MONOTONICITY PROPERTIES AND INEQUALITIES OF FUNCTIONS RELATED TO MEANS

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Abstract. In the paper, monotonicity properties of functions related to means are discussed and some inequalities are established.

1. Introduction

The extended logarithmic mean (Stolarsky mean) $L_r(a,b)$ of two positive numbers a and b is defined in [1, 2] for $a=b$ by $L_r(a,b)=a$ and for $a\neq b$ by

$$\begin{aligned} L_r(a,b) &= \left(\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, \quad r\neq -1,0;\\ L_{-1}(a,b) &= \frac{b-a}{\ln b-\ln a} \triangleq L(a,b); \\ L_0(a,b) &= \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \triangleq I(a,b). \end{aligned} \tag{1}$$

When $a\neq b$, $L_r(a,b)$ is a strictly increasing function of r. Clearly,

$$L_1(a,b)=A(a,b), \quad L_{-2}(a,b)=G(a,b),$$

where A and G are the arithmetic and geometric means, respectively.

The logarithmic mean $L(a,b)$ is generalized to the one-parameter mean in [3]:

$$\begin{aligned} J_r(a,b) &= \frac{r(b^{r+1}-a^{r+1})}{(r+1)(b^r-a^r)}, \quad a\neq b \text{ and } r\neq 0,-1;\\ J_0(a,b) &= L(a,b);\\ J_{-1}(a,b) &= \frac{[G(a,b)]^2}{L(a,b)};\\ J_r(a,a) &= a. \end{aligned}$$

When $a\neq b$, $J_r(a,b)$ is a strictly increasing function of r. Clearly,

$$J_{-2}(a,b)=H(a,b), \quad J_{-1/2}(a,b)=G(a,b), \quad J_1(a,b)=A(a,b), \tag{2}$$

where H is the harmonic mean.

For $a\neq b$, the following well known inequality holds clearly:

$$H(a,b)<G(a,b)<L(a,b)<I(a,b)<A(a,b). \tag{3}$$

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2. Lemmas

Lemma 1. *Let $a > 0$ and $b > 0$, then we have*

$$\begin{aligned} & J_{-1/2}^2(a,b)\left(\frac{1}{J_{-1}(a,b)} - \frac{2}{J_0(a,b)} + \frac{1}{J_1(a,b)}\right) \\ &= J_{-2}(a,b) - 2J_{-1}(a,b) + J_0(a,b) \end{aligned} \tag{4}$$

and

$$\begin{aligned} & J_{-1/2}^2(a,b)\left(\frac{1}{J_{-2}(a,b)} - \frac{2}{J_{-1}(a,b)} + \frac{1}{J_0(a,b)}\right) \\ &= J_{-1}(a,b) - 2J_0(a,b) + J_1(a,b). \end{aligned} \tag{5}$$

Proof. Notice that $J_{-2}(a,b) = H(a,b)$, $J_{-1}(a,b) = G^2(a,b)/L(a,b)$, $J_{-1/2}(a,b) = G(a,b)$, $J_0(a,b) = L(a,b)$ and $J_1(a,b) = A(a,b)$, we obtain

$$\begin{aligned} & J_{-1/2}^2(a,b)\left(\frac{1}{J_{-1}(a,b)} - \frac{2}{J_0(a,b)} + \frac{1}{J_1(a,b)}\right) \\ &= G^2(a,b)\left(\frac{L(a,b)}{G^2(a,b)} - \frac{2}{L(a,b)} + \frac{1}{A(a,b)}\right) \\ &= L(a,b) - \frac{2G^2(a,b)}{L(a,b)} + \frac{G^2(a,b)}{A(a,b)} \\ &= L(a,b) - \frac{2G^2(a,b)}{L(a,b)} + H(a,b) \\ &= J_0(a,b) - 2J_{-1}(a,b) + J_{-2}(a,b) \end{aligned}$$

and

$$\begin{aligned} & J_{-1/2}^2(a,b)\left(\frac{1}{J_{-2}(a,b)} - \frac{2}{J_{-1}(a,b)} + \frac{1}{J_0(a,b)}\right) \\ &= G^2(a,b)\left(\frac{1}{H(a,b)} - \frac{2L(a,b)}{G^2(a,b)} + \frac{1}{L(a,b)}\right) \\ &= \frac{G^2(a,b)}{H(a,b)} - 2L(a,b) + \frac{G^2(a,b)}{L(a,b)} \\ &= A(a,b) - 2L(a,b) + \frac{G^2(a,b)}{L(a,b)} \\ &= J_1(a,b) - 2J_0(a,b) + J_{-1}(a,b). \end{aligned}$$

The proof is complete. □

Corollary 1. *Let $a > 0$ and $b > 0$, then we have*

$$\begin{aligned} & [J_{-2}(a,b) - 2J_{-1}(a,b) + J_0(a,b)]\left(\frac{1}{J_{-2}(a,b)} - \frac{2}{J_{-1}(a,b)} + \frac{1}{J_0(a,b)}\right) \\ &= [J_{-1}(a,b) - 2J_0(a,b) + J_1(a,b)]\left(\frac{1}{J_{-1}(a,b)} - \frac{2}{J_0(a,b)} + \frac{1}{J_1(a,b)}\right). \end{aligned} \tag{6}$$

Proof. By (4) and (5), we have

$$\begin{aligned}&\frac{J_{-2}(a,b)-2J_{-1}(a,b)+J_0(a,b)}{J_{-1}^{-1}(a,b)-2J_0^{-1}(a,b)+J_1^{-1}(a,b)}\\ &=\frac{J_{-1}(a,b)-2J_0(a,b)+J_1(a,b)}{J_{-2}^{-1}(a,b)-2J_{-1}^{-1}(a,b)+J_0^{-1}(a,b)}\\ &=J_{-1/2}^2(a,b).\end{aligned}$$

Hence, (6) holds. □

Lemma 2. *Let $a>0$, $b>0$ and $a\neq b$, then we have for $r=-1,0$,*

$$\frac{1}{J_{r-1}(a,b)}+\frac{1}{J_{r+1}(a,b)}>\frac{2}{J_r(a,b)}. \tag{7}$$

Proof. Since a and b are symmetric, without loss of generality, assume $b>a>0$. For $r=-1$, (7) becomes

$$\frac{1}{H(a,b)}+\frac{1}{L(a,b)}>\frac{2L(a,b)}{G^2(a,b)},$$

which is equivalent to

$$\frac{2ab(\ln b-\ln a)^2+(b^2-a^2)(\ln b-\ln a)-4(b-a)^2}{2ab(b-a)(\ln b-\ln a)}>0.$$

Clearly, $2ab(b-a)(\ln b-\ln a)>0$, thus it is sufficient to prove that

$$\phi(x)\triangleq 2ax(\ln x-\ln a)^2+(x^2-a^2)(\ln x-\ln a)-4(x-a)^2>0$$

for $x>a>0$. Simple computation reveals that

$$\begin{aligned}\phi'(x)&=2a(\ln x-\ln a)^2+(2x+4a)(\ln x-\ lna)-7x-\frac{a^2}{x}+8a,\\ x\phi''(x)&=(2x+4a)(\ln x-\ln a)-5x+\frac{a^2}{x}+4a\triangleq\psi(x),\\ \psi'(x)&=\frac{4a}{x}+2(\ln x-\ln a)-\frac{a^2}{x^2}-3,\\ \psi''(x)&=\frac{2(x-a)^2}{x^3}>0.\end{aligned}$$

Hence, we have for $x>a$,

$$\begin{aligned}\psi'(x)>\psi'(a)=0\Longrightarrow\psi(x)>\psi(a)=0\Longrightarrow\phi''(x)>0\\ \Longrightarrow\phi'(x)>\phi'(a)=0\Longrightarrow\phi(x)>\phi(a)=0.\end{aligned}$$

Thus, (7) holds for $r=-1$.

For $r=0$, (7) becomes

$$\frac{L(a,b)}{G^2(a,b)}+\frac{1}{A(a,b)}>\frac{2}{L(a,b)},$$

which is equivalent to

$$\frac{-2ab(b+a)(\ln b-\ln a)^2+2ab(b-a)(\ln b-\ln a)+(b-a)^2(b+a)}{ab(b+a)((b-a))(\ln b-\ln a)}>0.$$

Clearly, $ab(b+a)(b-a)(\ln b-\ln a)>0$, thus it is sufficient to prove that

$$u(x)\triangleq -2ax(x+a)(\ln x-\ln a)^2+2ax(x-a)(\ln x-\ln a)+(x-a)^2(x+a)>0$$

for $x > a > 0$. Simple computation reveals that

$$\begin{aligned}
u'(x) &= -(4ax + 2a^2)(\ln x - \ln a)^2 - 6a^2(\ln x - \ln a) + 3(x^2 - a^2),\\
xu''(x) &= -4ax(\ln x - \ln a)^2 - 4a(2x + a)(\ln x - \ln a) + 6(x^2 - a^2) \triangleq v(x),\\
v'(x) &= -4a(\ln x - \ln a)^2 - 16a(\ln x - \ln a) - 8a - \frac{4a^2}{x} + 12x,\\
xv''(x) &= -8a(\ln x - \ln a) - 16a + \frac{4a^2}{x} + 12x \triangleq w(x),\\
w'(x) &= \frac{4(3x + a)(x - a)}{x^2} > 0.
\end{aligned}$$

Hence, we have for $x > a$,

$$\begin{aligned}
w(x) > w(a) = 0 \Longrightarrow v''(x) > 0 \Longrightarrow v'(x) > v'(a) = 0 \Longrightarrow v(x) > v(a) = 0\\
\Longrightarrow u''(x) > 0 \Longrightarrow u'(x) > u'(a) = 0 \Longrightarrow u(x) > u(a) = 0.
\end{aligned}$$

Thus, (7) holds for $r = 0$. The proof is complete. □

By Lemma 1 and Lemma 2, the following corollary is obvious.

Corollary 2. *Let $a > 0$, $b > 0$ and $a \neq b$, then*

$$J_{-1}(a, b) + J_1(a, b) > 2J_0(a, b), \tag{8}$$

$$J_{-2}(a, b) + J_0(a, b) > 2J_{-1}(a, b). \tag{9}$$

Lemma 3. *Let $a > 0$, $r \in (-\infty, \infty)$, define for $x > 0$,*

$$R_r(x) = \begin{cases} \dfrac{L_r^2(a, x)}{L_{r-1}(a, x)L_{r+1}(a, x)}, & x \neq a,\\ 1, & x = a. \end{cases} \tag{10}$$

Then we have for $x \neq a$,

$$\frac{1}{R_r(x)}\frac{\mathrm{d}R_r(x)}{dx} = \frac{a}{x - a}\left(-\frac{2}{J_r(a, x)} + \frac{1}{J_{r-1}(a, x)} + \frac{1}{J_{r+1}(a, x)}\right). \tag{11}$$

Proof. Taking the logarithm and differentiation yields

$$\begin{aligned}
&\frac{x - a}{R_r(x)}\frac{\mathrm{d}R_r(x)}{dx}\\
&= \frac{2(rx^{r+1} - (r + 1)ax^r + a^{r+1})}{r(x^{r+1} - a^{r+1})} - \frac{(r - 1)x^r - rax^{r-1} + a^r}{(r - 1)(x^r - a^r)}\\
&\quad - \frac{(r + 1)x^{r+2} - (r + 2)ax^{r+1} + a^{r+2}}{(r + 1)(x^{r+2} - a^{r+2})}\\
&= 2\left(\frac{rx^{r+1} - (r + 1)ax^r + a^{r+1}}{r(x^{r+1} - a^{r+1})} - 1\right) - \left(\frac{(r - 1)x^r - rax^{r-1} + a^r}{(r - 1)(x^r - a^r)} - 1\right)\\
&\quad - \left(\frac{(r + 1)x^{r+2} - (r + 2)ax^{r+1} + a^{r+2}}{(r + 1)(x^{r+2} - a^{r+2})} - 1\right)\\
&= -\frac{2a(r + 1)(x^r - a^r)}{r(x^{r+1} - a^{r+1})} + \frac{ar(x^{r-1} - a^{r-1})}{(r - 1)(x^r - a^r)} + \frac{a(r + 2)(x^{r+1} - a^{r+1})}{(r + 1)(x^{r+2} - a^{r+2})}\\
&= -\frac{2a}{J_r(a, x)} + \frac{a}{J_{r-1}(a, x)} + \frac{a}{J_{r+1}(a, x)}.
\end{aligned}$$

The proof is complete. □

3. Main Results

Theorem 1. *Let $a > 0$, define for $x > 0$,*

$$f(x) = \begin{cases} \dfrac{G^2(a,x)}{H(a,x)L(a,x)}, & x \neq a, \\ 1, & x = a. \end{cases} \tag{12}$$

Then $f(x)$ is strictly decreasing in $(0, a)$ and strictly increasing in (a, ∞).

Proof. Taking the logarithm and differentiation yields

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{1}{x+a} - \frac{x(\ln x - \ln a) - (x-a)}{x(x-a)(\ln x - \ln a)} \\ &= \frac{2a\left[\frac{x^2-a^2}{2ax} - (\ln x - \ln a)\right]}{(x+a)(x-a)(\ln x - \ln a)} \\ &= \frac{2a}{(x+a)(x-a)} \frac{x-a}{\ln x - \ln a} \left(\frac{x+a}{2ax} - \frac{\ln x - \ln a}{x-a}\right) \\ &= \frac{2aL(a,x)}{(x+a)(x-a)} \left(\frac{1}{H(a,x)} - \frac{1}{L(a,x)}\right) \\ &= \frac{2a[L(a,x) - H(a,x)]}{(x+a)(x-a)H(a,x)}. \end{aligned}$$

Since $L(a,x) > H(a,x)$, it is clear that $f'(x) < 0$ for $0 < x < a$ and $f'(x) > 0$ for $x > a$. The proof is complete. □

Corollary 3. *Let $c > b > a > 0$, then*

$$\left(\frac{G(a,b)}{G(a,c)}\right)^2 < \frac{H(a,b)L(a,b)}{H(a,c)L(a,c)}. \tag{13}$$

The inequality in (13) is reversed for $0 < b < c < a$.

Since $f(x)$ is continuous on $(0, \infty)$ and takes its only minimum $f(a) = 1$ at $x = a$, we get

Corollary 4. *Let $a > 0, b > 0$ and $a \neq b$, then*

$$G^2(a,b) > H(a,b)L(a,b). \tag{14}$$

Theorem 2. *Let $a > 0$, define for $x > 0$,*

$$g(x) = \begin{cases} \dfrac{L^2(a,x)}{G(a,x)I(a,x)}, & x \neq a, \\ 1, & x = a; \end{cases} \tag{15}$$

$$h(x) = \begin{cases} \dfrac{I^2(a,x)}{L(a,x)A(a,x)}, & x \neq a, \\ 1, & x = a. \end{cases} \tag{16}$$

Then both $g(x)$ and $h(x)$ are strictly decreasing in $(0, a)$ and strictly increasing in (a, ∞).

Proof. By Lemma 3 (taking $r = -1, 0$, respectively), we have for $x \neq a$,

$$\frac{g'(x)}{g(x)} = \frac{a}{x-a}\left(-\frac{2}{J_{-1}(a,x)} + \frac{1}{J_{-2}(a,x)} + \frac{1}{J_0(a,x)}\right),$$

$$\frac{h'(x)}{h(x)} = \frac{a}{x-a}\left(-\frac{2}{J_0(a,x)} + \frac{1}{J_{-1}(a,x)} + \frac{1}{J_1(a,x)}\right).$$

By Lemma 2, we have for $x \neq a$,

$$-\frac{2}{J_{-1}(a,x)} + \frac{1}{J_{-2}(a,x)} + \frac{1}{J_0(a,x)} > 0,$$
$$-\frac{2}{J_0(a,x)} + \frac{1}{J_{-1}(a,x)} + \frac{1}{J_1(a,x)} > 0.$$

Hence, it is clear that $g'(x) < 0$ and $h'(x) < 0$ for $0 < x < a$, and $g'(x) > 0$ and $h'(x) > 0$ for $x > a$. The proof is complete. □

Corollary 5. *Let $c > b > a > 0$, then*

$$\left(\frac{L(a,b)}{L(a,c)}\right)^2 < \frac{G(a,b)I(a,b)}{G(a,c)I(a,c)}, \tag{17}$$

$$\left(\frac{I(a,b)}{I(a,c)}\right)^2 < \frac{L(a,b)A(a,b)}{L(a,c)A(a,c)}. \tag{18}$$

The inequalities in (17) and (18) are reversed for $0 < b < c < a$.

Since both g and h are continuous on $(0,\infty)$ and take their unique minimum $g(a) = h(a) = 1$ at $x = a$, we get

Corollary 6. *Let $a > 0, b > 0$ and $a \neq b$, then*

$$L^2(a,b) > G(a,b)I(a,b), \tag{19}$$

$$I^2(a,b) > L(a,b)A(a,b). \tag{20}$$

References

[1] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92.

[2] K. B. Stolarsky, *The power and generalized logarithmic means*, Amer. Math. Monthly **87** (1980), 545–548.

[3] R.-Er Yang, D.-J. Cao, *Generalizations of the logarithmic mean*, J. Ningbo Univ. **2** (1989), no. 2, 105–108.

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