ \left(\sum_{i=1}^{n-1}\left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}a_k & \sum_{k=1}^{n}a_k\end{pmatrix}\right|^q\right)^{1/q}\cdot\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^p\right)^{1/p} \\ \text{for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1}\left|\det\begin{pmatrix} i & n \\ \sum_{k=1}^{i}a_k & \sum_{k=1}^{n}a_k\end{pmatrix}\right|\cdot\max_{1\leq j\leq n-1}\|\Delta x_j\|. \end{cases}\end{aligned}$$

The following result may be stated as well.

Theorem 5. *With the assumptions of Theorem 4 and if* $P_i \neq 0\,(i = 1, ..., n)$, *then we have the inequalities*

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \tag{3.3}$$

$$\leq |P_n| \times \begin{cases} \max_{1\leq i\leq n-1} \left| \frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{Pi} \right| \cdot \sum_{i=1}^{n-1} |P_i| \, \|\Delta x_i\| \, ; \\ \left(\sum_{i=1}^{n-1} |P_i| \left| \frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{Pi} \right|^q \right)^{1/q} \cdot \left(\sum_{i=1}^{n-1} |P_i| \, \|\Delta x_i\| \right)^{1/p} \\ \textit{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} |P_i| \left| \frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{Pi} \right| \cdot \max_{1\leq i\leq n-1} \|\Delta x_i\| \, . \end{cases}$$

All the inequalities in (3.3) *are sharp in the sense that the constant* 1 *cannot be replaced by a smaller constant.*

Proof. Follows by the second identity in (2.1) and taking into account that

$$\|T_n(\mathbf{p}; \mathbf{a}, \mathbf{x})\| \leq |P_n| \sum_{i=1}^{n-1} \left| \frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{Pi} \right| \cdot |P_i| \, \|\Delta x_i\| \, .$$

Using Hölder's weighted inequality, we easily deduce (3.3).

The sharpness of the constant may be shown in a similar manner. We omit the details. □

The following corollary containing the unweighted inequalities holds.

Corollary 3. *With the above assumptions for* **a** *and* **x** *one, has*

$$\|T_n(\mathbf{a}, \mathbf{x})\| \tag{3.4}$$

$$\leq \frac{1}{n} \times \begin{cases} \max_{1\leq i\leq n-1} \left| \frac{1}{n} \sum_{k=1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right| \cdot \sum_{i=1}^{n-1} i \, \|\Delta x_i\| \, ; \\ \left(\sum_{i=1}^{n-1} i \left| \frac{1}{n} \sum_{k=1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right|^q \right)^{1/q} \cdot \left(\sum_{i=1}^{n-1} i \, \|\Delta x_i\|^p \right)^{1/p} \\ \textit{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} i \left| \frac{1}{n} \sum_{k=1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right| \cdot \max_{1\leq i\leq n-1} \|\Delta x_i\| \, . \end{cases}$$

The inequalities in (3.4) *are sharp in the sense mentioned above.*

Another type of inequalities may be stated if one uses the third identity in (2.1).

Theorem 6. *With the assumptions in Theorem 4 and if* $P_i, \overline{P}_i \neq 0, i \in \{1, ..., n-1\}$, *then we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq \begin{cases} \max_{1\leq i\leq n-1} \left| \frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i} - \frac{A_i(\mathbf{p})}{Pi} \right| \cdot \sum_{i=1}^{n-1} |P_i| \left|\overline{P}_i\right| \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} |P_i| \left|\overline{P}_i\right| \left| \frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i} - \frac{A_i(\mathbf{p})}{Pi} \right|^q \right)^{1/q} \cdot \left(\sum_{i=1}^{n-1} |P_i| \left|\overline{P}_i\right| \|\Delta x_i\|^p \right)^{1/p} \\ \text{for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1} |P_i| \left|\overline{P}_i\right| \left| \frac{\overline{A}_i(\mathbf{p})}{\overline{P}_i} - \frac{A_i(\mathbf{p})}{Pi} \right| \cdot \max_{1\leq i\leq n-1} \|\Delta x_i\|. \end{cases} \tag{3.5}$$

In particular, if $p_i = \frac{1}{n}, i \in \{1, ..., n\}$, *then we have*

$$\|T_n(\mathbf{a},\mathbf{x})\| \leq \frac{1}{n^2} \cdot \begin{cases} \max_{1\leq i\leq n-1} \left| \frac{1}{n-i} \sum_{k=i+1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right| \cdot \sum_{i=1}^{n-1} i(n-i) \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} i(n-i) \left| \frac{1}{n-i} \sum_{k=i+1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right|^q \right)^{1/q} \\ \times \left(\sum_{i=1}^{n-1} i(n-i) \|\Delta x_i\|^p \right)^{1/p} \text{ for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1} i(n-i) \left| \frac{1}{n-i} \sum_{k=i+1}^{n} a_k - \frac{1}{i} \sum_{k=1}^{i} a_k \right| \cdot \max_{1\leq i\leq n-1} \|\Delta x_i\|. \end{cases} \tag{3.6}$$

The inequalities in (3.5)*and* (3.6) *are sharp in the above mentioned sense.*

A different approach may be considered if one uses the representation in terms of double sums for the Čebyšev functional provided by the Theorem 3.

The following result holds.

Theorem 7. *With the assumptions in Theorem 4, we have the inequalities*

$$\|T_n(\mathbf{p};\mathbf{a},\mathbf{x})\| \leq \begin{cases} \max_{1\leq i,j\leq n-1} \left\{ \left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| \right\} \cdot \sum_{i=1}^{n-1} |\Delta a_i| \sum_{i=1}^{n-1} \|\Delta x_i\|; \\ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right|^q \left|\bar{P}_{\max\{i,j\}}\right|^q \right)^{1/q} \\ \times \left(\sum_{i=1}^{n-1} |\Delta a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n-1} \|\Delta x_i\|^p \right)^{1/p} \text{ for } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left|P_{\min\{i,j\}}\right| \left|\bar{P}_{\max\{i,j\}}\right| \\ \times \max_{1\leq i\leq n-1} |\Delta a_i| \max_{1\leq i\leq n-1} \|\Delta x_i\|. \end{cases} \tag{3.7}$$

The inequalities are sharp in the sense mentioned above.

The proof follows by the identity (2.7) on using Hölder's inequality for double sums and we omit the details.

Now, define

$$k_{\infty} := \max_{1\leq i,j\leq n-1}\left\{\frac{\min\{i,j\}}{n}\left(1-\frac{\max\{i,j\}}{n}\right)\right\}, n\geq 2.$$

Using the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2, \quad a,b\in R;$$

we deduce

$$\begin{aligned}\min\{i,j\}\cdot(n-\max\{i,j\}) &\leq \frac{1}{4}(n+\min\{i,j\}-\max\{i,j\})^2 \\ &= \frac{1}{4}(n-|i-j|)^2, 1\leq i,j\leq n-1.\end{aligned}$$

Consequently, we observe that

$$k_{\infty} \leq \frac{1}{4n^2}\max_{1\leq i,j\leq n-1}\left\{(n-|i-j|)^2\right\} = \frac{1}{4}.$$

We may state now the following corollary of Theorem 7.

Corollary 4. *Let* $(X,\|.\|)$ *be a normed linear space,* $\mathbf{a}=(a_1,...,a_n)\in\mathbb{K}^n$ *and* $\mathbf{x}=(x_1,...,x_n)\in\mathbb{X}^n$*. Then we have the inequality*

$$\|T_n(\mathbf{a},\mathbf{x})\| \leq k_{\infty}\sum_{i=1}^{n-1}|\Delta a_i|\sum_{i=1}^{n-1}\|\Delta x_i\| \leq \frac{1}{4}\sum_{i=1}^{n-1}|\Delta a_i|\sum_{i=1}^{n-1}\|\Delta x_i\|. \tag{3.8}$$

The constant $\frac{1}{4}$ *cannot be replaced in general by a smaller constant.*

Remark 2. *The inequality* (3.8) *is better than the second inequality in Corollary 1.*

Consider now, for $q>1$, the number

$$k_q := \frac{1}{n^2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}[\min\{i,j\}\cdot(n-\max\{i,j\})]^q\right)^{1/q}.$$

We observe, by the symmetry of the terms under the sums symbol, we have that

$$k_q = \frac{1}{n^2}\left(2\sum_{1\leq i<j\leq n-1} i^q(n-j)^q+\sum_{i=1}^{n-1} i^q(n-i)^q\right)^{1/q},$$

that may be computed exactly if $q=2$ or another natural number.

Since, as above,

$$[\min\{i,j\}\cdot(n-\max\{i,j\})]^q \leq \frac{1}{4^q}(n-|i-j|)^{2q}$$

we deduce

$$\begin{aligned}k_q &\leq \frac{1}{4n^2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(n-|i-j|)^{2q}\right)^{1/q} \\ &\leq \frac{1}{4n^2}\left[(n-1)^2 n^{2q}\right]^{1/q} = \frac{1}{4}(n-1)^{2/q}.\end{aligned}$$

Consequently, we may state the following corollary as well.

Corollary 5. *With the assumption in Corollary 4, we have the inequalities*

$$\begin{aligned}\|T_n(\mathbf{a},\mathbf{x})\| &\leq k_q\left(\sum_{i=1}^{n-1}|\Delta a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n-1}\|\Delta x_i\|^p\right)^{1/p} \\ &\leq \frac{1}{4}(n-1)^{2/q}\left(\sum_{i=1}^{n-1}|\Delta a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n-1}\|\Delta x_i\|^p\right)^{1/p};\end{aligned}$$

provided $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. *The constant* $\frac{1}{4}$ *cannot be replaced in general by a smaller constant.*

Finally, if we denote

$$k_1 := \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left[\min\{i,j\}\cdot(n-\max\{i,j\})\right],$$

then we observe, for $\mathbf{u}=\left(\frac{1}{n},...,\frac{1}{n}\right)$, $\mathbf{e}=(1,2,...,n)$, that

$$k_1 = T_n(\mathbf{u};\mathbf{e},\mathbf{e}) = \frac{1}{n}\sum_{i=1}^{n}i^2-\left(\frac{1}{n}\sum_{i=1}^{n}i\right)^2 = \frac{1}{12}\left(n^2-1\right),$$

and by Theorem 7, we deduce the inequality

$$\|T_n(\mathbf{a},\mathbf{x})\| \leq \frac{1}{12}\left(n^2-1\right)\max_{1\leq j\leq n-1}|\Delta a_j|\max_{1\leq j\leq n-1}\|\Delta x_j\|.$$

Note that, the above inequality, has been discovered with a different method in [1]. The constant $\frac{1}{12}$ is best possible.

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