MONOTONICITY AND CONVEXITY OF THE FUNCTION $\sqrt[x]{\Gamma(x+1)}\,/\,\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}$

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ABSTRACT. For $\alpha > 0$ a real number, the function $\frac{\sqrt[x]{\Gamma(x+1)}}{\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}}$ is increasing with $x \in (x_0, \infty)$ and logarithmically concave with $x \in [1, \infty)$, where $x_0 \in (0, 1)$ is a constant. Moreover, some monotonicity and convexity results and inequalities of functions involving the gamma function and polygamma functions are obtained as corollaries and by-products.

1. Introduction

In this section, we first state some known results: monotonicity of the geometric mean sequence and some sequences involving geometric means, inequalities of ratio between geometric means and ratio between power means, and monotonicity and convexity of ratio between two gamma functions and functions involving the gamma function.

1.1. Inequalities of ratio between power means.

H. Minc and L. Sathre in [27] gave the lower and upper bounds of ratio between two geometric means of natural numbers:

$$\frac{n-1}{n} < \frac{\sqrt[n-1]{(n-1)!}}{\sqrt[n]{n!}} < 1. \tag{1}$$

The right hand side inequality in (1) also reveals that the geometric mean sequence $\left\{\sqrt[n]{n!}\right\}_{n\in\mathbb{N}}$ is strictly increasing and the sequence $\left\{\frac{\sqrt[n]{n!}}{n}\right\}_{n\in\mathbb{N}}$ is strictly decreasing. Note that $\Gamma(n+1) = n!$, where the gamma function is usually defined [19, 54] for $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t. \tag{2}$$

In [2, 26], H. Alzer and J. S. Martins refined the left hand side inequality in (1) and showed that, if n is a positive integer, then, for all positive real numbers r,

$$\frac{n}{n+1} < \left(\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r}\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{3}$$

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Both bounds in (3) are the best possible. The middle term in (3) is indeed a ratio between the power means $\left(\frac{1}{n}\sum_{i=1}^{n} i^r\right)^{1/r}$ and $\left(\frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r}$. The inequality (3) implies that the sequence $\left\{\frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n} i^r\right)^{1/r}\right\}_{n\in\mathbb{N}}$ is decreasing strictly and the sequence $\left\{\frac{1}{\sqrt[n]{n!}}\left(\frac{1}{n}\sum_{i=1}^{n} i^r\right)^{1/r}\right\}_{n\in\mathbb{N}}$ is increasing strictly for given $r>0$.

The integral form of inequality (3) was established in [29, 41] by the authors: Let $b>a>0$ and $\delta>0$ be real numbers. Then, for any positive $r\in\mathbb{R}$, we have

$$\begin{aligned}\frac{b}{b+\delta} &< \left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r}\\ &= \left(\frac{\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,\mathrm{d}x}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \end{aligned} \tag{4}$$

The lower and upper bounds in (4) are the best possible. The inequality (4) can be restated as monotonicity results: The function $\frac{1}{x}\left(\frac{x^{r+1}-a^{r+1}}{x-a}\right)^{1/r}$ is decreasing and the function $\frac{1}{[x^x/a^a]^{1/(x-a)}}\left(\frac{x^{r+1}-a^{r+1}}{x-a}\right)^{1/r}$ is increasing with $x>0$ for given $r>0$. Notice that $\frac{1}{e}\left[\frac{x^x}{a^a}\right]^{1/(x-a)}$ is called the identric or exponential mean.

After obtaining the following generalization of the left hand side inequality of (3):

$$\frac{n+k}{n+m+k} < \left(\frac{\frac{1}{n}\sum_{i=k+1}^{n+k} i^r}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r}\right)^{1/r}, \tag{5}$$

where r is a given positive real number, n and m are natural numbers and k is a nonnegative integer, the first author in [31] asked as an open problem the validity of an inequality below:

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n} a_i^r}{\frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r}\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{6}$$

where r is a positive number, $a_n!$ denotes the sequence factorial defined by $\prod_{i=1}^{n} a_i$. The upper bound in (6) is the best possible. Inequality (5) means that the sequence $\left\{\frac{1}{n+k}\left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r\right)^{1/r}\right\}_{n\in\mathbb{N}}$ is decreasing for given $r>0$ and nonnegative integer k.

Inequality (6) has been researched in [3, 36, 55], some sufficient conditions are found. The first author in [36] obtained: Let $n,m\in\mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m-1}$ is increasing, then inequality (6) holds. In particular, let a be a positive real numbers, b a nonnegative real number, k a nonnegative integer, and $m,n\in\mathbb{N}$, then, for any real number $r>0$, we have

$$\left(\frac{\frac{1}{n}\sum_{i=k+1}^{n+k}(ai+b)^r}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k}(ai+b)^r}\right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(ai+b)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(ai+b)}}. \tag{7}$$

The authors and a coworker in [47] give a lower bound for ratio betweem two power means: Let n and m be natural numbers, suppose $\{a_i\}_{i=1}^{n+m}$ is an increasing, logarithmically convex, and positive sequence, then

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n}a_i^r}{\frac{1}{n+m}\sum_{i=1}^{n+m}a_i^r}\right)^{1/r}\ge\frac{a_n}{a_{n+m}}. \tag{8}$$

The lower bound in (8) is the best possible.

Remark 1. Indeed, the inequalities (3) to (7) are also valid for negative power r. For more information, please refer to [4, 5] and some unpublished papers.

1.2. **Inequalities of ratio between geometric means.** The inequalities in (1) were also refined and generalized in [33, 45, 48] and the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1}<\frac{\left(\prod_{i=k+1}^{n+k}i\right)^{1/n}}{\left(\prod_{i=k+1}^{n+m+k}i\right)^{1/(n+m)}}\le\sqrt{\frac{n+k}{n+m+k}}\,, \tag{9}$$

where k is a nonnegative integer, n and m are natural numbers. For $n=m=1$, the equality in (9) is valid.

In [15, 34], inequalities in (9) were generalized and obtained the following inequalities on the ratio for the geometric means of a positive arithmetic sequence:

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b}\le\frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{\frac{1}{n+m}}}\le\sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}\,, \tag{10}$$

where a and b are positive constants, k is a nonnegative integer, n and m are natural numbers.

In [43, 44], the following general monotonicity properties are established: Let f be a positive function defined on $[1,\infty)$ such that $\frac{f(x+2)}{f(x+1)}\ge\frac{x+2}{x+1}\left[\frac{x(x+2)}{(x+1)^2}\right]^{\frac{x}{2}}$ for $x\ge0$, then the sequence $\left\{\frac{\sqrt[n]{\prod_{i=1}^{n}f(i)}}{\sqrt{n}}\right\}_{n\in\mathbb{N}}$ is increasing; if $\frac{f(x+2)}{f(x+1)}\le\left(\frac{x+3}{x+2}\right)^2\left[\frac{(x+1)(x+3)}{(x+2)^2}\right]^x$ holds on $[0,\infty)$, then the sequence $\left\{\frac{\sqrt[n]{\prod_{i=1}^{n}f(i)}}{(n+1)}\right\}_{n\in\mathbb{N}}$ is decreasing. Let f be a positive function such that $x\left[\frac{f(x+1)}{f(x)}-1\right]$ is increasing on $[1,\infty)$, then the sequence $\left\{\frac{\sqrt[n]{\prod_{i=1}^{n}f(i)}}{f(n+1)}\right\}_{n=1}^{\infty}$ is decreasing; if f is a logarithmically concave and positive function defined on $[1,\infty)$, then the sequence $\left\{\frac{\sqrt[n]{\prod_{i=1}^{n}f(i)}}{\sqrt{f(n)}}\right\}_{n=1}^{\infty}$ is increasing. As consequences of these monotonicities, the lower and upper bounds for the ratio $\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}f(i)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+k+m}f(i)}}$ are obtained, where k is a nonnegative integer and m a natural number.

As lemmas in [36], the following results were showed: Let $n,m\in\mathbb{N}$, and $\{a_i\}_{i=1}^{n+m+1}$ a nonconstant positive sequence such that the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m}$ is increasing, then the sequence $\left\{\frac{\sqrt[i]{a_i!}}{a_{i+1}}\right\}_{i=1}^{n+m}$ is decreasing, and then $\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}>\frac{a_{n+1}}{a_{n+m+1}}$. Let $n>1$ be a positive integer and $\{a_i\}_{i=1}^{n}$ an increasing nonconstant positive sequence such that $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n-1}$ is increasing, then the sequence

$\left\{\frac{a_i}{(a_i!)^{1/i}}\right\}_{i=1}^{n}$ is increasing, and then $\frac{a_\ell}{a_n} < \frac{\sqrt[\ell]{a_\ell!}}{\sqrt[n]{a_n!}}$ for any positive integer ℓ satisfying $1 \le \ell < n$, where $a_n!$ denotes the sequence factorial $\prod_{i=1}^{n} a_i$.

In [42] and a subsequent paper [7], some inequalities for ratios of geometric means of positive sequence are obtained as applications: If $\{a_i\}_{i\in\mathbb{N}}$ is an increasing, positive sequence such that $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ increases, then we have

$$\frac{a_n}{a_{n+1}} \le \frac{\sqrt[n]{\prod\limits_{i=1}^{n}(a_i+a_n)}}{\sqrt[n+1]{\prod\limits_{i=1}^{n+1}(a_i+a_{n+1})}} \le \frac{\sqrt[n]{\prod\limits_{i=1}^{n}a_i}}{\sqrt[n+1]{\prod\limits_{i=1}^{n+1}a_i}}. \tag{11}$$

If φ is increasing, convex, positive and defined on $(0,\infty)$ with $\left\{\varphi(i)\left[\frac{\varphi(i)}{\varphi(i+1)}-1\right]\right\}_{i\in\mathbb{N}}$ decreases, then

$$\frac{[\varphi(n)]^{n/\varphi(n)}}{[\varphi(n+1)]^{(n+1)/\varphi(n+1)}} \le \frac{\sqrt[\varphi(n)]{\prod\limits_{i=1}^{n}[\varphi(i)+\varphi(n)]}}{\sqrt[\varphi(n+1)]{\prod\limits_{i=1}^{n+1}[\varphi(i)+\varphi(n+1)]}}. \tag{12}$$

There are much literature devoted to research of ratios of mean values, for example [50]. For more detailed information, please refer to references in this paper and references therein.

1.3. **Monotonicity and convexity of functions involving gamma functions and ratio of gamma functions.** It is well-known that the incomplete gamma function is defined and denoted for $\operatorname{Re} z > 0$ by

$$\Gamma(z,x) = \int_x^\infty t^{z-1}e^{-t}\,\mathrm{d}t, \quad \gamma(z,x) = \int_0^x t^{z-1}e^{-t}\,\mathrm{d}t, \tag{13}$$

with $\Gamma(z,0)=\Gamma(z)$ and $\Gamma(0,x)=E_1(x)$ is called the exponential integral.

In [18], the following monotonicity results for the gamma function were established: The function $[\Gamma(1+\frac{1}{x})]^x$ decreases with $x>0$ and $x[\Gamma(1+\frac{1}{x})]^x$ increases with $x>0$, which recover the inequalities (1), which refer to integer values of n. These are equivalent to the function $[\Gamma(1+x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0,\infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})^x]$ decreases for $0<x<1$, where $\gamma=0.57721566\cdots$ denotes the Euler-Mascheroni constant, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In [6, 39], it is proved that the function $f(x)=\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing and strictly logarithmically convex in $(0,\infty)$ and the function $g(x)=\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing and strictly logarithmically concave in $(0,\infty)$. Moreover, if s is a positive real number, then for all real numbers $x>0$,

$$\frac{e^{-\gamma}}{[\Gamma(s+1)]^{1/s}} < \frac{[\Gamma(x+1)]^{1/x}}{[\Gamma(x+s+1)]^{1/(x+s)}} < 1, \tag{14}$$

$\lim_{x\to0} f(x)=e^{-\gamma}$ and $\lim_{x\to\infty} f(x)=e^{-1}$.

Using monotonicity properties and inequalities of the generalized weighted mean values (see [13, 30, 32, 37, 51]), the first author proved [35] that the functions $\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}$, $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$ and $\left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$ are increasing in $r>0$, $s>0$ and $x>0$; for any given $x>0$, the function $\frac{s\gamma(s,x)}{x^s}$ is decreasing in $s>0$. These generalize and extend the related results in [10, 11, 17, 18, 27] for the range of the argument. For more inequalities of quotients between gamma functions can be found in [16], [28, p. 526] and [52, pp. 442–443].

Using the approach by A. Laforgia and S. Sismondi in [20], some more general inequalities of the functions $\int_0^x e^{p^t}\,\mathrm{d}t$ and $\int_0^x e^{-p^t}\,\mathrm{d}t=\frac{\Gamma(1/p)-\Gamma(1/p,x^p)}{p}$ for $p>0$ and $x>0$ are obtained in [49]. These two functions are also been investigated by utilizing Tchebysheff integral inequality and Hermite-Hadamard integral inequality in [40, 46] by the first author and coworkers. For more information, please refer to [8].

In [9], Elezović, Giordana and Pečarić, among others, verified the convexity with respect to variable x of the function $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}$ for $|t-s|<1$, obtained the best bounds for $\frac{\Gamma(x+1)}{\Gamma(x+s)}$, where $s\in(0,1)$ and $x\ge1$, given some different approach from Gautschi's in [11], proved several new simple inequalities for digamma function, and improved related results by D. Kershaw in [17].

In [53], it is shown that the function $1+\frac{1}{x}\ln\Gamma(x+1)-\ln(x+1)$ is strictly completely monotone on $(-1,\infty)$ and tends to 1 as $x\to-1$ and to 0 as $x\to\infty$.

In [14], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{15}$$

is decreasing in $x\ge1$ for fixed $y\ge0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2}\le\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{16}$$

Inequality (16) extends and generalizes inequality (9), since $\Gamma(n+1)=n!$.

In this article, we are about to prove monotonicity and convexity properties of ratio between $\sqrt[x]{\Gamma(x+1)}$ and $\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}$ which are generalizations of the geometric means. Our main results are as follows.

Theorem 1. *For $\alpha>0$ a real number, the function $\frac{\sqrt[x]{\Gamma(x+1)}}{\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}}$ is increasing with $x\in(x_0,\infty)$, where $x_0\in(0,1)$ is a constant.*

Theorem 2. *For $\alpha>0$ a real number, the function $\frac{\sqrt[x]{\Gamma(x+1)}}{\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}}$ is logarithmically concave with $x\in[1,\infty)$.*

Remark 2. Basing on the graph of $\frac{\sqrt[x]{\Gamma(x+1)}}{\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}}$ pictured by Mathematica, we conjecture that the function $\frac{\sqrt[x]{\Gamma(x+1)}}{\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}}$ is increasing and logarithmically concave with $x\in(-1,\infty)$ for a given $\alpha>0$.

2. Lemma

It is well known that the Bernoulli numbers B_n is defined ([1] and [54, p. 1]) in general by

$$\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^{2n}}{(2n)!}B_n. \tag{17}$$

In particular, we have the following

$$B_1=\frac{1}{6}, \quad B_2=\frac{1}{30}, \quad B_3=\frac{1}{42}, \quad B_4=\frac{1}{30}. \tag{18}$$

In [54, p. 45], the following summation formula is given

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}}=\frac{\pi^{2k+1}E_k}{2^{2k+2}(2k)!} \tag{19}$$

for nonnegative integer k, where E_k denotes the Euler number, which implies

$$B_n=\frac{2(2n)!}{(2\pi)^{2n}}\sum_{m=1}^{\infty}\frac{1}{m^{2n}}, \quad n\in\mathbb{N}. \tag{20}$$

Remark 3. Recently, the Bernoulli and Euler numbers and polynomials are generalized in [12, 21, 22, 23, 24, 25] and some unpublished papers by the authors and coworkers.

Lemma 1. *For real number $x>0$ and natural number m, we have*

$$\begin{aligned}\ln\Gamma(x)&=\frac{1}{2}\ln(2\pi)+\Big(x-\frac{1}{2}\Big)\ln x-x+\sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{2(2n-1)n}\cdot\frac{1}{x^{2n-1}}\\&\quad+(-1)^m\theta_1\cdot\frac{B_{m+1}}{(2m+1)(2m+2)}\cdot\frac{1}{x^{2m+1}}, \quad 0<\theta_1<1;\end{aligned} \tag{21}$$

$$\begin{aligned}\psi(x)&=\ln x-\frac{1}{2x}+\sum_{n=1}^{m}(-1)^n\frac{B_n}{2n}\cdot\frac{1}{x^{2n}}+(-1)^{m+1}\theta_2\cdot\frac{B_{m+1}}{2m+2}\cdot\frac{1}{x^{2m+2}};\\&\quad 0<\theta_2<1,\end{aligned} \tag{22}$$

$$\psi'(x)=\frac{1}{x}+\frac{1}{2x^2}+\sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{x^{2n+1}}+(-1)^m\theta_3\cdot\frac{B_{m+1}}{x^{2m+3}}, \quad 0<\theta_3<1; \tag{23}$$

$$\begin{aligned}\psi''(x)&=-\frac{1}{x^2}-\frac{1}{x^3}+\sum_{n=1}^{m}(-1)^n(2n+1)\frac{B_n}{x^{2n+3}}\\&\quad+(-1)^{m+1}(2m+3)\theta_4\cdot\frac{B_{m+1}}{x^{2m+4}}, \quad 0<\theta_4<1.\end{aligned} \tag{24}$$

Proof. Let $x>0$, then we have

$$\begin{aligned}\ln\Gamma(x)&=\frac{1}{2}\ln(2\pi)+\Big(x-\frac{1}{2}\Big)\ln x-x-\int_{-\infty}^{0}\Big(\frac{1}{2}-\frac{1}{t}-\frac{1}{1-e^t}\Big)\frac{e^{xt}}{t}\,\mathrm{d}t\\&\triangleq\frac{1}{2}\ln(2\pi)+\Big(x-\frac{1}{2}\Big)\ln x-x-\omega(x),\end{aligned} \tag{25}$$

which is called the first Binet's formula. See [1] and [54, p. 106].

It is well-known that

$$\coth x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 + k^2\pi^2}$$

for $x \neq 0$, and

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \coth \frac{x}{2},$$

therefore

$$\frac{1}{x}\left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) = 2\sum_{k=1}^{\infty} \frac{1}{x^2 + 4\pi^2k^2}. \tag{26}$$

For any given natural number m, we have

$$\frac{1}{x^2 + 4\pi^2k^2} = \sum_{n=1}^{m}(-1)^{n-1}\frac{x^{2(n-1)}}{(4\pi^2k^2)^n} + (-1)^m\frac{x^{2m}}{(4\pi^2k^2)^{m+1}} \cdot \frac{1}{1 + \frac{x^2}{4\pi^2k^2}}, \tag{27}$$

for $1 \le n \le m$, we have

$$\sum_{k=1}^{\infty}(-1)^{n-1}\frac{x^{2(n-1)}}{(4\pi^2k^2)^n} = (-1)^{n-1}\frac{x^{2(n-1)}}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}}. \tag{28}$$

Substituting (20) into (28) leads to

$$\sum_{k=1}^{\infty}(-1)^{n-1}\frac{x^{2(n-1)}}{(4\pi^2k^2)^n} = (-1)^{n-1}\frac{B_n}{2(2n)!}x^{2(n-1)}, \quad 1 \le n \le m. \tag{29}$$

Summing up on both sides of (27) over $k \in \mathbb{N}$ yields

$$\sum_{k=1}^{\infty}(-1)^m\frac{x^{2m}}{(4\pi^2k^2)^{m+1}} \cdot \frac{1}{1 + \frac{x^2}{4\pi^2k^2}} = (-1)^m\tilde{\theta}\frac{B_{m+1}}{2(2m+2)!}x^{2m}, \tag{30}$$

where $\tilde{\theta}$ is a positive proper fraction (This means that $0 < \tilde{\theta} < 1$) and dependent on x. Hence, we have the following

$$\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} = \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{(2n)!}x^{2n-1} + (-1)^m\tilde{\theta}\frac{B_{m+1}}{(2m+2)!}x^{2m+1}. \tag{31}$$

It is easy to see that

$$\int_{-\infty}^{0} e^{xt}t^{2n-2}\,\mathrm{d}t = \int_{0}^{\infty} e^{-xt}t^{2n-2}\,\mathrm{d}t = \frac{(2n-2)!}{x^{2n-1}}, \tag{32}$$

$$\int_{-\infty}^{0} \tilde{\theta}e^{xt}t^{2m}\,\mathrm{d}t = \theta_1\int_{-\infty}^{0} e^{xt}t^{2m}\,\mathrm{d}t = \theta_1\frac{(2m)!}{x^{2m+1}}, \tag{33}$$

where $0 < \theta_1 < 1$ and θ_1 is indepent on x.

Substituting (32) and (33) into $\omega(x)$ reveals

$$\omega(x) = \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{2n(2n-1)} \cdot \frac{1}{x^{2n-1}} + (-1)^m\theta_1\frac{B_{m+1}}{(2m+1)(2m+2)} \cdot \frac{1}{x^{2m+1}}, \tag{34}$$

and then formula (21) follows.

Differentiating on both sides of (25) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x) = \ln x - \frac{1}{2x} + \omega'(x), \tag{35}$$

Easy computation gives

$$\int_{-\infty}^{0} e^{xt}t^{2n-1}\,\mathrm{d}t = -\frac{1}{x^{2n}}\int_{0}^{\infty} e^{-t}t^{2n-1}\,\mathrm{d}t = -\frac{(2n-1)!}{x^{2n}}, \tag{36}$$

$$\int_{-\infty}^{0} \tilde{\theta}e^{xt}t^{2m+1}\,\mathrm{d}t = \theta_2\int_{-\infty}^{0} e^{xt}t^{2m+1}\,\mathrm{d}t = -\theta_2\frac{(2m+1)!}{x^{2m+2}}, \tag{37}$$

where θ_2 is independent of x and $0<\theta_2<1$.

Substituting (31) into $\omega'(x)$ and utilizing (36) and (37) shows

$$\omega'(x) = \sum_{n=1}^{m}(-1)^n\frac{B_n}{2n}\cdot\frac{1}{x^{2n}} + (-1)^{m+1}\theta_2\frac{B_{m+1}}{2m+2}\cdot\frac{1}{x^{2m+2}}, \quad 0<\theta_2<1. \tag{38}$$

Formula (22) follows from combinating of (38) with $\omega'(x)$.

Differentiating with x on both sides of (35) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x) = \frac{1}{x} + \frac{1}{2x^2} + \omega''(x), \tag{39}$$

substituting (31) into $\omega''(x)$ and integrating directly produces

$$\begin{aligned}
\omega''(x) &= \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{(2n)!}\int_{-\infty}^{0} t^{2n}e^{xt}\,\mathrm{d}t + (-1)^m\frac{B_{m+1}}{(2m+2)!}\int_{-\infty}^{0}\tilde{\theta}t^{2m+2}e^{xt}\,\mathrm{d}t\\
&= \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{x^{2n+1}} + (-1)^m\frac{B_{m+1}}{(2m+2)!}\theta_3\int_{-\infty}^{0} t^{2m+2}e^{xt}\,\mathrm{d}t\\
&= \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{x^{2n+1}} + (-1)^m\theta_3\frac{B_{m+1}}{x^{2m+3}},
\end{aligned}$$

where θ_3 is independent of x and $0<\theta_3<1$. Formula (23) follows.

By the same argument as above, we obtain

$$\omega'''(x) = \sum_{n=1}^{m}\frac{(-1)^n(2n+1)B_n}{x^{2n+3}} + \frac{(-1)^{m+1}\theta_4(2m+3)B_{m+1}}{x^{2m+4}}, \tag{40}$$

where θ_4 is independent of x and $0<\theta_4<1$. Then formula (24) is proved. □

3. Proofs of Theorem 1 and 2

Proof of Theorem 1. For $\alpha>0$, let

$$f_\alpha(x) = \frac{\sqrt[x]{\Gamma(x+1)}}{\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}} \tag{41}$$

for $x>-1$. By direct calculation, we obtain

$$\ln f_\alpha(x) = \frac{\ln\Gamma(x+1)}{x} - \frac{\ln\Gamma(x+\alpha+1)}{x+\alpha}, \tag{42}$$

$$[\ln f_\alpha(x)]' = \left[\frac{\psi(x+1)}{x} - \frac{\ln\Gamma(x+1)}{x^2}\right] - \left[\frac{\psi(x+\alpha+1)}{x+\alpha} - \frac{\ln\Gamma(x+\alpha+1)}{(x+\alpha)^2}\right] \tag{43}$$

$$\triangleq g(x) - g(x+\alpha), \tag{44}$$

and

$$g'(x) = \frac{2\ln\Gamma(x+1) - 2x\psi(x+1) + x^2\psi'(x+1)}{x^3} \triangleq \frac{h(x)}{x^3}, \tag{45}$$

$$g''(x)=\frac{x^3\psi''(x+1)-3x^2\psi'(x+1)+6x\psi(x+1)-6\ln\Gamma(x+1)}{x^4}\triangleq\frac{p(x)}{x^4},\quad(46)$$

where $\psi(x)=\frac{[\Gamma(x)]'}{\Gamma(x)}$ is known as the digamma function, the logarithmic derivative of $\Gamma(x)$. Therefore, it is sufficient to verify $h(x)<0$ for $x>0$ and $h(x)>0$ for $-1<x<0$.

Using the inequality

$$\ln(1+t)\le\frac{t(2+t)}{2(1+t)}\quad(47)$$

for $t\ge0$ in [38] and the special cases $m=2$ of formulas (21), (22) and (23), we have

$$\begin{aligned}
h(x)&=2\ln\Gamma(x+1)-2x\psi(x+1)+x^2\psi'(x+1)\\
&<2\biggl[\frac12\ln(2\pi)+\biggl(x+\frac12\biggr)\ln(x+1)-(x+1)+\frac{1}{12(x+1)}\\
&\quad-\frac{1}{360(x+1)^3}+\frac{1}{1260(x+1)^5}\biggr]-2x\biggl[\ln(x+1)-\frac{1}{2(x+1)}\\
&\quad-\frac{1}{12(x+1)^2}+\frac{1}{120(x+1)^4}-\frac{1}{252(x+1)^6}\biggr]+x^2\biggl[\frac{1}{x+1}\\
&\quad+\frac{1}{2(x+1)^2}+\frac{1}{6(x+1)^3}-\frac{1}{30(x+1)^5}+\frac{1}{42(x+1)^7}\biggr]\\
&=\ln(2\pi)-2+\ln(x+1)-2x+\frac{6x^2+6x+1}{6(x+1)}\\
&\quad+\frac{3x^2+x}{6(x+1)^2}+\frac{30x^2-1}{180(x+1)^3}-\frac{x}{60(x+1)^4}\\
&\quad+\frac{1-21x^2}{630(x+1)^5}+\frac{x}{126(x+1)^6}+\frac{x^2}{42(x+1)^7}\\
&<\ln(2\pi)-2-2x+\frac{9x^2+12x+1}{6(x+1)}+\frac{3x^2+x}{6(x+1)^2}+\frac{30x^2-1}{180(x+1)^3}\\
&\quad-\frac{x}{60(x+1)^4}+\frac{1-21x^2}{630(x+1)^5}+\frac{x}{126(x+1)^6}+\frac{x^2}{42(x+1)^7}\\
&=\ln(2\pi)-1-\frac{x}{2}-\frac{1}{x+1}+\frac{1}{9(x+1)^3}\\
&\quad+\frac{1}{12(x+1)^4}-\frac{1}{18(x+1)^6}+\frac{1}{42(x+1)^7}\\
&\triangleq\ln(2\pi)-\frac12+\phi\biggl(\frac{1}{x+1}\biggr),
\end{aligned}\quad(48)$$

and, for $y\in(0,1]$,

$$\begin{aligned}
\phi'(y)&=-1+\frac{1}{2y^2}+\frac{y^2}{3}+\frac{y^3}{3}-\frac{y^5}{3}+\frac{y^6}{6},\\
\phi''(y)&=-\frac{1}{y^3}+\frac{2y}{3}+y^2-\frac{5y^4}{3}+y^5,\\
\phi^{(3)}(y)&=\frac23+\frac{3}{y^4}+2y-\frac{20y^3}{3}+5y^4,
\end{aligned}$$

$$\begin{aligned}
\phi^{(4)}(y)&=2-\frac{12}{y^5}-20y^2+20y^3,\\
\phi^{(5)}(y)&=\frac{60}{y^6}-40y+60y^2,\\
\phi^{(6)}(y)&=-40-\frac{360}{y^7}+120y,\\
\phi^{(7)}(y)&=120+\frac{2520}{y^8}.
\end{aligned}$$

It is clear that $\phi^{(7)}(y)>0$ and $\phi^{(6)}(y)$ is increasing. Since $\phi^{(6)}(1)=-280$ and $\lim_{y\to0}\phi^{(6)}(y)=-\infty$, we have $\phi^{(6)}(y)<0$ and $\phi^{(5)}(y)$ is decreasing. It is easy to see that $\lim_{y\to0}\phi^{(5)}(y)=\infty$ and $\phi^{(5)}(1)=80$, thus $\phi^{(5)}(y)>0$ and then $\phi^{(4)}(y)$ is increasing. From $\lim_{y\to0}\phi^{(4)}(y)=-\infty$ and $\phi^{(4)}(1)=-10$, it is deduced that $\phi^{(4)}(y)<0$, hence $\phi^{(3)}(y)$ decreases. From $\lim_{y\to0}\phi^{(3)}(y)=\infty$ and $\phi^{(3)}(1)=4$, it is concluded that $\phi^{(3)}(y)>0$, therefore $\phi''(y)$ increases. Because of $\lim_{y\to0}\phi''(y)=-\infty$ and $\phi''(1)=0$, we obtain $\phi''(y)\le0$, then $\phi'(y)$ decreases. By $\lim_{y\to0}\phi'(y)=\infty$ and $\phi'(1)=0$, it follows that $\phi'(y)\ge0$, and then $\phi(y)$ is increasing in $(0,1]$.

Utilizing monotonicity property of $\phi(y)$ and the inequality

$$h(x)<\ln(2\pi)-\frac12+\phi\Big(\frac1{x+1}\Big) \tag{49}$$

with

$$\ln(2\pi)-\frac12+\phi\Big(\frac1{0+1}\Big)=\ln(2\pi)-\frac12-\frac{337}{252}=\ln(2\pi)-\frac{463}{252}>0 \tag{50}$$

and

$$\ln(2\pi)-\frac12+\phi\Big(\frac1{1+1}\Big)=\ln(2\pi)-\frac12-\frac{2655}{1792}=\ln(2\pi)-\frac{3551}{1792}<0, \tag{51}$$

we conclude that there exists a point $x_0\in(0,1)$ such that $h(x)<0$ in $x\in(x_0,\infty)$. This implies $g'(x)<0$ for $x\in(x_0,\infty)$ and $g(x)$ is decreasing in (x_0,∞). Hence $[\ln f_\alpha(x)]'>0$ in (x_0,∞), and then $\ln f_\alpha(x)$ is increasing in (x_0,∞), that is, $f_\alpha(x)$ is increasing in (x_0,∞). The proof is complete. □

Proof of Theorem 2. Using the inequality (47) for $t\ge0$ in [38] and the special cases $m=2$ of formulas (21), (22), (23) and (24), we obtain

$$\begin{aligned}
p(x)&=x^3\psi''(x+1)-3x^2\psi'(x+1)+6x\psi(x+1)-6\ln\Gamma(x+1)\\
&>x^3\bigg[-\frac1{(x+1)^2}-\frac1{(x+1)^3}-\frac1{2(x+1)^5}+\frac1{6(x+1)^7}-\frac1{6(x+1)^8}\bigg]\\
&\quad-3x^2\bigg[\frac1{x+1}+\frac1{2(x+1)^2}+\frac1{6(x+1)^3}-\frac1{30(x+1)^5}+\frac1{42(x+1)^7}\bigg]\\
&\quad+6x\bigg[\ln(x+1)-\frac1{2(x+1)}-\frac1{12(x+1)^2}+\frac1{120(x+1)^4}-\frac1{252(x+1)^6}\bigg]\\
&\quad-6\bigg[\frac12\ln(2\pi)+\bigg(x+\frac12\bigg)\ln(x+1)-(x+1)+\frac1{12(x+1)}\\
&\quad-\frac1{360(x+1)^3}+\frac1{1260(x+1)^5}\bigg]
\end{aligned}$$

$$
\begin{aligned}
&> x^3\left[-\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}-\frac{1}{2(x+1)^5}+\frac{1}{6(x+1)^7}-\frac{1}{6(x+1)^8}\right]\\
&\quad-3x^2\left[\frac{1}{x+1}+\frac{1}{2(x+1)^2}+\frac{1}{6(x+1)^3}-\frac{1}{30(x+1)^5}+\frac{1}{42(x+1)^7}\right]\\
&\quad+6x\left[-\frac{1}{2(x+1)}-\frac{1}{12(x+1)^2}+\frac{1}{120(x+1)^4}-\frac{1}{252(x+1)^6}\right]\\
&\quad-6\left[\frac{1}{2}\ln(2\pi)-(x+1)+\frac{1}{12(x+1)}-\frac{1}{360(x+1)^3}+\frac{1}{1260(x+1)^5}\right]\\
&\quad-\frac{3x(2+x)}{2(1+x)}\\
&= 4-3\ln(2\pi)+\frac{x}{2}+\frac{3}{x+1}-\frac{5}{2(x+1)^2}+\frac{13}{6(x+1)^3}-\frac{19}{12(x+1)^4}\\
&\quad-\frac{1}{6(x+1)^5}+\frac{7}{6(x+1)^6}-\frac{31}{42(x+1)^7}+\frac{1}{6(x+1)^8}\\
&\triangleq (x+1)q\left(\frac{1}{x+1}\right),
\end{aligned}
$$

and, for $t\in[0,\frac{1}{2}]$,

$$
\begin{aligned}
q'(t)&=\frac{7}{2}-3\ln(2\pi)+6t-\frac{15t^2}{2}+\frac{26t^3}{3}-\frac{95t^4}{12}-t^5+\frac{49t^6}{6}-\frac{124t^7}{21}+\frac{3t^8}{2},\\
q''(t)&=6-15t+26t^2-\frac{95t^3}{3}-5t^4+49t^5-\frac{124t^6}{3}+12t^7,\\
q^{(3)}(t)&=-15+52t-95t^2-20t^3+245t^4-248t^5+84t^6,\\
q^{(4)}(t)&=52-190t-60t^2+980t^3-1240t^4+504t^5,\\
q^{(5)}(t)&=-190-120t+2940t^2-4960t^3+2520t^4,\\
q^{(6)}(t)&=-120+5880t-14880t^2+10080t^3,\\
q^{(7)}(t)&=5880-29760t+30240t^2,\\
q^{(8)}(t)&=-29760+60480t.
\end{aligned}
$$

It is easy to see that $t=\frac{31}{63}$ is an unique minimum point of $q^{(7)}(t)$ on $[0,\frac{1}{2}]$. Since $q^{(7)}(0)=5880$ and $q^{(7)}(\frac{1}{2})=-1440$, thus $q^{(6)}(t)$ has an unique maximum on $[0,\frac{1}{2}]$. Since $q^{(6)}(0)=-120$ and $q^{(6)}(\frac{1}{2})=360$, then $q^{(5)}(t)$ has an unique minimum on $[0,\frac{1}{2}]$. Because of $q^{(5)}(0)=-190$ and $q^{(5)}(\frac{1}{2})=\frac{45}{2}$, therefore $q^{(4)}(t)$ has an unique minimum on $[0,\frac{1}{2}]$. The unique zero point of $q^{(5)}(t)$ in $[0,\frac{1}{2}]$ is

$$
\begin{aligned}
t_0&=\frac{31}{63}+\frac{1}{756}\sqrt{\frac{1}{7}\left[381528-1323\kappa-7938\mu+\frac{294848}{\nu}\right]}-\frac{\nu}{2} \qquad (52)\\
&=0.4437889482188733\cdots,
\end{aligned}
$$

where

$$\kappa=\sqrt[3]{5600664-1296\sqrt{17855817}}, \qquad (53)$$

$$\mu=\sqrt[3]{3\left(8643+2\sqrt{17855817}\right)}, \qquad (54)$$

$$\nu=\sqrt{\frac{757}{3969}+\frac{1}{756}\sqrt[3]{5600664-1296\sqrt{17855817}}+\frac{\sqrt[3]{8643+2\sqrt{17855817}}}{42\sqrt[3]{9}}}\,, \tag{55}$$

the minimum $q^{(4)}(t_0)=0.03717920\cdots$. Hence $q^{(4)}(t)>0$ and $q^{(3)}(t)$ is increasing on $[0,\frac12]$. From $q^{(3)}(\frac12)=-\frac{51}{8}$, it follows that $q^{(3)}(t)<0$ and $q''(t)$ is decreasing on $[0,\frac12]$. From $q''(\frac12)=\frac{41}{24}$, it is deduced that $q''(t)>0$ and $q'(t)$ is increasing on $[0,\frac12]$. Since $q'(\frac12)=\frac{56659}{10752}-3\ln(2\pi)<0$, we have $q'(t)<0$ and $q(t)$ is decreasing on $[0,\frac12]$. From $q(\frac12)=\frac{22093}{21504}+\frac12(\frac72-3\ln(2\pi))>0$, it is concluded that the function $q(t)>0$ on $[0,\frac12]$.

Note that $q(t)>0$ with $t\in(0,\frac12]$ is equivalent to $q\bigl(\frac{1}{x+1}\bigr)>0$ with $x\in[1,\infty)$. This implies that $p(x)>0$ and $g''(x)>0$ with $x\in[1,\infty)$, then $g'(x)$ is increasing and $g(x)$ is convex with $x\in[1,\infty)$. Therefore $[\ln f_\alpha(x)]''=g'(x)-g'(x+\alpha)<0$, that is, the function $f_\alpha(x)$ is logarithmically concave on $[1,\infty)$. The proof is complete. □

Remark 4. To prove that $\frac{\sqrt[x]{\Gamma(x+1)}}{\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}}$ is increasing and logarithmically concave with $x\in(-1,\infty)$ for a given $\alpha>0$, it is sufficient to verify

$$h(x)=x^3\left(\tau''(x)-\frac{1}{x^2+1}\right)\le 0, \tag{56}$$

$$p(x)=x^4\tau'''(x)-12+5x+\frac{2}{(1+x)^3}-\frac{11}{(1+x)^2}+\frac{21}{1+x}\ge 0, \tag{57}$$

where

$$\tau(x)=\frac1x\int_0^\infty\left(\frac1t-\frac{1}{e^t-1}\right)e^{-t}\frac{1-e^{-xt}}{t}\,\mathrm{d}t. \tag{58}$$

We will give proofs of (56) and (57) in a subsequent article.

4. Corollaries

As by-products, from Theorem 1 and 2, the following corollaries are deduced.

Corollary 1. *The function* $\frac{\psi(x+1)}{x}-\frac{\ln\Gamma(x+1)}{x^2}$ *is decreasing and convex on* $[1,\infty)$.

Corollary 2. *For* $x\in[1,\infty)$, *we have*

$$\ln\Gamma(x+1)<x\psi(x+1)-\frac{x^2}{2}\psi'(x+1), \tag{59}$$

$$\ln\Gamma(x+1)<x\psi(x+1)-\frac{x^2}{2}\psi'(x+1)+\frac{x^3}{6}\psi''(x+1). \tag{60}$$

Remark 5. It is conjectured that Corallary 1 and 2 are valid on the interval $(-1,\infty)$.

Corollary 3. *Let* n *be natural number. Then the sequence*

$$\frac{\sqrt[n]{n!}}{\sqrt[n+k]{(n+k+1)!}} \tag{61}$$

are increasing with $k\in\mathbb{N}$.

5. Open problems

The function $\frac{\sqrt[x]{\Gamma(x+1)}}{\sqrt[x+\alpha]{\Gamma(x+\alpha+1)}}$ can be expressed as

$$\frac{\sqrt[x]{\int_0^\infty t^x e^{-t}\,\mathrm{d}t}}{\sqrt[x+\alpha]{\int_0^\infty t^{x+\alpha} e^{-t}\,\mathrm{d}t}}, \tag{62}$$

where $\int_0^\infty e^{-t}\,\mathrm{d}t = 1$. Then we propose the following

Open Problem 1. *Let $w(x) \geq 0$ be a nonnegative weight defined on a domain Ω with $\int_\Omega w(x)\,\mathrm{d}x = 1$. Find conditions about $w(x)$ and $f(x) \geq 0$ such that the ratio between two power means*

$$\mathcal{Q}(t) = \frac{\left[\int_\Omega w(x) f^t(x)\,\mathrm{d}x\right]^{1/t}}{\left[\int_\Omega w(x) f^{t+\alpha}(x)\,\mathrm{d}x\right]^{1/(t+\alpha)}} \tag{63}$$

is monotonic or convex with $t \in \mathbb{R}$ for a given $\alpha > 0$.

Open Problem 2. *Find conditions about the positive sequence $\{a_i\}_{n\in\mathbb{N}}$ such that the function*

$$\mathcal{F}(r) = \left(\frac{\frac{1}{n}\sum_{i=1}^n a_i^r}{\frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r}\right)^{1/r} \tag{64}$$

is monotonic or convex with $r \in \mathbb{R}$, where n and m are two given natural numbers. In particular, for $\{a_i\}_{n\in\mathbb{N}}$ being the natural number sequence (that is, $a_i = i$), show that the function $\mathcal{F}(r)$ is decreasing strictly with $r \in \mathbb{R}$.

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