MONOTONICITY PROPERTIES AND INEQUALITIES OF FUNCTIONS RELATED TO MEANS

CHAO-PING CHEN AND FENG QI

Abstract. In the paper, monotonicity properties of functions related to means are discussed and some inequalities are established.

1. Introduction

The extended logarithmic mean (Stolarsky mean) $L_r(a,b)$ of two positive numbers $a$ and $b$ is defined in [1, 2] for $a = b$ by $L_r(a,b) = a$ and for $a \neq b$ by

$$L_r(a,b) = \left( \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right)^{1/r}, \quad r \neq -1, 0;$$

$$L_{-1}(a,b) = \frac{b-a}{\ln b - \ln a} \triangleq L(a,b);$$

$$L_0(a,b) = \frac{1}{e} \left( \frac{b^a}{a^b} \right)^{1/(b-a)} \triangleq I(a,b).$$

When $a \neq b$, $L_r(a,b)$ is a strictly increasing function of $r$. Clearly,

$$J_{-2}(a,b) = H(a,b), \quad J_{-1/2}(a,b) = G(a,b), \quad J_1(a,b) = A(a,b),$$

where $A$ and $G$ are the arithmetic and geometric means, respectively.

The logarithmic mean $L(a,b)$ is generalized to the one-parameter mean in [3]:

$$J_r(a,b) = \frac{r(b^{r+1} - a^{r+1})}{(r+1)(b^r - a^r)}, \quad a \neq b \text{ and } r \neq 0,-1;$$

$$J_0(a,b) = L(a,b);$$

$$J_{-1}(a,b) = \frac{[G(a,b)]^2}{L(a,b)};$$

$$J_r(a,a) = a.$$

When $a \neq b$, $J_r(a,b)$ is a strictly increasing function of $r$. Clearly,

$$J_{-2}(a,b) = H(a,b), \quad J_{-1/2}(a,b) = G(a,b), \quad J_1(a,b) = A(a,b),$$

where $H$ is the harmonic mean.

For $a \neq b$, the following well known inequality holds clearly:

$$H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b).$$

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2. Lemmas

**Lemma 1.** Let \( a > 0 \) and \( b > 0 \), then we have

\[
J_{1/2}^2(a, b) \left( \frac{1}{J_{-1}(a, b)} - \frac{2}{J_0(a, b)} + \frac{1}{J_1(a, b)} \right)
= J_{-2}(a, b) - 2J_{-1}(a, b) + J_0(a, b)
\]

\( \quad \) \( (4) \)

and

\[
J_{-1/2}^2(a, b) \left( \frac{1}{J_{-2}(a, b)} - \frac{2}{J_{-1}(a, b)} + \frac{1}{J_0(a, b)} \right)
= J_{-1}(a, b) - 2J_0(a, b) + J_1(a, b).
\]

**Proof.** Notice that \( J_{-2}(a, b) = H(a, b), \ J_{-1}(a, b) = G^2(a, b)/L(a, b), \ J_{-1/2}(a, b) = G(a, b), \ J_0(a, b) = L(a, b) \) and \( J_1(a, b) = A(a, b) \), we obtain

\[
J_{-1/2}^2(a, b) \left( \frac{1}{J_{-1}(a, b)} - \frac{2}{J_0(a, b)} + \frac{1}{J_1(a, b)} \right)
= G^2(a, b) \left( \frac{L(a, b)}{G^2(a, b)} - \frac{2}{L(a, b)} + \frac{1}{A(a, b)} \right)
= L(a, b) - \frac{2G^2(a, b)}{L(a, b)} + \frac{G^2(a, b)}{A(a, b)}
= L(a, b) - \frac{2G^2(a, b)}{L(a, b)} + H(a, b)
= J_0(a, b) - 2J_{-1}(a, b) + J_{-2}(a, b)
\]

\( \quad \) \( (5) \)

and

\[
J_{-1/2}^2(a, b) \left( \frac{1}{J_{-2}(a, b)} - \frac{2}{J_{-1}(a, b)} + \frac{1}{J_0(a, b)} \right)
= G^2(a, b) \left( \frac{1}{H(a, b)} - \frac{2L(a, b)}{G^2(a, b)} + \frac{1}{L(a, b)} \right)
= \frac{G^2(a, b)}{H(a, b)} - 2L(a, b) + \frac{G^2(a, b)}{L(a, b)}
= A(a, b) - 2L(a, b) + \frac{G^2(a, b)}{L(a, b)}
= J_1(a, b) - 2J_0(a, b) + J_{-1}(a, b).
\]

The proof is complete. \( \square \)

**Corollary 1.** Let \( a > 0 \) and \( b > 0 \), then we have

\[
\left[ J_{-2}(a, b) - 2J_{-1}(a, b) + J_0(a, b) \right] \left( \frac{1}{J_{-2}(a, b)} - \frac{2}{J_{-1}(a, b)} + \frac{1}{J_0(a, b)} \right)
= \left[ J_{-1}(a, b) - 2J_0(a, b) + J_1(a, b) \right] \left( \frac{1}{J_{-1}(a, b)} - \frac{2}{J_0(a, b)} + \frac{1}{J_1(a, b)} \right).
\]

\( \quad \) \( (6) \)
Proof. By (4) and (5), we have
\[
\frac{J_{-2}(a, b) - 2J_{-1}(a, b) + J_0(a, b)}{J_{-1}(a, b) - 2J_0(a, b) + J_1(a, b)} = \frac{J_{-1}(a, b) - 2J_0(a, b) + J_1(a, b)}{J_{-1}(a, b) - 2J_0(a, b) + J_1(a, b)} = \frac{J_1(a, b) - 2J_0(a, b) + J_1(a, b)}{J_{-1}(a, b) - 2J_0(a, b) + J_1(a, b)} = J_{1/2}^2(a, b).
\]
Hence, (6) holds. \(\square\)

Lemma 2. Let \(a > 0, b > 0\) and \(a \neq b\), then we have for \(r = -1, 0,\)
\[
\frac{1}{J_{r-1}(a, b)} + \frac{1}{J_{r+1}(a, b)} > \frac{2}{J_r(a, b)}.
\]
(7)
Proof. Since \(a\) and \(b\) are symmetric, without loss of generality, assume \(b > a > 0\).
For \(r = -1, (7)\) becomes
\[
\frac{1}{H(a, b)} + \frac{1}{L(a, b)} > \frac{2L(a, b)}{G^2(a, b)},
\]
which is equivalent to
\[
\frac{2ab(\ln b - \ln a)^2 + (b^2 - a^2)(\ln b - \ln a) - 4(b - a)^2}{2ab(b - a)(\ln b - \ln a)} > 0.
\]
Clearly, \(2ab(b - a)(\ln b - \ln a) > 0\), thus it is sufficient to prove that
\[
\phi(x) \triangleq 2ax(\ln x - \ln a)^2 + (x^2 - a^2)(\ln x - \ln a) - 4(x - a)^2 > 0
\]
for \(x > a > 0\). Simple computation reveals that
\[
\phi'(x) = 2a(\ln x - \ln a)^2 + (2x + 4a)(\ln x - \ln a) - 7x - \frac{a^2}{x} + 8a,
\]
\[
x\phi''(x) = (2x + 4a)(\ln x - \ln a) - 5x + \frac{a^2}{x} + 4a \triangleq \psi(x),
\]
\[
\psi'(x) = \frac{4a}{x} + 2(\ln x - \ln a) - \frac{a^2}{x^2} - 3,
\]
\[
\psi''(x) = \frac{2(x - a)^2}{x^3} > 0.
\]
Hence, we have for \(x > a,\)
\[
\psi'(x) > \psi'(a) = 0 \implies \psi(x) > \psi(a) = 0 \implies \phi''(x) > 0
\]
\[
\implies \phi'(x) > \phi'(a) = 0 \implies \phi(x) > \phi(a) = 0.
\]
Thus, (7) holds for \(r = -1.\)
For \(r = 0, (7)\) becomes
\[
\frac{L(a, b)}{G^2(a, b)} + \frac{1}{A(a, b)} > \frac{2}{L(a, b)},
\]
which is equivalent to
\[
\frac{-2ab(b + a)(\ln b - \ln a)^2 + 2ab(b - a)(\ln b - \ln a) + (b - a)^2(b + a)}{ab(b + a)((b - a))(\ln b - \ln a)} > 0.
\]
Clearly, \(ab(b + a)(b - a)(\ln b - \ln a) > 0\), thus it is sufficient to prove that
\[
u(x) \triangleq -2ax(x + a)(\ln x - \ln a)^2 + 2ax(x - a)(\ln x - \ln a) + (x - a)^2(x + a) > 0
\]
Then we have for 

\[
x>v(a) = 0 \Rightarrow v''(x) < 0 \Rightarrow v'(x) > v'(a) = 0 \Rightarrow v(x) > v(a) = 0 \Rightarrow u''(x) > 0 \Rightarrow u'(x) > u'(a) = 0 \Rightarrow u(x) > u(a) = 0.
\]

Thus, (7) holds for \( r = 0 \). The proof is complete. \( \square \)

By Lemma 1 and Lemma 2, the following corollary is obvious.

**Corollary 2.** Let \( a > 0, b > 0 \) and \( a \neq b \), then

\[
J_{-1}(a,b) + J_1(a,b) > 2J_0(a,b),
\]

\[
J_{-2}(a,b) + J_0(a,b) > 2J_{-1}(a,b).
\]

**Lemma 3.** Let \( a > 0, r \in (-\infty, \infty) \), define for \( x > 0 \),

\[
R_r(x) = \begin{cases} 
\frac{L_r^2(a,x)}{L_{r-1}(a,x)L_{r+1}(a,x)}, & x \neq a, \\
1, & x = a.
\end{cases}
\]

Then we have for \( x \neq a \),

\[
\frac{1}{R_r(x)} \frac{dR_r(x)}{dx} = \frac{a}{x-a} \left( -\frac{2}{J_r(a,x)} + \frac{1}{J_{r-1}(a,x)} + \frac{1}{J_{r+1}(a,x)} \right).
\]

**Proof.** Taking the logarithm and differentiation yields

\[
\frac{x-a}{R_r(x)} \frac{dR_r(x)}{dx} = \frac{2(r+1)x^{r-1} + r^2 + r}{r(x^{r+1} - a^{r+1})} - \frac{(r-1)x^{r-1} - rax^{r-1} + a^{r-1}}{(r-1)(x^{r} - a^{r})} - \frac{(r+1)x^{r+2} - (r+2)ax^{r+1} + a^{r+2}}{(r+1)(x^{r+2} - a^{r+2})} = 2 \left( \frac{r+1}{r(x^{r+1} - a^{r+1})} + \frac{1}{r(x^{r} - a^{r})} - \frac{1}{(r+1)(x^{r+2} - a^{r+2})} \right) = \frac{2a(r+1)(x^{r} - a^{r}) + a(r+2)(x^{r+1} - a^{r+1})}{(r-1)(x^{r} - a^{r})} + \frac{a(r+1)(x^{r+2} - a^{r+2})}{(r+1)(x^{r+2} - a^{r+2})} = \frac{2a}{J_r(a,x)} + \frac{a}{J_{r-1}(a,x)} + \frac{a}{J_{r+1}(a,x)}.
\]

The proof is complete. \( \square \)
3. Main Results

**Theorem 1.** Let $a > 0$, define for $x > 0$,

$$f(x) = \begin{cases} 
G^2(a, x) & x \neq a, \\
(H(a, x)L(a, x), & x = a.
\end{cases}$$

(12)

Then $f(x)$ is strictly decreasing in $(0, a)$ and strictly increasing in $(a, \infty)$.

**Proof.** Taking the logarithm and differentiation yields

$$f'(x) = \frac{1}{x + a} - \frac{x(\ln x - \ln a) - (x - a)}{x(x - a)(\ln x - \ln a)} = 2a \left[ \frac{x^2 - a^2 - (\ln x - \ln a)}{(x + a)(x - a)(\ln x - \ln a)} \right] = 2a \frac{x - a}{(x + a)(x - a)(\ln x - \ln a)} \left( \frac{1}{H(a, x)} - \frac{1}{L(a, x)} \right) = \frac{2a[H(a, x) - L(a, x)]}{(x + a)(x - a)H(a, x)}.$$ 

Since $L(a, x) > H(a, x)$, it is clear that $f'(x) < 0$ for $0 < x < a$ and $f'(x) > 0$ for $x > a$. The proof is complete. \[ \square \]

**Corollary 3.** Let $c > b > a > 0$, then

$$\left( \frac{G(a, b)}{G(a, c)} \right)^2 < \frac{H(a, b) L(a, b)}{H(a, c) L(a, c)}. \quad (13)$$

The inequality in (13) is reversed for $0 < b < c < a$.

Since $f(x)$ is continuous on $(0, \infty)$ and takes its only minimum $f(a) = 1$ at $x = a$, we get

**Corollary 4.** Let $a > 0, b > 0$ and $a \neq b$, then

$$G^2(a, b) > H(a, b) L(a, b). \quad (14)$$

**Theorem 2.** Let $a > 0$, define for $x > 0$,

$$g(x) = \begin{cases} 
L^2(a, x) & x \neq a, \\
\frac{G(a, x)I(a, x)}{G(a, x)} & x = a;
\end{cases}$$

(15)

$$h(x) = \begin{cases} 
L^2(a, x) & x \neq a, \\
\frac{I^2(a, x)}{L(a, x)A(a, x)} & x = a.
\end{cases}$$

(16)

Then both $g(x)$ and $h(x)$ are strictly decreasing in $(0, a)$ and strictly increasing in $(a, \infty)$.

**Proof.** By Lemma 3 (taking $r = -1, 0$, respectively), we have for $x \neq a$,

$$\frac{g'(x)}{g(x)} = \frac{a}{x - a} \left( -\frac{2}{J_1(a, x)} + \frac{1}{J_2(a, x)} + \frac{1}{J_0(a, x)} \right),$$
\[
\frac{h'(x)}{h(x)} = \frac{a}{x-a} \left( -\frac{2}{J_0(a,x)} + \frac{1}{J_{-1}(a,x)} + \frac{1}{J_1(a,x)} \right).
\]

By Lemma 2, we have for \(x \neq a\),
\[
-\frac{2}{J_{-1}(a,x)} + \frac{1}{J_0(a,x)} > 0,
\]
\[
-\frac{2}{J_0(a,x)} + \frac{1}{J_{-1}(a,x)} + \frac{1}{J_1(a,x)} > 0.
\]

Hence, it is clear that \(g'(x) < 0\) and \(h'(x) < 0\) for \(0 < x < a\), and \(g'(x) > 0\) and \(h'(x) > 0\) for \(x > a\). The proof is complete. \(\square\)

**Corollary 5.** Let \(c > b > a > 0\), then
\[
\left( \frac{L(a,b)}{L(a,c)} \right)^2 < \frac{G(a,b)I(a,b)}{G(a,c)I(a,c)}, \tag{17}
\]
\[
\left( \frac{I(a,b)}{I(a,c)} \right)^2 < \frac{L(a,b)A(a,b)}{L(a,c)A(a,c)}. \tag{18}
\]

The inequalities in (17) and (18) are reversed for \(0 < b < c < a\).

Since both \(g\) and \(h\) are continuous on \((0, \infty)\) and take their unique minimum \(g(a) = h(a) = 1\) at \(x = a\), we get

**Corollary 6.** Let \(a > 0, b > 0\) and \(a \neq b\), then
\[
L^2(a,b) > G(a,b)I(a,b), \tag{19}
\]
\[
I^2(a,b) > L(a,b)A(a,b). \tag{20}
\]

**References**


(Ch.-P. Chen) **DEPARTMENT OF APPLIED MATHEMATICS AND INFORMATICS, JIAOZUO INSTITUTE OF TECHNOLOGY, JIAOZUO CITY, HENAN 454000, CHINA**

*E-mail address:* chenchaoping@sohu.com

(F. Qi) **DEPARTMENT OF APPLIED MATHEMATICS AND INFORMATICS, JIAOZUO INSTITUTE OF TECHNOLOGY, JIAOZUO CITY, HENAN 454000, CHINA**

*E-mail address:* qifeng@jzit.edu.cn, fengqi618@member.ams.org